# p-Strong Roman Domination in Graphs

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*Abstract:* Domination in graphs is a widely studied field, where many different definitions have been introduced in the last years to respond to different network requirements. This paper presents a new dominating parameter based on the well-known strong Roman domination model. Given a positive integer $p$, we call a $p$-strong Roman domination function ($p$-StRDF) in a graph $G$ to a function $f : V(G) \to \{0, 1, 2, \ldots, \left\lceil \frac{\Delta+p}{p} \right\rceil\}$ having the property that if $f(v) = 0$, then there is a vertex $u \in N(v)$ such that $f(u) \geq 1 + \left\lceil \frac{|B_0 \cap N(u)|}{p} \right\rceil$, where $B_0$ is the set of vertices with label 0. The $p$-strong Roman domination number $\gamma^p_{StR}(G)$ is the minimum weight (sum of labels) of a $p$-StRDF on $G$. We study the NP-completeness of the $p$-*StRD*-problem, we also provide general and tight upper and lower bounds depending on several classical invariants of the graph and, finally, we determine the exact values for some families of graphs.

*Key-Words:* graph; NP-complete problem; domination; Roman domination; strong Roman domination; p-strong Roman domination.

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## 1 Introduction and Notation

Throughout this paper, $G$ is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of $v$ is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d_G(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We denote by $P_n$ ~~the~~ *path* of order $n$, $C_n$ for the *cycle* of length $n$ and $K_n$ for the edgeless graph with $n$ vertices.

A *leaf* of $G$ is a vertex of degree one, while a *support vertex* of $G$ is a vertex adjacent to a leaf. An $S$-*external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ adjacent to $v$ but no other vertex of $S$. The set of all $S$-external private neighbors of $v \in S$ is called the $S$-*external private neighborhood* of $v$ and is denoted $\text{epn}(v, S)$.

A *tree* is a connected graph containing no cycles. A tree $T$ is called a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively $p$ and $q$ leaves attached at each support vertex is denoted by $S_{p,q}$.

Given two different graphs $G$ and $H$, let us denote by $G \vee H$ the graph obtained by adding to $G \cup H$ all possible edges joining a vertex in $G$ with a vertex in $H$.

A set $S \subseteq V$ in a graph $G$ is called a *dominating set* if every vertex of $G$ is either in $S$ or adjacent to a vertex of $S$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in $G$.

The concept of Roman domination has arisen as a solution to a classic problem of military defensive strategy introduced by [1], [2], having its origin in the time of Emperor Constantine I. At that moment, the Roman Empire had more conquered cities than legions for their defense in case of an attack.

Defending a city was enough with a legion, which might be positioned in such a city or moved from another neighboring city. Then, Emperor Constantine I decreed a strategy based on two facts: first, any unprotected city should be able to be defended by a neighboring city and, second, no legion could come to defend an attacked neighboring city if such legion left unguarded its original location. The goal was to minimize the costs of settlement and mobilization of the legions while guaranteeing the possibility of defense of each position of the empire. For this, they could place up to two legions in each military settlement.

The first formal definition of Roman domination was introduced in 2004 by [3], inspired by the works mentioned above. A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on $G$ if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on $G$. Afterwards, the properties of this invariant have been extensively studied.

In recent years, other variations of Roman domination have been introduced, generally modifying the conditions in which the vertices are dominated, or adding some additional property to the classic version of the Roman domination. We highlight, for example: the *independent Roman domination*, [4], the *maximal Roman domination*, [5], the *weak Roman domination*, [6], the *edge Roman domination*, [7], the *total Roman domination*, [8], the *signed Roman domination*, [9], the *mixed Roman domination*, the *Roman $\{2\}$-domination*, [10], or the work by [11], on parallel algorithms for connected domination problems on interval and circular-arc graphs.

In all previous variants of Roman domination, it is assumed that a legion is enough to defend a position from individual attacks. However, there may be situations in which, even for individual attacks, this defensive strategy is insufficient. Other new variants are defined in [12], [13], to contemplate other situations. This approach can have vast applications in service network modeling such as distribution, maintenance or provisioning.

The attack capacity is increasingly wider, giving rise to new situations, for instance, when the attacks occur simultaneously. In such cases, the previous defensive strategies are weak and insufficient. Some works try to solve this problem. The study, [14], based on weak Roman domination, provides a new version of the defense of the Roman Empire against multiple and sequential attacks. Recently, in [15], the protection of a graph against sequential attacks on its vertices or edges is studied, by positioning mobile guards on vertices according to certain structures, for example, an eternal dominating set.

However, many real situations remain unresolved with these models, since nowadays, the attacks can be multiple and also simultaneous as, for instance, fires with several sources, synchronous natural disasters in different areas, joint attacks in cybernetics or security systems, etc. Several works address this approach. In 2009, the *Roman k-domination*, for $k \geq 1$, was defined, [16], to provide a reply against $k$ attacks in different vertices of a graph. A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman k-dominating function* on $G$ if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least $k$ vertices, $v_1, \ldots, v_k$, with $f(v_i) = 2$, for $i = 1, \ldots, k$. This defensive strategy is dependable, as a defenseless vertex is covered by $k$ neighbors, but it can sometimes be excessive or unnecessary.

In 2017, the *strong Roman domination* was introduced, [17], as a reinforcement of the Roman domination against multiple and simultaneous attacks, where legions are placed in strong vertices to defend themselves and, at least, half of its unsafe neighbors. For a graph $G$ of order $n$ and maximum degree $\Delta$, let $f : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ be a function that labels the vertices of $G$. Let $B_j = \{v \in V : f(v) = j\}$ for $j = 0, 1$ and let $B_2 = V \setminus (B_0 \cup B_1) = \{v \in V : f(v) \geq 2\}$. Then, $f$ is a strong Roman dominating function (StRDF) on $G$, if every $v \in B_0$ has a neighbour $u$, such that $u \in B_2$ and $f(u) \geq 1 + \left\lceil \frac{|N(u) \cap B_0|}{2} \right\rceil$. The minimum weight, $w(f) = f(V) = \sum_{u \in V} f(u)$, over all the strong Roman dominating functions for $G$, is called the *strong Roman domination number* of $G$ and we denote it by $\gamma_{StR}(G)$. An StRDF of minimum weight is called a $\gamma_{StR}(G)$-function. After this work, many relevant contributions on the strong Roman domination, [18], [19], [20], [21], [22], have been

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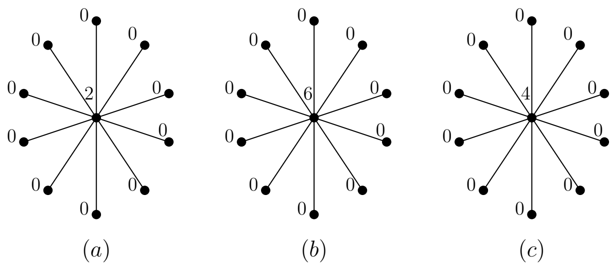


Fig. 1: For a star graph: (a) a RDF, (b) an StRDF, and (c) a 4-StRDF

provided.

Despite the particular interest of the strong Roman domination strategy and the further development of this kind of study, we may observe that the definition is certainly restrictive. In the StRD model, we consider simultaneous attacks to unprotected neighbors of strong vertices under the condition that the stronger vertex may defend, at least, one-half of its neighbors.

In this paper, we introduce *$p$-strong Roman domination*, a refined strategy of strong Roman domination that relaxes the definition, allowing for the development of less expensive defensive strategies.

**Definition 1.** *Given a positive integer $p$, a function $f : V(G) \to \{0, 1, \ldots, \left\lceil \frac{\Delta+p}{p} \right\rceil\}$ is a* $p$-strong Roman dominating function *(*$p$-StRDF*) if for every vertex $u \in B_0$ there is a vertex $v \in N(u)$ such that $f(v) \geq 1+\left\lceil \frac{|N(v)\cap B_0|}{p} \right\rceil$. The minimum weight of such a function is called the* $p$-strong Roman domination number *of the graph and it is denoted by $\gamma^p_{StR}(G)$.*

In other words, the strong Roman domination model ensures that each strong vertex is capable of defending at least half of its undefended neighbors without leaving its own location unprotected. This means that it has one unit to protect every group of two weak neighbors. In this case, the $p$-StRD model ensures that each strong position has at least one legion to defend each group of $p$ undefended neighbors. Figure 1 shows a clarifying example.

## 2 Complexity Results

This section aims to establish the NP-completeness of the $p$-StRD problem for bipartite and chordal graphs.

The following decision problem is associated with the optimization problem of calculating the $p$-StRD number of a given graph.

***pStRD-Number Problem***
**Instance**: Graph $G = (V, E)$ and a positive integer $r$.
**Question**: Does $G$ have a $p$-StRD function $f$ with $f(V) \leq r$?

We make use of the *Exact Cover by 3-Sets (X3C) problem* [23] to demonstrate that *pStRD-Number Problem* is NP-complete. Namely, an instance of X3C is the following

**EXACT 3-Cover (X3C) Problem**
**Instance**: A collection $C$ of 3-element subsets of a finite set $X$ with $|X| = 3q$.
**Question**: Does $X$ have an *exact cover* in $C$, that is, a subcollection $C' \subseteq C$ that contains every element of $X$ in exactly one member?

**Example.** Let $q = 2$ and $X = \{x_1, x_2, \ldots, x_6\}$ be a set of $3q$ literals, and let $C = \{(x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_5, x_6), (x_2, x_3, x_4), (x_3, x_5, x_6)\}$ be a collection of clauses *(*subsets of cardinality 3*)* of $X$. Clearly, $C' = \{(x_1, x_2, x_4), (x_3, x_5, x_6)\} \subseteq C$ is an exact cover of $C$, because each and every element of $X$ belongs to exactly one clause in $C'$. Note that $|C'| = q = 2$.

The main result of this section is presented in the following theorem.

**Theorem 2.** *The* $p$-StRD *number problem is NP-complete, even when restricted to bipartite or chordal graphs.*

**Proof.**

First, this problem belongs to the class of NP problems since we could verify, in polynomial time concerning the size $n$, whether a given possible solution is indeed a solution or not.

Next, we prove that *pStRD*-Number Problem is NP-complete for bipartite graphs by constructing a polynomial-time transformation from X3C problem, which is a well-known NP-problem [23].

Assume that $I = (X, C)$ is an arbitrary instance of X3C, with $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$.

The key of this proof is that we construct a bipartite graph $B(I)$ starting from $I$ and provide a positive integer $r$ such that $I$ contains an *exact cover* by 3-sets if and only if $B(I)$ has a $p$-StRD function $f$ having weight $w(f) \leq r = 2q + 3t$

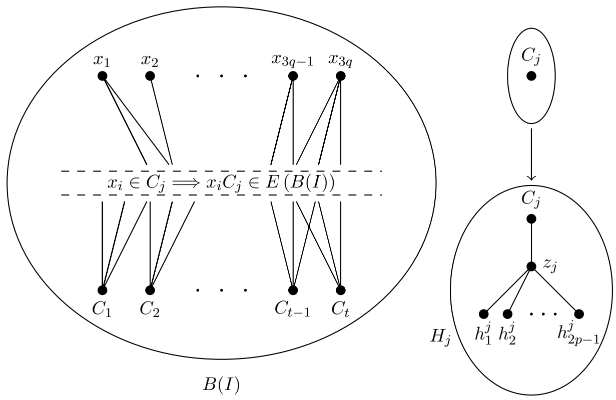


Fig 2: Constructing $B(I)$ and $C_j$-gadgets

Let $B(I)$ the bipartite graph with classes $X = \{x_i : 1 \leq i \leq 3q\}$ and $C = \{C_j : 1 \leq j \leq t\}$ being $x_i$ adjacent to $C_j$ if $x_i \in C_j$. Next, the vertices $C_j$ of the bipartite graph will be swapped out for an appropriate gadget.

For each $C_j \in C$, let $H_j$ be the star with central vertex $z_j$ and leaves $\{C_j, h_1^j, \ldots, h_{2p-1}^j\}$. We replace $C_j$ with $H_j$ by identifying $C_j$ with the corresponding leaf $C_j$ of $H_j$. The new graph, denoted by $B(I)$, is bipartite with classes $\{x_i : 1 \leq i \leq 3q\} \cup \{z_j : 1 \leq j \leq t\}$ and $\{C_j : 1 \leq j \leq t\} \cup \{h_l^j : 1 \leq j \leq t,\ 1 \leq l \leq 2p-1\}\}$. We set $r = 2q + 3t$. Of course, we can construct $B(I)$ in polynomial time on the size of the given instance. Figure 2 shows a diagram of the described construction.

By assuming that $C'$ is an exact cover for $X$ in $C$, we define the following function over $V(B(I))$:

$$f(v) = \begin{cases} 0 & \text{if} \quad v \in \{x_i : 1 \leq i \leq 3q\} \\ 0 & \text{if} \quad v \in \{h_l^j : 1 \leq j \leq t,\ 1 \leq l \leq 2p-1\} \\ 0 & \text{if} \quad v \in \{C_j : C_j \notin C', 1 \leq j \leq t\} \\ 2 & \text{if} \quad v \in \{C_j : C_j \in C', 1 \leq j \leq t\} \\ 3 & \text{if} \quad v \in \{z_j : 1 \leq j \leq t\} \end{cases}$$

Clearly, $C'$ has cardinality equal to $q$, because $C'$ covers $C$, $|C| = 3q$, and each clause in $C'$ has cardinality equal to 3. Besides, $f(\{C_j : C_j \in C', 1 \leq j \leq t\}) = 2q$.

Since $C'$ is an exact cover for $X$ in $C$, we know that for all $1 \leq i \leq 3q$ there exists $C_j \in C'$ with $x_i \in C_j$. Therefore $f(C_j) = 2 \geq 1 + \left\lceil \frac{|N(C_j) \cap B_0|}{p} \right\rceil = 1 + \left\lceil \frac{3}{p} \right\rceil = 2$ for $p \geq 3$. Furthermore, for each $v \in \{C_j : C_j \notin C', 1 \leq j \leq t\} \cup \{h_l^j : 1 \leq j \leq t,\ 1 \leq l \leq 2p-1\}$, there exists $k \in \{1 \ldots, t\}$ with $z_k \in N(v)$ and $f(z_k) = 3 \geq 1 + \left\lceil \frac{|N(z_k) \cap B_0|}{p} \right\rceil = 1 + 2$.

So, $f$ is a $p$-StRD function with $f(V(B(I))) = 2q + 3t = r$.

On the other hand, suppose now that there exists a $p$-StRD function $f$ with $f(V(B(I))) \leq r$. Without loss of generality, let $f$ be one of those functions that assigns as much label value as feasible to the set $\{z_j : 1 \leq j \leq t\}$.

Under these conditions, we may readily verify that $f(z_j) = 3$ and $f(h_l^j) = 0$ for all $1 \leq j \leq t$ and $1 \leq l \leq 2p-1$. On the contrary, if there exists $j \in \{1, \ldots, t\}$ and $l \in \{1, \ldots, 2p-1\}$ with $f(h_l^j) \geq 1$ then either $f(h_l^j) \geq 1$ for all $1 \leq l \leq 2p-1$ or $f(z_j) \geq 2$ because not all $f(h_l^j) \geq 1$. Anyhow, we can define $f^*$ such that $f^*(z_j) = 3$ and $f^*(h_l^j) = 0$.

Hence, it follows that

$$f\left(\{z_j : 1 \leq j \leq t\}) \cup \{h_l^j : 1 \leq j \leq t,\ 1 \leq l \leq 2p-1\}\right)$$

is equal to $3t$, and then
$f(\{c_j : 1 \leq j \leq t\}) \cup \{x_i : 1 \leq i \leq 3q\}) \leq r - 3t$.

Let us denote by $A_i = \{x \in X : f(x) = i\}$ and $a_i = |A_i|$, for $i = 0, 1$; $A_2 = \{x \in X : f(x) \geq 2\}$ and $a_2 = |A_2|$; $D_i = \{C_j \in C : f(C_j) = i\}$ and $d_i = |D_i|$, for $i = 0, 1$; $D_2 = \{C_j \in C : f(C_j) \geq 2\}$ and $d_2 = |D_2|$.

The following equalities, $a_0 + a_1 + a_2 = 3q$ and $d_0 + d_1 + d_2 = t$, are an immediate consequence of this notation. Note that $a_1 + 2a_2 + d_1 + 2d_2 \leq a_1 + f(A_2) + d_1 + f(D_2) \leq 2q$ because $f(V) \leq r$. Additionally, for all $x \in A_0$ there exists $C_j \in D_2$ with $x \in N(C_j)$ and then $d_2 \geq \frac{a_0}{3}$. We have that $2q \geq a_1 + 2a_2 + d_1 + 2d_2 = 3q - a_0 + a_2 + d_1 + 2d_2 \geq 3q - 3d_2 + a_2 + d_1 + 2d_2$ and therefore $d_2 - (a_2 + d_1) \geq q$. So, $d_2 \geq q$, and since $a_1 + 2a_2 + d_1 + 2d_2 \leq 2q$, it is verified that $d_2 = q$ and $f(C_j) = 2$ for all $C_j \in D_2$ and, finally $a_1 = a_2 = d_1 = 0$. As a result, we get that $d_1 = 0$ and $d_0 = t - q$. Since $|X| = 3q = 3|D_2|$, we have that the set $C' = \{C_j : C_j \in D_2\}$ is a subcollection $C' \subseteq C$ that contains every element of

$X$ in exactly one member of $C'$, and the result follows for bipartite graphs.

By adding all the edges between the vertices $C_j$'s, we obtain a chordal graph. Consequently, by using a similar proof to the one developed to arrive at the previous result, we can derive the one for chordal graphs.

□

## 3 General bounds

This section is dedicated to presenting general and different bounds for the $p$-strong Roman domination number in graphs, which is a natural step after checking the NP-completeness of the $p$-StRD Roman domination problem in the previous section.

First of all, let us see which values of parameter $p$ have to be considered.

Let $G$ be any graph of order $n$. Let $p$ be a positive integer and let $f$ be a $p$-StRD function having minimum weight in the graph $G$. Let us see that $3 \leq p \leq \Delta - 1$ have to be assumed.

If $p = 1$, then

$$\begin{aligned} w(f) &= \sum_{v\in B_1\cup B_2} f(v) = \sum_{v\in B_1} f(v) + \sum_{v\in B_2} f(v) \\ &\geq |B_1| + \sum_{v\in B_2}\left(1 + \left\lceil \frac{|N(v)\cap B_0|}{p} \right\rceil\right) \\ &= |B_1| + |B_2| + \sum_{v\in B_2} |N(v)\cap B_0|. \end{aligned}$$

Since $f$ is a $p$-StRD function of minimum weight, each $v \in B_2$ must have a private neighbor in $B_0$ and therefore $w(f) \geq |B_1| + |B_2| + |B_0| = |V(G)| = n$. Hence, for $p = 1$, the function $f(u) = 1$ for all $u \in V(G)$ is a 1-strong Roman domination function of minimum weight and $\gamma^p_{StR}(G) = n$.

If $p = 2$, taking into account the definition of a $p$-StRD function, we may derive that the strategy of 2-strong Roman domination is just the same as the one of the strong Roman domination model.

Finally, if $p \geq \Delta$, then $\left\lceil \frac{\Delta}{p} \right\rceil = 1$. Hence we have that $f : V(G) \to \{0,1,2\}$ and the condition $f(v) \geq 1 + \left\lceil \frac{|N(v)\cap B_0|}{p} \right\rceil$ is equivalent to $f(v) = 2$, because

$$1 < 1 + \left\lceil \frac{|N(v)\cap B_0|}{p} \right\rceil \leq 1 + \left\lceil \frac{\Delta}{p} \right\rceil = 2$$

Therefore, $p$-strong Roman domination corresponds to the original Roman domination model when $p \geq \Delta$.

Summing up, the $p$-StRD model is trivial for $p = 1$, matches the StRD strategy for $p = 2$ and coincides with the original Roman domination problem for all $p \geq \Delta$. Therefore, from now on, we will only consider $3 \leq p \leq \Delta - 1$. Observe that the latter implies that $\Delta \geq 4$.

As an immediate consequence of the definition, we can point out the following remark.

**Remark 3.** *Let $G$ be a connected graph having maximum degree $\Delta \geq 4$. Let $p, q$ be positive integers such that $3 \leq p \leq q \leq \Delta - 1$. Then,*

$$\gamma^p_{StR}(G) \geq \gamma^q_{StR}(G).$$

Of course, it is not difficult to relate our new parameter to some of the most well-known parameters in domination. As an initial bound for the $p$-StRD number, we prove the following result.

**Remark 4.** *Let $G$ be a connected graph having maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$. Then,*

$$\gamma_R(G) \leq \gamma^p_{StR}(G) \leq \left(\left\lceil \frac{\Delta}{p} \right\rceil + 1\right)\gamma(G).$$

**Proof.** For the lower bound, let $f = (B_0, B_1, B_2)$ be any $\gamma^p_{StR}(G)$-function on $G$ and define the function $g : V(G) \to \{0,1,2\}$, such that $g(u) = 2$ whenever $u \in B_2$ and $g(u) = f(u)$ otherwise. Hence, $\gamma_R(G) = |V_1| + 2|V_2| = |B_1| + 2|B_2| \leq \gamma^p_{StR}(G)$. On the other hand, let $D$ be a dominating set and let $f : V(G) \to \{0, 1, \ldots, \left\lceil \frac{\Delta}{p} \right\rceil + 1\}$ be the function defined as follows $f(u) = \left\lceil \frac{\Delta}{p} \right\rceil + 1$ for all $u \in D$ and $f(u) = 0$ otherwise. The function $f$ is a $p$-StRD function, which leads us to the upper bound.

□

Next, we prove an upper bound that only depends on the order and the maximum degree of the graph.

**Proposition 5.** *Let $G$ be a graph with order $n$ and maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$. Then*

$$\gamma^p_{StR}(G) \leq n - \Delta + \left\lceil \frac{\Delta}{p} \right\rceil$$

**Proof.** Let $u$ be a vertex with degree $\Delta$. Let us define the function $f: V(G) \to \{0, 1, \ldots, \left\lceil \frac{\Delta}{p} \right\rceil + 1\}$ as follows: $f(u) = \left\lceil \frac{\Delta}{p} \right\rceil + 1$, $f(v) = 0$ for all $v \in N(u)$ and $f(v) = 1$ otherwise. Taking into account that $B_2 = \{u\}, B_0 = N(u)$ and $B_1 = V \setminus N[u]$, then $f$ is a $p$-StRD function and therefore

$$\begin{aligned}\gamma^p_{StR}(G) &\leq w(f)\\ &= |B_1| + \sum_{x \in B_2} f(x)\\ &= |V \setminus N[u]| + f(u)\\ &= (n - \Delta - 1) + \left\lceil \tfrac{\Delta}{p} \right\rceil + 1\\ &= n - \Delta + \left\lceil \tfrac{\Delta}{p} \right\rceil.\end{aligned}$$

□

It is worth noting that the upper bound given by Proposition 5 is sharp, for example, for every star $K_{1,n-1}$ with $3 \leq p \leq n-2$.

**Corollary 6.** *Let $G$ be a graph with order $n$ and maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$. Then*

$$\gamma^p_{StR}(G) \leq n - 2$$

**Proof.** By applying Proposition 5, we have that

$$\begin{aligned}\gamma^p_{StR}(G) &\leq n - \Delta + \left\lceil \tfrac{\Delta}{p} \right\rceil \leq n - \Delta + \tfrac{\Delta}{p} + 1\\ &\leq n + 1 - \tfrac{2\Delta}{3} \leq n + 1 - \tfrac{8}{3} \leq n - 2.\end{aligned}$$

□

Our next result concerns improving the previous bound for $r$-regular graphs.

**Proposition 7.** *Let $3 \leq p \leq \Delta - 1$ be a positive integer and let $G$ be a $r$-regular graph, with $r \geq p+1$ and girth $g \geq 5$. Then*

$$\gamma^p_{StR}(G) \leq n - r^2 + \left( \left\lceil \frac{r-1}{p} \right\rceil + 1 \right) r$$

**Proof.** Consider any vertex $u \in V(G)$. Since $G$ is an $r$-regular graph, we have that $N(u)$ is a set of $r$ vertices, say $N(u) = \{w_1, \ldots, w_r\}$. Moreover, each one of the sets $N(w_j) - u$, for $j = 1 \ldots r$, is formed by $r-1$ different vertices, say $N(w_j) - u = \{z^j_1, \ldots, z^j_{r-1}\}$. Note that, due to the girth of $G$, $N(u)$ is an independent set; each set $N(w_j) - u$ is also an independent set; and they are disjoint set of vertices of $V(G)$.

Let us define a function $f$ as follows: $f(u) = 1$, $f(w_j) = 1 + \left\lceil \frac{r-1}{p} \right\rceil$ for all $1 \leq j \leq r$, $f(z^j_k) = 0$ for all $1 \leq k \leq r-1$, and $f(v) = 1$ for any non yet labelled vertex $v$, if any. As we have observed before, the vertices $u, w_j, z^j_k$ are all different, since the girth of $G$ is at least 5. Finally, any vertex $z^j_k$ is dominated by a vertex $w_j$ labelled with a label equal to $1 + \left\lceil \frac{r-1}{p} \right\rceil = 1 + \left\lceil \frac{|N(w_j) \cap B_0|}{p} \right\rceil$, therefore, the defined function $f = (B_0, B_1, B_2)$ is a $p$-StRD function on $G$ and it holds

$$\begin{aligned}\gamma^p_{StR}(G) &\leq w(f) = |B_1| + \sum_{x \in B_2} f(x)\\ &\leq 1 + [n - (1 + r + (r-1)r)]\\ &\quad + \left(1 + \left\lceil \tfrac{r-1}{p} \right\rceil\right) r\\ &= n - r^2 + \left( \left\lceil \tfrac{r-1}{p} \right\rceil + 1 \right) r.\end{aligned}$$

□

**Corollary 8.** *Let $3 \leq p \leq \Delta - 1$ be a positive integer and let $G$ be a $(p+1)$-regular graph with girth $g \geq 5$. Then*

$$\gamma^p_{StR}(G) \leq n - p^2 + 1$$

To check the tightness of this upper bound for regular graphs, we first prove a technical result which will be useful later.

**Lemma 9.** *Let $G$ be a graph with order $n$ and maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$. Let $f = (B_0, B_1, B_2)$ be any $p$-StRD function on $G$. Hence*

$$\gamma^p_{StR}(G) \geq n + \left\lceil \frac{1-p}{p} |B_0| \right\rceil.$$

**Proof.** Observe that each vertex in $B_1 \cup B_2$ adds one unit, by itself, to the weight of $f$. In addition, since every vertex in $B_0$ has, at least, a neighbor in $B_2$, each vertex in $B_0$ adds, at least, $\frac{1}{p}$ units to the weight of $f$. Therefore

$$\begin{aligned}\gamma^p_{StR}(G) &= w(f) \geq |B_1| + |B_2| + \left\lceil \tfrac{1}{p} \right\rceil |B_0|\\ &= n - |B_0| + \left\lceil \tfrac{|B_0|}{p} \right\rceil \geq n + \left\lceil \tfrac{1-p}{p} |B_0| \right\rceil\end{aligned}$$

□

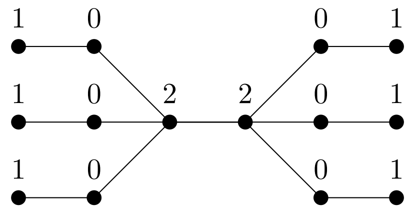


Fig 3: A graph for which the lower bound (Lemma 9) is attained, with $p = 3$.

Notice that this lower bound is sharp, as seen in the graph of Figure 3.

**Corollary 10.** *Let $G$ be a graph with order $n$ and maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$. Let $f = (B_0, B_1, B_2)$ be any $p$-StRD function on $G$. Then*

$$|B_0| \geq \frac{p}{p-1}\left(n - \gamma^p_{StR}(G)\right)$$

Now, we prove the tightness of the upper bound provided by Corollary 8. To do that, it is sufficient to consider the $(4, 5)$-cage graph, known as the Robertson graph. It is a 4-regular graph with $n = 19$, and girth $g = 5$. Since $\Delta = 4$ then $p$ must be 3. The next example shows that $\gamma^3_{StR}(G) = n-p^2+1 = 11$.

**Example 11.** *Let $G$ be the $(4, 5)$-cage, the Robertson graph. For this graph, it can be shown that $\gamma^3_{StR}(G) = 11$.*

**Proof.** It is readily to prove that $\gamma_{3StR}(G) \leq 11$ by following the construction described in the proof of Proposition 7.

To see that $\gamma^3_{StR}(G) \geq 11$, we reasoning by contradiction. Assume that $\gamma^3_{StR}(G) \leq 10$. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma^3_{StR}(G)$-function such that $V_1$ has maximum cardinality. By Proposition 13, we have that $\gamma^3_{GStR}(G) \geq 7$. Therefore, $\gamma^3_{StR}(G) \in \{7, 8, 9, 10\}$. Since $n = 19$ and $p = 3$, by Corollary 10, we deduce that

$$|V_0| \geq \left\lceil \frac{3}{2}\left(19 - \gamma^3_{StR}(G)\right) \right\rceil .$$

Clearly, $|V_1| + |V_2| + |V_3| = 19 - |V_0|$. If $7 \leq \gamma^3_{StR}(G) \leq 8$ then $17 \leq |V_0| \leq 18$ which implies $|V_1| + |V_2| + |V_3| \leq 2$ and hence $\gamma^3_{StR}(G) \leq 6$, a contradiction. Therefore, $9 \leq \gamma^3_{StR}(G) \leq 10$.

Since $f$ is a $\gamma^3_{StR}(G)$-function such that $V_1$ has maximum cardinality, the only possibilities are either $\gamma^3_{StR}(G) = 10$ with $|V_1| = |V_3| = 2, |V_2| = 1, |V_0| = 14$ or either $\gamma^3_{StR}(G) = 9$ with $|V_1| = |V_2| = 1, |V_3| = 2, |V_0| = 15$.

Clearly, every vertex with a label 0 must have a strong neighbor because $f$ is a 3-StRDF. Besides, each vertex with a label 3 is adjacent to, at most, 4 vertices labeled with 0 and each vertex with a label 2 is adjacent to, at most, 3 vertices labeled with 0. If $|V_1| = |V_3| = 2, |V_2| = 1, |V_0| = 14$ then we have that $n = 19 \leq |N[V_3]| + |N[V_2]| + |V_1| \leq (2 + 8) + (1 + 3) + 2 = 16$, a contradiction. In other case, if $|V_1| = |V_2| = 1, |V_3| = 2, |V_0| = 15$ then $n = 19 \leq |N[V_3]|+|N[V_2]|+|V_1| \leq (2+8)+(1+3)+1 = 15$, again a contradiction.

□

Next, we present a result with a probabilistic approach providing an upper bound. It is described in terms of the order, the maximum and minimum degree of the graph and the value of $p$.

**Proposition 12.** *Let $G$ be a graph with order $n$, minimum degree $\delta$ and maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$, such that $\left\lceil \frac{\Delta}{p} \right\rceil < \delta$. Then,*

$$\gamma^p_{StR}(G) \leq \frac{\left(1 + \left\lceil \frac{\Delta}{p} \right\rceil\right) n}{1+\delta}\left(\ln\left(\frac{1+\delta}{1 + \left\lceil \frac{\Delta}{p} \right\rceil}\right) + 1\right).$$

**Proof.** Let $A \subseteq V(G)$ be a subset of vertices of $G$ and let $\xi \in (0, 1)$ be the probability that a vertex $v \in V(G)$ belongs to the set $A$. We assume that two vertices can independently belong to the set $A$. Let $B \subseteq V(G)$ be the subset of vertices of $G$ such that do not belong to set $A$ neither have neighbors in $A$, that is $B = V(G) - N[A] = (N[A])^c = A^c \cap N(A)^c$. Then, for each vertex $v \in V(G)$ we have that

$$\begin{aligned} P[v \in B] &= (1-\xi)(1-\xi)^{d(v)} \\ &= (1-\xi)^{1+d(v)} \leq (1-\xi)^{1+\delta(G)}, \end{aligned}$$

since $0 < \xi < 1$ and $\delta(G) \leq d(v)$, for any $v \in V(G)$.

Now, for each vertex $v \in V(G)$, we define the following random variable

$$X(v) = \begin{cases} 1 + \left\lceil \frac{\Delta}{p} \right\rceil & \text{si } v \in A, \\ 0 & \text{si } v \in N(A) - A, \\ 1 & \text{si } v \in B = V(G) - N[A]. \end{cases}$$

It is not difficult to upper bound its expected value, for any $v \in V(G)$, as follows

$$\begin{aligned} E[X(v)] &= (1+\left\lceil \tfrac{\Delta}{p}\right\rceil)P[v\in A]+P[v\in B] \\ &= (1+\left\lceil \tfrac{\Delta}{p}\right\rceil)\xi+P[v\in B] \\ &\le (1+\left\lceil \tfrac{\Delta}{p}\right\rceil)\xi+(1-\xi)^{1+\delta(G)} \end{aligned}$$

Then, the value that $X(v)$ assigns to each vertex $v \in V(G)$, leads us to a function $f : V(G) \to \{0,1,\dots,1+\left\lceil \frac{\Delta}{p}\right\rceil\}$, such that $f(v)=X(v)$ for each $v\in V(G)$. Since for every vertex $w \in V(G)$, with $f(w)=0$, it has at least one neighbor $u$ in $A$ such that $f(u)=1+\left\lceil \frac{\Delta}{p}\right\rceil \ge 1+\left\lceil \frac{1}{p}|N(u)\cap B_0|\right\rceil$ the $f$ is a $p$-StRD function. As a consequence, we have that

$$\begin{aligned} E[f(V)] &= \sum_{v\in V(G)} E[f(v)] = \sum_{v\in V(G)} E[X(v)] \\ &\le \sum_{v\in V(G)} \left((1+\left\lceil \frac{\Delta}{p}\right\rceil)\xi+(1-\xi)^{1+\delta(G)}\right) \\ &= (1+\left\lceil \tfrac{\Delta}{p}\right\rceil)n\xi+n(1-\xi)^{1+\delta(G)} \end{aligned}$$

Since $0<\xi<1$, it follows that $(1-\xi)<\mathrm{e}^{-\xi}$ and then

$$E[f(V)] \le \left(1+\left\lceil \frac{\Delta}{p}\right\rceil\right) n\xi + n\mathrm{e}^{-\xi(1+\delta(G))} \quad (1)$$

For each value $\xi\in(0,1)$ minimizing the value of the expression (1) it must be

$$\left(1+\left\lceil \frac{\Delta}{p}\right\rceil\right) n - n(1+\delta(G))\mathrm{e}^{-\xi(1+\delta(G))}=0.$$

Therefore, $\mathrm{e}^{-\xi(1+\delta(G))}=\frac{1+\left\lceil \frac{\Delta}{p}\right\rceil}{1+\delta(G)}$ and we deduce that $\xi=\frac{1}{1+\delta(G)}\ln\left(\frac{1+\delta(G)}{1+\left\lceil \frac{\Delta}{p}\right\rceil}\right)$.

It is readily to see that $\xi<1$ because $\ln\left(\frac{1+\delta(G)}{1+\left\lceil \frac{\Delta}{p}\right\rceil}\right)<\ln\left(\frac{1+\delta(G)}{2}\right)<1+\delta(G)$, for any $\delta(G)$. Observe also that $\xi>0$ since $\left\lceil \frac{\Delta}{p}\right\rceil<\delta(G)$. Finally, since $n(1+\delta(G))^2\mathrm{e}^{-\xi(1+\delta(G))}>0$, we may derive that the critical value of $\xi$ is a local minimum. Hence, by using (1), we obtain

$$\begin{aligned} \gamma_{StR}^p(G) \le\ & \left(1+\left\lceil \tfrac{\Delta}{p}\right\rceil\right)\tfrac{n}{1+\delta}\ln\left(\tfrac{1+\delta}{1+\left\lceil \frac{\Delta}{p}\right\rceil}\right) \\ &+\left(1+\left\lceil \tfrac{\Delta}{p}\right\rceil\right)\tfrac{n}{1+\delta}, \end{aligned}$$

which concludes the proof.

□

Let us conclude this section with a lower bound expressed in terms of $p$ and the order of the graph and a direct consequence for graphs containing a universal vertex.

**Proposition 13.** *Let $G$ be a connected graph with order $n$ and maximum degree $\Delta \ge 4$. Let $p$ be a positive integer such that $3\le p\le \Delta-1$. Then*

$$\gamma_{StR}^p(G) \ge \left\lceil \frac{n+p-1}{p}\right\rceil.$$

*If $n\equiv 1 \pmod p$ then equality holds if and only if $\Delta=n-1$.*

**Proof.** Let $f=(B_0,B_1,B_2)$ be a $\gamma_{StR}^p(G)$-funcion. Let us denote by $B_0^1$ the set of vertices in $B_0$ that have, at most, $p-1$ neighbors in $B_2$ and $B_0^2=B_0-B_0^1$. Clearly, $n=|B_0^1|+|B_0^2|+|B_1|+|B_2|$. Observe that each vertex in $B_1\cup B_2$ contributes with one unit, by itself, to the weight of $f$ and each vertex $v\in B_0$ contributes with $\frac{|N(v)\cap B_2|}{p}$ to the total weight of $f$. Hence

$$\begin{aligned} \gamma_{StR}^p(G) &\ge |B_1|+|B_2|+\sum_{v\in B_0^1}\frac{|N(v)\cap B_2|}{p} \\ &\quad +\sum_{v\in B_0^2}\frac{|N(v)\cap B_2|}{p} \\ &\ge |B_1|+|B_2|+\frac{1}{p}|B_0^1|+|B_0^2| \\ &= n-|B_0^1|+\tfrac{1}{p}|B_0^1| \\ &= n-\left(1-\frac{1}{p}\right)|B_0^1| \\ &\ge n-\tfrac{p-1}{p}(n-1)=\frac{n+p-1}{p}, \end{aligned}$$

since $|B_0^1|\le|B_0|\le n-1$ and $\gamma_{StR}^p(G)$ is an integer. Now, let us assume that $n\equiv 1 \pmod p$. On the one hand, if $\gamma_{StR}^p(G)=\frac{n+p-1}{p}$, then all previous inequalities became equalities and therefore $|B_0^1|=n-1$ and $|B_2|=1$, which implies that $\Delta=n-1$. On the other hand, if $\Delta=n-1$ we know that $\gamma_{StR}^p(G)\ge\left\lceil\frac{n+p-1}{p}\right\rceil$. To see the other inequality we define the function $f$ such that $f(u)=\left\lceil\frac{n-1}{p}\right\rceil+1$, for a vertex $u$ such that $d_G(u)=n-1$, and $f(v)=0$ for all

$v \in N(u)$. Then, $\gamma^p_{StR}(G) \le w(f) = \left\lceil \frac{n-1}{p} \right\rceil + 1 = \left\lceil \frac{n+p-1}{p} \right\rceil$.

□

**Corollary 14.** *Let $G$ be a connected graph with order $n$ and maximum degree $\Delta \ge 4$. Let $p$ be a positive integer such that $3 \le p \le \Delta - 1$. If $\Delta = n-1$, then*

$$\gamma^p_{StR}(G) = \left\lceil \frac{n+p-1}{p} \right\rceil = n - \left\lfloor \frac{p-1}{p}\Delta \right\rfloor.$$

## 4 Exact values

This section is devoted to studying the exact value of the $p$-strong Roman domination number in certain families of graphs of interest. We start with the complete bipartite graphs.

**Proposition 15.** *Let $p$ be a positive integer such that $3 \le p \le \Delta - 1$. Let $2 \le r \le s$ be two positive integers such that $s \ge 4$. Then*

$$\gamma^p_{StR}(K_{r,s}) = \begin{cases} 2 + \left\lceil \frac{s}{p} \right\rceil & \text{if } r = 2, \\ \left\lceil \frac{r+p-1}{p} \right\rceil + \left\lceil \frac{s+p-1}{p} \right\rceil & \text{if } r \ge 3. \end{cases}$$

**Proof.** Let us denote by $n = n(K_{2,s}) = s + 2$. To begin with, let us assume that $r = 2$. By applying Proposition 5 we have that $\gamma^p_{StR}(K_{2,s}) \le n - \Delta(K_{2,s}) + \left\lceil \frac{\Delta(K_{2,s})}{p} \right\rceil = s+2-s+\left\lceil \frac{s}{p} \right\rceil = \left\lceil \frac{s+2p}{p} \right\rceil$. If $\gamma^p_{StR}(K_{2,s}) \le \left\lceil \frac{s}{p} \right\rceil + 1$ then there must be a $p$-StRD function $f$ having weight $w(f) \le \left\lceil \frac{s}{p} \right\rceil + 1$. Then, by Corollary 10, the number of vertices labeled with a 0 can be bounded as follows

$$\begin{aligned} |B_0| &\ge \frac{p}{p-1}\left(n - \left\lceil \frac{s}{p} \right\rceil - 1\right) \\ &= \frac{p}{p-1}\left(\left\lfloor \frac{p-1}{p}s \right\rfloor + 1\right) \\ &\ge \frac{p}{p-1}\,\frac{p-1}{p}s = n-2. \end{aligned}$$

Hence, $|B_1| + |B_2| \le 2$, with $B_2 \ne \emptyset$ because $B_0$ is non-empty. We have to consider several situations.

**Case 1.** If $|B_1| = 0$ and $|B_2| = 1$ then $|B_0| = n-1$ and $\Delta(K_{2,s}) = n-1$, a contradiction.

**Case 2.** If $|B_1| = |B_2| = 1$ then $|B_0| = s$ and $N(B_2) = B_0$ which implies that $w(f) \ge f(B_1) + f(B_2) = 1 + 1 + \left\lceil \frac{s}{p} \right\rceil = 2 + \left\lceil \frac{s}{p} \right\rceil$, again a contradiction.

**Case 3.** Assume that $|B_1| = 0, |B_2| = 2$ and let us denote by $B_2 = \{u, v\}$. If $u, v$ are adjacent then $f(u) + f(v) \ge 1 + \left\lceil \frac{s-1}{p} \right\rceil + 2$ implying that $w(f) \ge 3 + \left\lceil \frac{s-1}{p} \right\rceil \ge 2 + \left\lceil \frac{s}{p} \right\rceil$ which is not possible. If $u, v$ are not adjacent then $|N(\{u,v\})| = s$ and $B_0 = N(u) = N(v)$, because $f$ is a $p$-StRDF. Hence, $f(u) + f(v) \ge 2\left(1 + \left\lceil \frac{s}{p} \right\rceil\right) > 2 + \left\lceil \frac{s}{p} \right\rceil$, which is a contradiction.

So, it must be $\gamma^p_{StR}(K_{2,s}) \ge 2 + \left\lceil \frac{s}{p} \right\rceil$ and the equality is proven.

Now, let us suppose that $r \ge 3$. Let $u$ be a vertex belonging to the $r$-class and $v$ be a vertex of the $s$-class. The function defined as $f(u) = 1 + \left\lceil \frac{s-1}{p} \right\rceil$, $f(v) = 1 + \left\lceil \frac{r-1}{p} \right\rceil$ is a $p$-StRD function in $K_{r,s}$. Therefore $\gamma^p_{StR}(K_{r,s}) \le \left\lceil \frac{r+p-1}{p} \right\rceil + \left\lceil \frac{s+p-1}{p} \right\rceil$.

Reasoning by contradiction, let us assume that $\gamma^p_{StR}(K_{r,s}) \le \left\lceil \frac{r-1}{p} \right\rceil + \left\lceil \frac{s-1}{p} \right\rceil + 1$. Let $f = (B_0, B_1, B_2)$ be a $\gamma^p_{StR}$-function. Again by Corollary 10, we have that

$$\begin{aligned} |B_0| \ge\ & \frac{p}{p-1}\left(r + s - \left\lceil \frac{r-1}{p} \right\rceil - \left\lceil \frac{s-1}{p} \right\rceil - 1\right) \\ =\ & \frac{p}{p-1}\left(\left\lfloor \frac{p-1}{p}(s-1) \right\rfloor + 1 \right. \\ & \left. + \left\lfloor \frac{p-1}{p}(r-1) \right\rfloor + 1 - 1\right) \\ \ge\ & \frac{p}{p-1}\left(\frac{p-1}{p}(s-1) + \frac{p-1}{p}(r-1) - 1\right) \\ =\ & r + s - 2 - \frac{p}{p-1} \end{aligned}$$

As the function $h(p) = \frac{p}{p-1}$ is a non-increasing function for positive values of $p$, and since $h(3) = \frac{3}{2}$ we deduce that $|B_0| \ge r + s - \frac{7}{2}$, which in turn lead us to $|B_1| + |B_2| \le 3$. Since $B_2 \ne \emptyset$ then we have to consider different cases: $|B_1| = j$ and $1 \le |B_2| \le 3 - j$, for all $j \in \{0, 1, 2\}$.

**Case 1.** If $|B_1| = 2, |B_2| = 1$ then there must be $r = 3$, $N(B_2) = B_0$ and the 3-class of $K_{3,s}$ coincides with $B_1 \cup B_2$. But in this case, it would be

$$w(f) = 2 + 1 + \left\lceil \frac{s}{p} \right\rceil > \left\lceil \frac{r-1}{p} \right\rceil + \left\lceil \frac{s-1}{p} \right\rceil + 1$$

**Case 2.** $|B_1| = 1$ and $1 \le |B_2| \le 2$. If $|B_1| = |B_2| = 1$ then $|B_0| = n-2$ which is impossible because $r \geq 3$ and $f$ is an StRDF. Hence, $|B_1| = 1$ and $|B_2| = 2$. Let us denote by $B_1 = \{z\}, B_2 = \{u, v\}$. If $u, v$ are not adjacent then it must be $r = 3$ and $B_1 \cup B_2$ is the 3-class of $K_{3,s}$, because $f$ is an StRDF. Then, we deduce that $w(f) = 2\left(1 + \left\lceil \frac{s}{p} \right\rceil\right) + 1 > \left\lceil \frac{r-1}{p} \right\rceil + \left\lceil \frac{s-1}{p} \right\rceil + 1$, a contradiction. If $u, v$ are adjacent then, without loss of generality, we may assume that $d(z) = s$. Hence, $w(f) = f(z) + f(u) + f(v) = 1 + \left\lceil \frac{s-1}{p} \right\rceil + 1 + \left\lceil \frac{r-2}{p} \right\rceil + 1 > \left\lceil \frac{r-1}{p} \right\rceil + \left\lceil \frac{s-1}{p} \right\rceil + 1$.

**Case 3.** $|B_1| = 0$ and $1 \leq |B_2| \leq 3$. As $r \geq 3$ then $\Delta \leq n - 3$ and therefore $B_2 \geq 2$. If the induced subgraph by the vertices of $B_2$ is an edgeless subgraph then $r = |B_2| = 3$ and $w(f) = 3\left(1 + \left\lceil \frac{s}{p} \right\rceil\right) + 1 > \left\lceil \frac{r-1}{p} \right\rceil + \left\lceil \frac{s-1}{p} \right\rceil + 1$. Then, there must be adjacent vertices in the set $B_2$. If $|B_2| = 2$ then $w(f) = f(B_2) \geq 1 + \left\lceil \frac{s-1}{p} \right\rceil + 1 + \left\lceil \frac{r-1}{p} \right\rceil$, a contradiction. If $|B_2| = 3$ then $w(f) = f(B_2) \geq 1 + \left\lceil \frac{s-1}{p} \right\rceil + 1 + \left\lceil \frac{r-2}{p} \right\rceil + 1 = 2 + \left\lceil \frac{s-1}{p} \right\rceil + \left\lceil \frac{r+p-2}{p} \right\rceil \geq 2 + \left\lceil \frac{s-1}{p} \right\rceil + \left\lceil \frac{r-1}{p} \right\rceil$, again a contradiction. So the result holds.

□

Our next result provides the exact value of the $p$-strong Roman domination number for bi-star graphs. The proof is quite similar to that of Proposition 15, so we leave the details to the reader.

**Proposition 16.** *Let $r, s$ be two integers such that $1 \leq r \leq s$. Let $T_{r,s}$ be the bi-star graph with order $n = r + s + 2$ and maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$. Then,*

$$\gamma_{StR}^{p}(T_{r,s}) = 2 + \left\lceil \frac{r}{p} \right\rceil + \left\lceil \frac{s}{p} \right\rceil .$$

We conclude by characterizing those graphs having the smallest possible values of the $p$-strong Roman domination number.

**Proposition 17.** *Let $G$ be a graph with order $n$ and maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$. Then $\gamma_{StR}^{p}(G) = 3$ if and only if $G = K_1 \vee H$, where $p + 2 \leq n \leq 2p + 1$, $\Delta = n - 1$ and $H$ is any graph with $n - 1$ vertices.*

**Proof.** Assume that $\gamma_{StR}^{p}(G) = 3$. Let $f = (B_0, B_1, B_2)$ be a $p$-StRD function on $G$ with minimum weight $w(f) = 3$. Since $w(f) = |B_1| + \sum_{v \in B_2} f(x)$, hence there are only two possibilities: (i) $B_1 = \emptyset$ and $B_2 = \{v\}$, with $f(v) = 3$ or (ii) $|B_1| = 1$ and $B_2 = \{v\}$, with $f(v) = 2$.
(i) If $B_1 = \emptyset$ and $B_2 = \{v\}$, with $f(v) = 3$, then $B_0 = V \setminus \{v\}$, that is, every vertex in $B_0$ must be adjacent to $v$ and $p + 1 \leq |B_0| \leq 2p$, since $f(v) = 3$. Therefore, $p + 2 \leq n \leq 2p + 1$ and $\Delta = n - 1$, and then, $G = \{v\} \vee H$, where $H$ is a subgraph on $p + 1 \leq |V(H)| \leq 2p$ vertices.
(ii) If $B_1 = \{u\}$ and $B_2 = \{v\}$, with $f(v) = 2$, then $B_0 = V \setminus \{u, v\}$ and all vertices in $B_0$ must be adjacent to $v$, hence $1 \leq |B_0| \leq p$, since $f(v) = 2$, and $3 \leq n \leq p + 2$. Now, we distinguish two subcases: $|N(v)| = n - 1$ or $|N(v)| = n - 2$.
Suppose that $|N(v)| = n - 1$. If $|B_0| < p$, then, we can label the vertex $u$ with 0 and we have $\gamma_{StR}^{p}(G) = 2$, which is a contradiction. Hence, $|B_0| = p$ and there exists a $p$-StRD function $f$ on $G$ with minimum weight $w(f) = 3$, where $f(v) = 3$, which lead us to case (i).
Suppose now that $|N(v)| = n - 2$. Since $1 \leq |B_0| \leq p$, then $n \leq p + 2$. Due to $3 \leq p \leq \Delta - 1$, then $p + 1 \leq \Delta$. We deduce that $n - 1 \leq \Delta$ and, therefore, $n - 1 = \Delta$, which means that there exists a vertex $w \in B_0$ such that $d(w) = \Delta = n - 1$, which describes the graph of the case (i). The reciprocal is trivial. □

**Proposition 18.** *Let $G$ be a graph with order $n$ and maximum degree $\Delta \geq 4$. Let $p$ be a positive integer such that $3 \leq p \leq \Delta - 1$. Then $\gamma_{StR}^{p}(G) = 4$ if and only if one of the following conditions hold*

1. *$\Delta = n - 1$, $2p + 2 \leq n \leq 3p + 1$ and $G = K_1 \vee H$ where $H$ is any graph on $2p + 1 \leq |V(H)| \leq 3p$ vertices.*

2. *$\Delta = n - 2$, $4 \leq n \leq 2p + 2$ and $G = H_1 \vee H_2$, where $H_1 \subseteq K_2$ and $H_2$ is a subgraph on $2 \leq |V(H_2)| \leq 2p$ vertices.*

3. *$\Delta = n - 2$, $p + 3 \leq n \leq 2p + 2$ and $G = K_1 \vee H$, where $K_1 = \{z\}$, with $d(z) = \Delta(G)$, and $H$ is a subgraph on $|V(H)| = n - 1$ vertices.*

**Proof.** Assume that $\gamma_{StR}^{p}(G) = 4$. Let $f = (B_0, B_1, B_2)$ be a $p$-StRD function on $G$ with minimum weight $w(f) = 4$. Since $w(f) = |B_1| + \sum_{v \in B_2} f(x)$, hence there exists different possibilities.

**Case 1:** Assume that $B_1 = \emptyset$.

**Subcases 1a:** Suppose that $B_2 = \{v\}$, with $f(v) = 4$. Then every vertex in $B_0$ must be adjacent to $v$ and $2p + 1 \leq |B_0| \leq 3p$, since $f(v) = 4$. Therefore, $2p + 2 \leq n \leq 3p + 1$ and $\Delta = n - 1$, and then, $G = \{v\} \vee H$, where $H$ is a subgraph on $2p + 1 \leq |V(H)| \leq 3p$.

**Subcase 1b:** Suppose that $B_2 = \{v, w\}$, with $f(v) = f(w) = 2$. Then every vertex in $B_0$ must be adjacent to $v$ and $w$, then $2 \leq |B_0| \leq 2p$. Therefore, $4 \leq n \leq 2p + 2$ and $\Delta = n - 2$, and then, $G = H_1 \vee H_2$, where $H_1 \subseteq K_2$ and $H_2$ is a subgraph on $2 \leq |V(H_2)| \leq 2p$ vertices.

**Case 2:** Assume that $B_1 \neq \emptyset$.

**Subcase 2a:** Suppose that $B_1 = \{u\}$ and $B_2 = \{v\}$, with $f(v) = 3$. Then every vertex in $B_0$ must be adjacent to $v$ and $p+1 \leq |B_0| \leq 2p$, since $f(v) = 3$, and $p + 3 \leq n \leq 2p + 2$. If $w \in N(v)$, then $\Delta = n - 1$ and there exists a $p$-StRD function $f$ on $G$ with minimum weight $w(f) = 4$, where $f(v) = 4$, which lead us to case (1a). If $w \notin N(v)$, then $|B_0| \leq n - 2$, since $f(v) = 3$, $\Delta = n - 2$ and $G = K_1 \vee H$, where $K_1 = \{z\}$, with $d(z) = \Delta(G) = n - 2$, and $H$ is a subgraph on $|V(H)| = n - 1$ vertices.

**Subcase 2b:** Suppose that $B_1 = \{u, w\}$ and $B_2 = \{v\}$, with $f(v) = 2$. Then every vertex in $B_0$ must be adjacent to $v$ and $1 \leq |B_0| \leq p$, since $f(v) = 2$, and $4 \leq n \leq p + 3$. If $u, w \in N(v)$, then $d(v) = n - 1 = p + 2$ and $n = p + 3$, therefore, $|B_0| = p$, which implies that there exists a $p$-StRD function $f$ on $G$ with minimum weight $w(f) = 3$, where $f(v) = 3$, which is a contradiction, since we are assuming that $\gamma^p_{StR}(G) = 4$. If $u, w \notin N(v)$, then $1 \leq |B_0| \leq p$ and $\Delta = n - 3$, hence, $p \leq \Delta - 1 = n - 4$, that is, $p + 4 \leq n$, which is a contradiction, since $4 \leq n \leq p + 3$. If $u \in N(v)$ and $w \notin N(v)$, then necessarily $|B_0| = p$, since $f(v) = 2$. Therefore $n = p + 3$ and $\Delta = p + 1 = n - 2$, which leads us to case (2a).

The reciprocals are trivial. □

# 5 Conclusion

Recently, many definitions of Roman domination models for graphs have been proposed. In this work, we introduce the concept of p-strong Roman domination, which enables the development of newer, more adaptable, and less expensive defensive strategies.

The NP-completeness of the problem has been explored for bipartite and chordal graphs by linking it to the Exact 3-Cover problem. Various general upper and lower bounds have been examined, along with an upper bound derived using probabilistic methods. Concerning the study of exact values, specific cases like Robertson's (4,5)-cage, where 3-StR equals 11, and extensive families of graphs such as complete bipartite graphs or bi-stars have been investigated.

# 6 Future research directions

This work opens several compelling avenues for future research, particularly in light of the NP-completeness of the p-strong Roman domination problem.

Several promising directions warrant investigation. Firstly, seeking tighter bounds, either by improving existing ones or by considering other graph invariants, would be valuable. Another interesting possibility is exploring new inequalities that relate this parameter to parameters of other domination types beyond those mentioned in Remark 4.

Regarding Remark 4, characterizing the graphs that achieve lower and upper bounds would be a significant contribution. Determining the exact value of the p-strong Roman domination number in other graph families and graph products is also a worthwhile pursuit.

Furthermore, studying the exact value of the p-strong Roman domination number in trees with specific structures is of interest. Based on these findings, an attempt to establish a general bound for any tree of order n≥5 could be undertaken.

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