Lagrange Multiplier Method for Variational Theory and Optimal Control and Beyond

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Abstract: - Lagrange multipliers are a commonly employed method for the elimination of constraints in both variational theory and optimal control. However, it should be noted that the validity of this approach may be compromised when the identified multiplier is equal to zero. The paper elucidates the phenomenon of a zero multiplier and its hidden mechanism, and it proffers two methodologies for surmounting the associated challenges.

Key-Words: - Lagrange multiplier method, variational theory, optimal control, semi-inverse method, Hamilton principle, deep learning, machine learning, variational iteration method.

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1 Introduction

The Lagrange multiplier method represents the primary tool in both variational theory, [1] [2] [3] and optimal control [4]. Furthermore, it has become a valuable tool for addressing constraints in deep learning [5] and machine learning [6] and it plays a significant role in the variational iteration method, [7], [8]. The identification of the Lagrange multiplier in the variational iteration method is a crucial process. Some novel approaches for identifying the multiplier have been proposed, such as the Aboodh transform [8], the calculus of variations [7], and the Laplace transform, [9], [10]. These make the method highly appealing for complex problems, particularly those involving fractional differential equations.

Nevertheless, in certain particular instances, the multipliers may potentially become null during the identification process, thereby rendering the Lagrange multiplier method inapplicable. In this paper, we will present the issue and propose two solutions to address it.

2 The Crisis of Lagrange Multiplier

When the Lagrange multiplier reaches a value of zero during the process of identification, we may refer to this as the crisis of the Lagrange multiplier.

To illustrate the phenomenon, we consider the following Laplace equation:

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \tag{1}$$

This equation can be used to describe the potential flow of inviscid fluids. The potential function, ϕ, satisfies the following relationships:

$$\phi_x=u \tag{2}$$

$$\phi_y=v \tag{3}$$

$$\phi_z=w \tag{4}$$

where u,v, and w are velocities in x-, y- and z-directions respectively.

Eq. (1) can be formulated as a variational problem in the form:

$$J(\phi)=\iiint\left\{\frac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)\right\}dxdydz \tag{5}$$

In order to establish a generalized variational principle, it is possible to utilize the Lagrange multiplier method.

$$J(\phi,u,v,z,\lambda_1,\lambda_2,\lambda_3)=\iiint\left\{\frac{\phi_x^2+\phi_y^2+\phi_z^2}{2}\right\}dxdydz$$
$$+\iiint\left\{\lambda_1(\phi_x-u)+\lambda_2(\phi_y-v)+\lambda_3(\phi_z-w)\right\}dxdydz \tag{6}$$

where $\lambda_i(i=1,2,3)$ are Lagrange multipliers.

The identification of the Lagrange multipliers yields the following result:

$$\lambda_i=0,\ (i=1,2,3) \tag{7}$$

This is called the crisis of Lagrange multiplier.

3 Explanation of the Crisis of Lagrange Multiplier

The zero multipliers in Eq. (7) indicate that the constraints cannot be eliminated. However, a detailed examination of the phenomenon suggests that Eq. (7) may involve hidden equations, namely, Eqs. (2), (3), and (4). Consequently, Eq. (7) can be interpreted as follows:

$$\lambda_1=a_1(\phi_x-u) \tag{8}$$

$$\lambda_2=a_2(\phi_y-v) \tag{9}$$

$$\lambda_3=a_3(\phi_z-w) \tag{10}$$

where $a_i(i=1,2,3)$ are nonzero constants. So we obtain the following variational formulation:

$$J(\phi,u,v,z)=\iiint\left\{\frac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)\right\}dxdydz$$
$$+\iiint\left\{a_1(\phi_x-u)^2+a_2(\phi_y-v)^2+a_3(\phi_z-w)^2\right\}dxdydz \tag{11}$$

This variational formulation bears resemblance to the penalty function method [11], yet it is a genuine variational principle. The proof is presented below.

Proof. Its stationary conditions of Eq.(11) are:

$$\begin{aligned}&-(\phi_{xx}+\phi_{yy}+\phi_{zz})-2a_1(\phi_x-u)_x\\&-2a_2(\phi_y-v)_y-2a_3(\phi_z-w)_z=0\end{aligned} \tag{12}$$

$$-2a_1(\phi_x-u)=0 \tag{13}$$

$$-2a_2(\phi_y-v)=0 \tag{14}$$

$$-2a_3(\phi_z-w)=0 \tag{15}$$

It is evident that equations (13), (14), and (15) are, in order, equations (2), (3), and (4). In light of the preceding equations, it is possible to transform equation (12) into equation (1). Consequently, the variational formulation of Eq. (11) allows us to obtain the equations (1) to (4).

Alternatively, the Lagrange multipliers in Eq.(7) can be identified as:

$$\lambda_1=b_1(\phi_y-v) \tag{16}$$

$$\lambda_2=b_2(\phi_z-w) \tag{17}$$

$$\lambda_3=b_3(\phi_x-u) \tag{18}$$

where $b_i(i=1,2,3)$ are nonzero constants. So we obtain the following variational formulation:

$$\begin{aligned}J(\phi,u,v,z)&=\iiint\left\{\frac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)\right\}dxdydz\\&+\iiint\left\{b_1(\phi_y-v)(\phi_x-u)\right\}dxdydz\\&+\iiint\left\{b_2(\phi_z-w)(\phi_y-v)\right\}dxdydz\\&+\iiint\left\{b_3(\phi_x-u)(\phi_z-w)\right\}dxdydz\end{aligned} \tag{19}$$

Its stationary conditions are:

$$\begin{aligned}&-(\phi_{xx}+\phi_{yy}+\phi_{zz})\\&-b_1(\phi_x-u)_y-b_1(\phi_y-v)_x\\&-b_2(\phi_y-v)_z-b_2(\phi_z-w)_y\\&-b_3(\phi_z-w)_x-b_3(\phi_x-u)_z=0\end{aligned} \tag{20}$$

$$-b_1(\phi_y-v)-b_3(\phi_z-w)=0 \tag{21}$$

$$-b_1(\phi_x-u)-b_2(\phi_z-w)=0 \tag{22}$$

$$-b_2(\phi_y-v)-b_3(\phi_x-u)=0 \tag{23}$$

In order to obtain Eqs.(2),(3) and (4) from Eqs.(21),(22) and (23), it requires:

$$\begin{vmatrix} & -b_1 & -b_3 \\ -b_1 & & -b_2 \\ -b_3 & -b_2 & \end{vmatrix} = 0 \quad (24)$$

Eq.(24) is an identity, that means for any nonzero $b_i(i=1,2,3)$, Eqs.(21),(22), and (23) can reduce to Eqs. (2), (3) and (4).

The third way to the identification of the multipliers in Eq.(7) is:

$$\lambda_1 = \frac{1}{3}a(\phi_y - v)(\phi_z - w) \quad (25)$$

$$\lambda_2 = \frac{1}{3}a(\phi_x - u)(\phi_z - w) \quad (26)$$

$$\lambda_3 = \frac{1}{3}a(\phi_x - u)(\phi_y - v) \quad (27)$$

where a is a nonzero constant. So the following variational formulation is obtained:

$$\begin{aligned} J(\phi,u,v,z) &= \iiint \left\{ \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) \right\} dxdydz \\ &+ \iiint \left\{ a(\phi_x - u)(\phi_y - v)(\phi_z - w) \right\} dxdydz \end{aligned} \quad (28)$$

Its stationary conditions are:

$$\begin{aligned} &-(\phi_{xx} + \phi_{yy} + \phi_{zz}) \\ &-a\left[(\phi_y - v)(\phi_z - w) \right]_x \\ &-a\left[(\phi_x - u)(\phi_z - w) \right]_y \\ &-a\left[(\phi_x - u)(\phi_y - v) \right]_z = 0 \end{aligned} \quad (29)$$

$$-a(\phi_y - v)(\phi_z - w) = 0 \quad (30)$$

$$-a(\phi_x - u)(\phi_z - w) = 0 \quad (31)$$

$$-a(\phi_x - u)(\phi_y - v) = 0 \quad (32)$$

It is readily apparent that the equations presented in Eqs. (29) to (32) are, in fact, equivalent to the equations presented in Eqs. (1) to (4).

4 Semi-inverse Method

In the preceding section, three methods for identifying the zero multipliers are presented. Nevertheless, this identification is also invalid in the case of complex constraints. In order to demonstrate this, we will consider the following constraints:

$$\phi_x = u + f_1(u,v,w) \quad (33)$$

$$\phi_y = v + f_2(u,v,w) \quad (34)$$

$$\phi_z = w + f_3(u,v,w) \quad (35)$$

where $f_i(i=1,2,3)$ are functions of u,v and w. By employing a similar manipulation as previously described, we can construct the following variational formulation:

$$\begin{aligned} J(\phi,u,v,z) &= \iiint \left\{ \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) \right\} dxdydz \\ &+ \iiint \left\{ c_1(\phi_x - u - f_1)^2 \right\} dxdydz \\ &+ \iiint \left\{ c_2(\phi_y - v - f_2)^2 \right\} dxdydz \\ &+ \iiint \left\{ c_3(\phi_z - w - f_3)^2 \right\} dxdydz \end{aligned} \quad (36)$$

where $c_i(i=1,2,3)$ are nonzero constants. Its stationary conditions with respect to u,v, and w are:

$$\begin{aligned} &-2a_1(\phi_x - u - f_1)(1 + \frac{\partial f_1}{\partial u}) \\ &-2a_2(\phi_y - v - f_2)\frac{\partial f_2}{\partial u} \\ &-2a_3(\phi_z - w - f_3)\frac{\partial f_3}{\partial u} = 0 \end{aligned} \quad (37)$$

$$\begin{aligned} &-2a_1(\phi_x - u - f_1)\frac{\partial f_1}{\partial v} \\ &-2a_2(\phi_y - v - f_2)(1 + \frac{\partial f_2}{\partial u}) \\ &-2a_3(\phi_z - w - f_3)\frac{\partial f_3}{\partial v} = 0 \end{aligned} \quad (38)$$

$$\begin{aligned} &-2a_1(\phi_x - u - f_1)\frac{\partial f_1}{\partial w} \\ &-2a_2(\phi_y - v - f_2)\frac{\partial f_2}{\partial w} \\ &-2a_3(\phi_z - w - f_3)(1 + \frac{\partial f_3}{\partial w}) = 0 \end{aligned} \quad (39)$$

In order to convert Eqs.(37)~(39) in to Eqs.(33)~(35), it requires:

$$\begin{vmatrix} 1 + \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \\ \frac{\partial f_1}{\partial v} & 1 + \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \\ \frac{\partial f_1}{\partial w} & \frac{\partial f_2}{\partial w} & 1 + \frac{\partial f_2}{\partial w} \end{vmatrix} = 0 \quad (40)$$

It is evident that the constraint of Eq. (40) cannot be satisfied in a multitude of practical applications. To illustrate this point, one need only consider a simple case:

$$\phi_x = u + \varepsilon_1 v \quad (41)$$

$$\phi_y = v + \varepsilon_2 w \quad (42)$$

$$\phi_z = w + \varepsilon_3 u \quad (43)$$

and we assume that

$$\varepsilon_1 \varepsilon_2 \varepsilon_3 \neq -1 \quad (44)$$

Here $\varepsilon_i (i = 1, 2, 3)$ are nonzero constants. According to Eq.(40), it requires:

$$\begin{vmatrix} 1 & 0 & \varepsilon_3 \\ \varepsilon_1 & 1 & 0 \\ 0 & \varepsilon_2 & 1 \end{vmatrix} = 0 \quad (45)$$

i.e.

$$\varepsilon_1 \varepsilon_2 \varepsilon_3 + 1 = 0 \quad (46)$$

So the variational formulation below is only valid when Eq.(46) holds.

$$\begin{aligned} J(\phi, u, v, z) &= \iiint \left\{ \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) \right\} dxdydz \\ &+ \iiint \left\{ c_1(\phi_x - u - \varepsilon_1 v)^2 \right\} dxdydz \\ &+ \iiint \left\{ c_2(\phi_y - v - \varepsilon_2 w)^2 \right\} dxdydz \\ &+ \iiint \left\{ c_3(\phi_z - w - \varepsilon_3 u)^2 \right\} dxdydz \end{aligned} \quad (47)$$

In this section, we demonstrate that when Eq. (46) is not satisfied, the variational formulation remains viable. By means of the semi-inverse method[12], a trial functional is constructed in the form of:

$$\begin{aligned} J(\phi, u, v, z) &= \iiint \left\{ \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + F \right\} dxdydz \\ &= \iiint \frac{(u + \varepsilon_1 v)^2 + (v + \varepsilon_2 w)^2 + (w + \varepsilon_3 u)^2}{2} dxdydz \\ &+ \iiint F dxdydz \end{aligned} \quad (48)$$

where F is an unknown function of u,v,w, ϕ and/or their derivatives.

The semi-inverse method [12] has achieved a prominent position in the mathematical toolkit, extending beyond the scope of the Lagrange multiplier method. By employing this method, a number of new variational principles were discovered, including formulations for the nonlinear optic model [13], the fractal modified KdV-Zakharov-Kuznetsov [14], the grinding technology [15], solitary waves [16], [17], and the Benjamin-Bona-Mahony equation [18]. It is of particular note that the semi-inverse method [12] has been successfully employed in the context of MEMS systems [19]. The variational formulation provides a means of studying the periodic solution [20] and its associated pull-in instability [21], [22] from an energy perspective.

The stationary conditions of Eq. (48) with respect to u, v, and w are, respectively, as follows:

$$(u + \varepsilon_1 v) + \varepsilon_3 (w + \varepsilon_3 u) + \frac{\delta F}{\delta u} = 0 \quad (49)$$

$$\varepsilon_1 (u + \varepsilon_1 v) + (v + \varepsilon_2 w) + \frac{\delta F}{\delta v} = 0 \quad (50)$$

$$\varepsilon_2 (v + \varepsilon_2 w) + (w + \varepsilon_3 u) + \frac{\delta F}{\delta w} = 0 \quad (51)$$

where $\delta F / \delta u$ is the variational derivative with respect to u [12]. In order to align with the established nomenclature, equations (49) through (51) should be converted to equations (41) through (43). This can be achieved by setting:

$$\frac{\delta F}{\delta u} = -\phi_x - \varepsilon_3 \phi_z \quad (52)$$

$$\frac{\delta F}{\delta v} = -\varepsilon_1 \phi_x - \phi_y \quad (53)$$

$$\frac{\delta F}{\delta w} = -\varepsilon_2 \phi_y - \phi_z \quad (54)$$

From equations (52) and (53), it can be demonstrated that F is equal to

$$F = -(u\phi_x + v\phi_y + w\phi_z) - \varepsilon_3 u\phi_z - \varepsilon_1 v\phi_x - \varepsilon_2 w\phi_y + f \quad (55)$$

where f is an unknown function of ϕ and/or their derivatives. The variational formulation of Eq.(48) is updated as:

$$\begin{aligned} &J(\phi, u, v, z) = \\ &\iiint \left\{ \frac{(u + \varepsilon_1 v)^2 + (v + \varepsilon_2 w)^2 + (w + \varepsilon_3 u)^2}{2} \right\} dxdydz \\ &+ \iiint \left\{ -(u\phi_x + v\phi_y + w\phi_z) \right\} dxdydz \\ &+ \iiint \left\{ -\varepsilon_3 u\phi_z - \varepsilon_1 v\phi_x - \varepsilon_2 w\phi_y + f \right\} dxdydz \end{aligned} \quad (56)$$

Now the stationary condition with respect to ϕ is:

$$u_x + v_y + w_z + \varepsilon_3 u_z + \varepsilon_1 v_x + \varepsilon_2 w_y + \frac{\delta f}{\delta \phi} = 0 \quad (57)$$

This equation should be converted to Eq.(1), so we set:

$$\frac{\delta f}{\delta\phi} = -(u+\varepsilon_1 v)_x - (v+\varepsilon_2 w)_y - (w+\varepsilon_3 u)_z \\ = -\phi_{xx} - \phi_{yy} - \phi_{zz} \tag{58}$$

From Eq.(58), f is determined as:

$$f = \frac{1}{2}\left[(\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2\right] \tag{59}$$

Finally we obtain the following variational formulation

$$J(\phi,u,v,z) = \\ \iiint \frac{(u+\varepsilon_1 v)^2 + (v+\varepsilon_2 w)^2 + (w+\varepsilon_3 u)^2}{2} dxdydz \\ +\iiint \left\{-(u\phi_x + v\phi_y + w\phi_z)\right\} dxdydz \\ +\iiint \left\{-\varepsilon_3 u\phi_z - \varepsilon_1 v\phi_x - \varepsilon_2 w\phi_y\right\} dxdydz \\ +\iiint \left\{\frac{1}{2}\left[(\phi_x)^2 + (\phi_y)^2 + (\phi_z)^2\right]\right\} dxdydz \tag{60}$$

Eq.(60) is valid for the cases when $\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$.

The variational principle offers a novel avenue for examining dispersive optical solitons [23], water waves [24], nonlinear optical solitary waves [25], [26] and the semi-inverse method can be employed to identify variational formulations for fluid mechanics [27], fractal variational principles [28], [29] and singular waves[30].

5 Conclusion

This paper demonstrates the phenomenon of zero Lagrange multipliers and elucidates the underlying mechanism, thereby facilitating the identification of the multipliers. Furthermore, the paper demonstrates that the semi-inverse method is a more flexible approach for deriving variational formulations. The paper provides an effective paradigm for addressing the issue of zero multipliers. In instances where the Lagrange multiplier method proves ineffective, the semi-inverse method offers a promising alternative.

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