A Boundary Value Problem with Strong Degeneracy and Local Splines

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Abstract: - A new method for solving the singular one-dimensional boundary value problem with a strong degeneracy is proposed in this paper. In the case of the strong degeneration of the differential equation, the boundary condition is set only at one end of the interval. This method is based on the use of the polynomial and non-polynomial Lagrangian and Hermitian type local splines and the variational method. The use of splines of Hermitian type with the first level is convenient if it is needed to obtain simultaneously a solution and the first derivative of the solution at the grid nodes. Next, it is possible to construct the solution between the grid nodes using the same spline approximation formulas. The non-polynomial splines help us to construct a more accurate solution. The results of solving a one-dimensional boundary value problem with strong degeneracy are presented in this paper.

Key-Words: - Boundary value problem, strong degeneracy, degenerate differential equation, variational method, local splines, polynomial splines, trigonometrical splines, Hermitian type splines.

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1 Introduction

Degenerate equations arise when solving many applied problems, for example, in gas dynamics [1], in modeling the motion of viscous droplets spreading over a solid surface [2], [3], the study of isoperimetric problems for probability measures, [4]. Results for a class of nonlinear degenerate Navier problems associated with the degenerate nonlinear elliptic equations are given in [5].

In paper [6], the nonlinear degenerate elliptic differential problem: $-x^2u'' = a\,u + |u|^{(p-1)}, u$ in (0, 1), $u(0) = u(1) = 0,$ is discussed. This equation is a simplified version of the nonlinear Wheeler DeWitt equation. The Wheeler DeWitt equation appears in quantum cosmological models and it is used to model quantum states of the universe and study the qualitative behavior of the universe wave function.

The question of the existence of solutions for degenerate equations was studied in detail in [7], [8].

Among the papers published recently, we note [9], [10].

The novelty of the paper [9] is that the authors find proper weights under which the existence, uniqueness, and regularity of solutions in Sobolev spaces are established.

The main result of the paper [10] is the establishment of the conditions for the existence or not of eigenvalues of the linearization.

Boundary value problems for a degenerate elliptic equation are often studied in the theory of partial differential equations. The study of differential equations with a coefficient at the highest derivative that vanishes has been carried out in many works. Such equations are obtained by studying variable-type partial differential equations, as well as by establishing asymptotic expansions of bisingular problems. In the paper [11], the authors consider a degenerate parabolic problem occurring in the spatial diffusion of a biological population. In paper [12], the authors consider a degenerate parabolic equation occurring in the gas filtration problems.

When modeling physical processes, we often come to the need to numerically solve boundary value problems and initial boundary value problems with degeneracy.

When solving boundary value problems, the B-splines or piecewise linear functions are often used, [13], [14], [15].

In paper [14] the solution to the one-dimensional Stationary Transport Equation is given. Here, for the construction of the approximate solution, a piecewise linear function and trigonometrical polynomials are used.

The [15], is devoted to the construction of the Hermitian- type-splines.

Paper [16] is devoted to the solution of the nonstationary integro-differential equation with a degenerate elliptic differential operator.

Special difficulties arise in the case when solving degenerate equations. Problems connected with solving degenerate equations are discussed in [17], [18], [19], [20].

In paper [21], a new approach to the local improvement of an approximate solution which has been obtained with the finite element method is developed.

In paper [22], an algorithm of the adaptive-grid for one-dimensional boundary value problems of the second order is constructed and the corresponding approximation theorems are established.

The cases are known when the difference approximation of an elliptic differential equation turns out to be non-elliptic. When applying projection and variational methods, orthogonal polynomials are often used as the basis functions. In this case, it is necessary to distinguish between the main and natural boundary conditions. The basis functions must satisfy the main boundary conditions but may not satisfy the natural ones.

We consider the approximation of the solution in the degenerate Sobolev spaces. This means that the weight function can vanish or go to infinity at one of the ends of the interval, [15]. In our work, we assume that the weight function can vanish at the left end of the interval.

In this paper, we discuss the numerical solving the problem:

$$\left(-k(x)u'(x)\right)' + q(x)u(x) = f(x),$$
$$x \in (0,1),\ u(1) = u_n.$$
$$k(x) = x^{\alpha}p(x),\ \alpha \in [1,2),$$
$$p(x) \geq \gamma_0 = \text{const} > 0, q(x) \geq 0, \quad f \in L_2.$$

In this equation, the function x^{α} vanishes only at $x = 0$. Thus, this equation degenerates at the point 0. If $1 \leq \alpha < 2$, then the degeneration is called the strong degeneration and the boundary condition must be set only at one end of the interval. So we put $u(1) = u_n$.

In our paper in Section 1.1 we give general remarks about the variational method, Section 2 is devoted to the local splines. Section 3 is about the construction of the solution of the boundary value problem with the variational method.

1.1 General Remarks

Consider the equation

$$-\frac{d}{dx}\, k(x)\frac{du(x)}{dx} + q(x)u = f(x)\,, 0 < x < 1,$$
$$f \in L_2(0,1).$$

Here $k \in C(0,1) \cap C^{(1)}(0,1)$, the function q is measurable, bounded and non-negative. Suppose that $k(0) = 0, k(x) > 0$ for $x > 0$**.** Of particular interest is the case $(x) = x^{\alpha}p(x)$, $\alpha = \text{const} > 0$, $p \in C^{(1)}[0,1]$, where:

$$p(x) \geq p_0 = \text{const} > 0.$$

Let us denote by A_0 the operator:

$$A_0 u = -\frac{d}{dx}\, k(x)\frac{du}{dx}.$$

The domain of the definition of this operator is taken to be a set of functions $u(x)$ satisfying the condition: $u(x)$ and $k(x)\frac{du}{dx}$ are absolutely continuous on any segment $[\delta, 1]$ where $0 < \delta < 1$. If $\alpha < 1$, then these functions are continuous on $[0, 1]$.

We will need the following facts, [16].

If the integral

$$\int_0^1 \frac{dx}{k(x)}$$

converges, then we take $u(0) = u(1) = 0$.

If the integral

$$\int_0^1 \frac{dx}{k(x)}$$

diverges, then we take $u(1) = 0$.

Let us denote by A the operator $A = A_0 + qu$.
If the integral

$$\int_0^1 \frac{xdx}{k(x)}$$

converges, then the operator A_0 (and the operator A) is positive definite, [16].

If the integral

$$\int_0^1 \frac{xdx}{k(x)}$$

diverges, then the operator A_0 is positive, [16].

It is known that if the operator A is a positive definite, then the equation Au=f has a unique generalized solution. This solution belongs to the energy space H_A. This solution also belongs to $L_2(0,1)$. In our case, H_A consists of functions that are absolutely continuous on any segment $[\delta, 1]$, $0 < \delta < 1$ and satisfy the condition $u(1) = 0$. The energy scalar production and norm are calculated using the formula

$$[u, v]_A = \int_0^1 (k\, u'v' + quv)dx \ ,$$

$$\| \ u \ \|_A^2 = \int_0^1 (k\, u'^2 + q\, u^2)dx \, .$$

On the interval (0,1) we will solve the following boundary value problem with strong degeneracy:

$$\big(-k(x)u'(x)\big)' + q(x)u(x) = f(x),$$
$$u(1) = 0, \qquad (1)$$

where

$$k(x) = x^\alpha p(x), \quad \alpha \in [1, 2),$$
$$p(x) \geq p_0 = \text{const} > 0, \qquad q(x) \geq 0,$$
$$f \in L_2.$$

To solve this problem, we construct a uniform grid $x_j, j = 0,1, \ldots, n$. Divide the interval [0,1] into n parts. Thus, we have constructed a grid of nodes x_j, $j = 0, 1, \ldots, n$,

$$a = x_0 < \cdots < x_{j-1} < x_j < x_{j+1} < \cdots < x_n = b.$$

We will look for an approximate solution $\tilde{u}(x)$ of problem (1) as shown below:

$$\tilde{u}(x) = \sum_{j=1}^{n} c_j \omega_j(x), \qquad (2)$$

where $\omega_j(x)$ are the basis splines, and c_j are some parameters determined from the condition of the minimum of the functional:

$$I = \int_0^1 (k(x)\tilde{u}'^2(x) + q(x)\tilde{u}^2(x) - 2f(x)\tilde{u}(x))dx. \qquad (3)$$

The problem of minimizing this functional (3) on the space of functions of the form (2) leads to the system of equations:

$$\sum_{i=1}^{N} c_i \, [\omega_k, \omega_j]_A = \big(f, \omega_j\big), j = 1,2, \ldots, n.$$

It can be written in the short form:

$$MC = F \, .$$

In more detail, this system can be written as:

$$\sum_{k=1}^{n} c_k \, m_{kj} = f_j, j = 1,2, \ldots, n,$$

The coefficients and the right side of which are calculated using the formulas:

$$m_{kj} = \int_0^1 \big[k(x)\omega_k{}'(x)\omega_j{}'(x) + q(x)\omega_k(x)\omega_j(x)\big]dx \, ,$$

$$f_j = \int_0^1 \big[f(x)\omega_j(x)\big]dx \ .$$

2 Local Spline Application

Next, we consider the application of Lagrangian-type splines and Hermitian-type splines to the solution of a boundary value problem with the strong degeneracy.

2.1 The Application of Splines of the Second Order of Approximation

First, we apply the polynomial basis splines of the second order of approximation.

2.1.1 The Polynomial Basis Splines

The support of the basis spline $\omega_j(x)$ consists of two parts: $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$:

$$\omega_j(x) = \begin{cases} \dfrac{x - x_{j+1}}{x_j - x_{j+1}}, & x \in [x_j, x_{j+1}], \\ \dfrac{x - x_{j-1}}{x_j - x_{j-1}} & x \in [x_{j-1}, x_j] \, , \end{cases}$$

and

$$\omega_j(x) = 0, x \notin [x_{j-1}, x_{j+1}].$$

Obviously, the derivatives of the basis splines ω_j have the form:

$$\omega'_j(x) = \begin{cases} \dfrac{1}{x_j - x_{j+1}}, & x \in [x_j, x_{j+1}], \\ \dfrac{1}{x_j - x_{j-1}} & x \in [x_{j-1}, x_j], \end{cases}$$

and $\omega'_j(x) = 0, x \notin [x_{j-1}, x_{j+1}]$.

The solution of the equation on the grid interval $[x_j, x_{j+1}]$ we take in the form:

$$\tilde{u}(x) = \sum_{k=j}^{j+1} c_k \, \omega_k(x) \, ,$$

where the polynomial basis splines of the second order of approximation $\omega_j(x), \omega_{j+1}(x)$ on every grid interval $[x_j, x_{j+1}]$ have the following form

$$\omega_j(x) = \frac{x - x_{j+1}}{x_j - x_{j+1}},$$

$$\omega_{j+1}(x) = \frac{x - x_j}{x_{j+1} - x_j}.$$

2.1.2 The Application of the Polynomial Basis Splines to the Boundary Value Problem

Let us take the equation:

$$-\frac{d}{dx} x^\alpha \frac{du(x)}{dx} + q(x)u(x) = f(x), \quad (4)$$

$0 < x < 1,\ 1 \le \alpha < 2,\ u(1) = 0$.

We construct the right part $f(x)$ of the equation in accordance with the exact solution $u(x) = ((x^{3-\alpha}) - 1)/(3 - \alpha)$. We are interested in the values of the function in the internal nodes x_j, $j = 1, \ldots, n - 1$. We know the value of the function in the node x_n. Let's consider the case, $q = 1, \alpha = 1$.

We need to solve the system $MC = F$, where the matrix M has elements m_{ji}. The matrix M turns out to be a symmetric one and it has a banded tridiagonal form.

In order to solve this problem with the variational-difference method, we have to calculate the integrals:

$$m_{jj} = [\omega_j, \omega_j] =$$

$$\int_{x_{j-1}}^{x_j} \left(\frac{1}{x_j - x_{j-1}}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{1}{x_j - x_{j+1}}\right)^2 dx +$$

$$\int_{x_{j-1}}^{x_j} \left(\frac{x - x_{j-1}}{x_j - x_{j-1}}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{x - x_{j+1}}{x_j - x_{j+1}}\right)^2 dx,$$

$$m_{jj+1} = [\omega_j, \omega_{j+1}]$$

$$= \int_{x_j}^{x_{j+1}} \omega'_j(x)\omega'_{j+1}(x)\, dx + \int_{x_j}^{x_{j+1}} \omega_j(x)\omega_{j+1}(x)\, dx$$

$$= \int_{x_j}^{x_{j+1}} \frac{1}{x_j - x_{j+1}} \frac{1}{x_{j+1} - x_j}\, dx + \int_{x_j}^{x_{j+1}} \frac{x - x_{j+1}}{x_j - x_{j+1}} \frac{x - x_{j+1}}{x_{j+1} - x_j}\, dx.$$

The right side F of the system $MC = F$ has the elements as followed:

$$f_j = \int_{x_{j-1}}^{x_{j+1}} f(x)\omega_j(x)\, dx.$$

Next, we need to solve the system of equations $MC = F$.

2.1.3 The Trigonometrical Basis Splines

The support of the basis spline $\omega_j(x)$ consists of the two parts: $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$:

$$w_j(x) = \begin{cases} \dfrac{\sin(x - x_{j+1})}{\sin(x_j - x_{j+1})}, & x \in [x_j, x_{j+1}], \\ \dfrac{\sin(x - x_{j-1})}{\sin(x_j - x_{j-1})}, & x \in [x_{j-1}, x_j], \end{cases}$$

and $\omega_j(x) = 0, x \notin [x_{j-1}, x_{j+1}]$.

Also, we have the relation:

$$\omega_{j+1}(x) = \frac{\sin(x - x_j)}{\sin(x_{j+1} - x_j)}, x \in [x_j, x_{j+1}].$$

The trigonometric splines are convenient for solving equations of the form:

$$-\frac{d}{dx} \sin^\alpha(x) \frac{du(x)}{dx} + u(x) = f(x),$$

$$0 < x < 1, \quad 1 < \alpha < 2, \quad u(1) = 0.$$

Next, we consider the application of Hermitian-type splines to the solution of a boundary value problem with strong degeneracy.

In the next section we will discuss the use of local splines of a non-zero level for solving boundary value problems with degeneracy. We would like to recall the construction of the fourth-order splines of the first level. By the level we mean the number of derivatives of a function used to construct the approximation. The peculiarity of using these local splines is that we obtain a continuously differentiable approximation to the solution to boundary value problems. In addition, it is easy to construct the first and the second derivatives of this approximate solution.

2.2 The Application of Splines of the Fourth Order of Approximation of the First Level

Let us recall how the approximation is constructed using local splines of a non-zero level. We call the approximation level the number of derivatives that are used to construct the approximation.

The approximation using local splines of a non-zero level is constructed on every grid interval in the following form:

$$\tilde{u}(x) = \sum_s \sum_j u^{(s)}(x_j)\, \omega_{j,s}(x), x \in [x_k, x_{k+1}].$$

Here $\omega_{j,s}(x)$ are the basis splines. If $s = 0,1, \ldots m$, then the level of the approximation is m.

We assume that the support of each of the basis splines consists of two grid intervals. Let us assume that at each grid node x_k the values of the function $u(x)$ and its first derivative are known.

In this case, we construct an approximation of function $u(x)$, $x \in [x_k, x_{k+1}]$, in the form:

$$\tilde{u}(x) = \sum_{j=k,k+1} u(x_j)\,\omega_{j,0}(x) + u'(x_j)\omega_{j,1}(x).$$

We assume that supp $\omega_{k,0}$=supp $\omega_{k,1} = [x_{k-1}, x_{k+1}]$. Let the basis functions $\omega_{k,i}$ be determined from the conditions:

$$u \equiv \tilde{u}, u = 1, x, x^2, x^3.$$

From these conditions we obtain a system of equations for determining the basis functions on the interval $[x_j, x_{j+1}]$:

$$\omega_{j,0}(x) + \omega_{j,1}(x) = 1\,,$$
$$x_j\omega_{j,0}(x) + x_{j+1}\omega_{j+1,0}(x) + \omega_{j,1}(x) + \omega_{j+1,1}(x) = x\,,$$
$$x_j^2\ \omega_{j,0}(x) + x_{j+1}^2\omega_{j+1,0}(x) + 2\,x_j\ \omega_{j,1}(x) + 2\,x_{j+1}\ \omega_{j+1,1}(x) = x^2\,,$$
$$x_j^3\ \omega_{j,0}(x) + x_{j+1}^3\omega_{j+1,0}(x) + 3\,x_j^2\ \omega_{j,1}(x) + 3\,x_{j+1}^2\ \omega_{j+1,1}(x) = x^3\,.$$

Having solved the system of equations, we find the basis splines on the grid interval $[x_j, x_{j+1}]$.
The basis splines of the first level have the form:

$$\omega_{j,0}(x) = \frac{(x - x_{j+1})^2}{(x_{j+1} - x_j)^2} + \frac{2(x - x_j)(x - x_{j+1})^2}{(x_{j+1} - x_j)^3},$$
$$\omega_{j+1,0}(x) = \frac{(x - x_j)^2}{(x_{j+1} - x_j)^2} + \frac{2(x_{j+1} - x)(x - x_j)^2}{(x_{j+1} - x_j)^3},$$
$$\omega_{j,1}(x) = \frac{(x - x_j)(x - x_{j+1})^2}{(x_{j+1} - x_j)^2},$$
$$\omega_{j+1,1}(x) = \frac{(x - x_{j+1})(x - x_j)^2}{(x_{j+1} - x_j)^2}.$$

The plots of basis splines $\omega_{j,i}(x)$ on the interval $[x_j, x_{j+1}]$ are presented in Figure 1, Figure 2, Figure 3 and Figure 4.

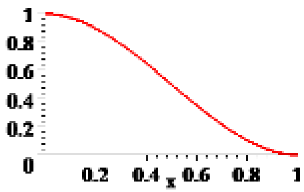

Fig. 1: The plot of the basis spline $\omega_{j,0}(x)$, $x \in [x_j, x_{j+1}]$

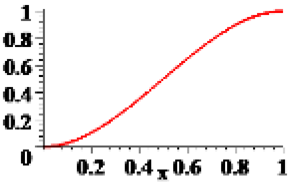

Fig. 2: The plot of the basis spline $\omega_{j+1,0}(x)$, $x \in [x_j, x_{j+1}]$

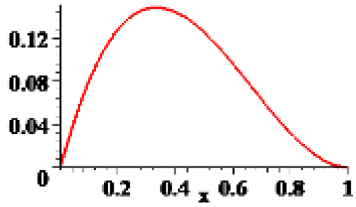

Fig. 3: The plot of the basis spline $\omega_{j,1}(x)$, $x \in [x_j, x_{j+1}]$

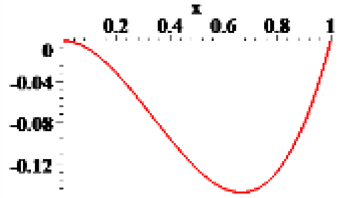

Fig. 4: The plot of the basis spline $\omega_{j+1,1}(x)$, $x \in [x_j, x_{j+1}]$

Now we get formulas for the approximation function $u(x)$,

$$\tilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x), \quad (5)$$
$$x \in [x_j, x_{j+1}].$$

Similarly, we construct basis splines on the adjacent grid interval $[x_{j-1}, x_j]$.

To construct a solution, we also need formulas for basis splines on the interval $x \in [x_{j-1}, x_j]$

$$\omega_{j-1,0}(x) + \omega_{j-1,1}(x) = 1\,,$$
$$x_{j-1}\omega_{j-1,0}(x) + x_j\omega_{j,0}(x) + \omega_{j-1,1}(x) + \omega_{j,1}(x) = x\,,$$
$$x_{j-1}^2\ \omega_{j-1,0}(x) + x_j^2\omega_{j,0}(x) + 2\,x_{j-1}\ \omega_{j-1,1}(x) + 2\,x_j\ \omega_{j,1}(x) = x^2\,,$$
$$x_{j-1}^3\ \omega_{j-1,0}(x) + x_j^3\omega_{j,0}(x) + 3\,x_{j-1}^2\ \omega_{j-1,1}(x) + 3\,x_j^2\ \omega_{j,1}(x) = x^3\,.$$

We assume that the support of the basis spline is supp$\omega_{j,0}$ = supp$\omega_{j,1} = [x_{j-1}, x_{j+1}]$.

The superlative point of the basis spline $\omega_{j,i}(x)$ will be called the point with coordinates $(x_j, \omega_{j,i}(x_j))$.

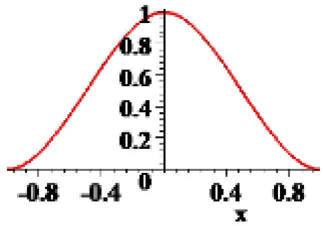

Fig. 5: The plot of the basis spline $\omega_{j,0}(x)$, $x \in [x_{j-1}, x_{j+1}]$

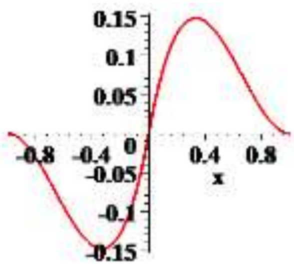

Fig. 6: The plot of the basis spline $\omega_{j,1}(x)$, $x \in [x_{j-1}, x_{j+1}]$

We have the formulas:

$$\omega_{j-1,1}(x) = \frac{(x - x_j)(x - x_{j-1})^2}{(x_{j-1} - x_j)^2},$$

$$\omega_{j-1,0}(x) = \frac{(x - x_j)^2}{(x_{j+1} - x_j)^2} + \frac{2(x_{j+1} - x)(x - x_j)^2}{(x_{j+1} - x_j)^3}.$$

By combining basis splines with a common superlative point and taking into account the support of the basis spline, we obtain local basis splines shown in Figure 5 and Figure 6.

We use the constructed splines to solve the boundary value problems.

If the function $u \in C^4[x_{j-1}, x_{j+1}]$, then the next estimation of the approximation of function $u(x)$ is valid when $x \in [x_j, x_{j+1}]$

$$|\tilde{u}(x) - u(x)| \leq Kh^4 \parallel u^{(4)} \parallel_{[x_{j-1}, x_{j+1}]}.$$

Here $h = x_{j+1} - x_j, K = 1/384$.

Splines of the fourth order of approximation and the first level were applied in the Least Squares Method, [23]. In the next section we discuss the application the splines of the fourth order of approximation and the first level for the solving the boundary value problem.

3 The Construction of the Solution of the Boundary Value Problem

Let us consider the following problem:

$$-\frac{d}{dx} x^\alpha p(x) \frac{du(x)}{dx} + q(x)u(x) = f(x),$$

$$0 < x < 1, \quad 1 \leq \alpha < 2, \quad u(1) = 0,$$

$$u'(1) = 0, \quad p(x) > 0.$$

Note that to solve this problem using splines of the first level, we will need the value of the first derivative at the right end of the interval [0, 1]: here we have $u'(1) = 0$.

On the interval [0, 1] we construct an ordered grid of nodes. First, let the grid nodes be equally spaced with step h. The matrix M of the system of equations $MC = F$ (according to (5)) will have a block form:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where every M_{ij} has a band structure. Non-zero elements are those lying on the main diagonal, supradiagonal and subdiagonal.

We take $p(x) = 1, q(x) = 1$. The elements of the matrix M_{11} have the following form:

$$[\omega_{j,0}, \omega_{j,0}] = \int_{x_{j-1}}^{x_j} (x^\alpha \ \omega'^2_{j-1,0}(x) + \omega^2_{j-1,0}(x))dx$$

$$+ \int_{x_j}^{x_{j+1}} (x^\alpha \ \omega'^2_{j,0}(x) \ + \omega^2_{j,0}(x))dx,$$

$$[\omega_{j,0}, \omega_{j+1,0}] = \int_{x_j}^{x_{j+1}} (x^\alpha \omega'_{j,0}(x) \ \omega'_{j+1,0}(x)$$

$$+\omega_{j,0}(x) \ \omega_{j+1,0}(x))dx.$$

Note that the minimum number n of nodes to solve our problem is $n = 2$. If $n = 2$ then we have:

$$M_{11} = \begin{pmatrix} [\omega_{1,0}, \omega_{1,0}] & [\omega_{1,0}, \omega_{2,0}] \\ [\omega_{2,0}, \omega_{1,0}] & [\omega_{2,0}, \omega_{2,0}] \end{pmatrix}.$$

In this case we have:

$$[\omega_{1,0}, \omega_{1,0}] = \int_{x_0}^{x_1} (x^\alpha \ \omega'^2_{1,0}(x) \ + \omega^2_{1,0}(x))dx,$$

If $n = 2$ then we have:

$$M_{12} = \begin{pmatrix} [\omega_{1,0}, \omega_{1,1}] & [\omega_{1,0}, \omega_{2,1}] \\ [\omega_{2,0}, \omega_{1,1}] & [\omega_{2,0}, \omega_{2,1}] \end{pmatrix}.$$

Similarly we have,

$$M_{21} = \begin{pmatrix} [\omega_{1,1}, \omega_{1,0}] & [\omega_{1,1}, \omega_{2,0}] \\ [\omega_{2,1}, \omega_{1,0}] & [\omega_{2,1}, \omega_{2,0}] \end{pmatrix}$$

and

$$M_{22} = \begin{pmatrix} [\omega_{1,1}, \omega_{1,1}] & [\omega_{1,1}, \omega_{2,1}] \\ [\omega_{2,1}, \omega_{1,1}] & [\omega_{2,1}, \omega_{2,1}] \end{pmatrix}.$$

We calculate the right side F of the system of equations using the formulas:

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.$$

The elements of vector $F_1 = (f_{1,0}, \dots, f_{n-1,0})^T$ have the form:

$$f_{i,0} = \int_{x_{j-1}}^{x_j} f(x)\omega_{j-1,0}(x)\,dx + \int_{x_j}^{x_{j+1}} f(x)\omega_{j,0}(x)\,dx\,.$$

The elements of vector $F_2 = (f_{1,1}, \dots, f_{n-1,1})^T$ have the form:

$$f_{i,1} = \int_{x_{j-1}}^{x_j} f(x)\omega_{j-1,1}(x)\,dx + \int_{x_j}^{x_{j+1}} f(x)\omega_{j,1}(x)\,dx\,.$$

Next, we have to solve the system

$$MC = F,$$

where

$$C = (c_{1,0}, \dots, c_{n,0}, c_{1,1}, \dots, c_{n,1})^T.$$

To construct a continuously differentiable smoothing solution of the boundary value problem, we use the solution $c_{j,i}$, $j = 1, 2, \dots, n, i = 0,1$, of and the splines of the fourth order of approximation of the first level $\omega_{j,i}$.

4 The Numerical Experiments

Example 1. We solve the boundary value problem

$$-\frac{d}{dx} x \frac{du(x)}{dx} + u(x) = f(x),$$
$$0 < x < 1, \quad u(1) = 0, \qquad u'(1) = 0.$$

where the exact solution is $u(x) = (1 - x)^3$.

We apply the splines of the fourth order of approximation and the first level. We take $n = 20$.

The plot of the error of the solution and its first derivative are given in Figure 7 and Figure 8. The numbers of the grid nodes are plotted along the axis x.

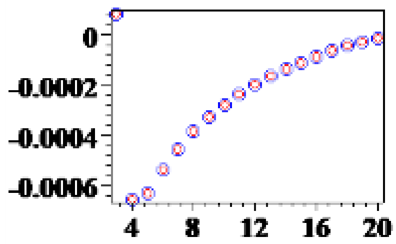

Fig. 7: The plot of the error of the solution

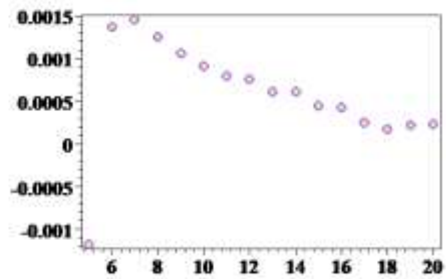

Fig. 8: The plot of the error of the first derivative of the solution

Example 2. We solve the boundary value problem:

$$-\frac{d}{dx} x^{\alpha} \frac{du(x)}{dx} + u(x) = f(x),$$
$$0 < x < 1, \quad u(1) = 0, \qquad u'(1) = 0.$$

where α=1.5 and the exact solution $u(x) = (1 - x)^3$. We apply the splines of the fourth order of approximation and the first level. After we solve the system of equations and obtain the values $c_{j,i}$ we can connect the points of the grid solution using the rule:

$$\widetilde{U}(x) = c_{j,0}\omega_{j,0}(x) + c_{j+1,0}\omega_{j+1,0}(x) + c_{j,1}\omega_{j,1}(x) + c_{j,1}\omega_{j+1,1}(x),$$
$$x \in [x_j, x_{j+1}].$$

The plot of the error of the solution and its first derivative are given in Figure 9 and Figure 10. The numbers of the grid nodes are plotted along the axis x.

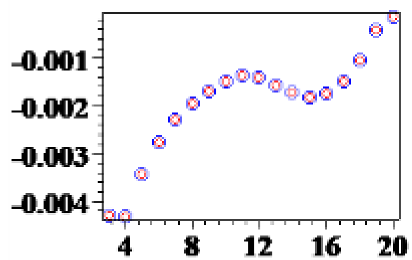

Fig. 9: The plot of the error of the solution

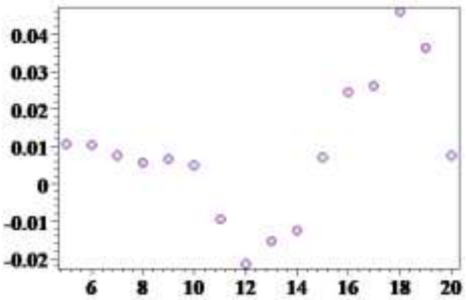

Fig. 10: The plot of the error of the first derivative of the solution

Example 3. We solve the boundary value problem

$$-\frac{d}{dx}(\sin x)^{\alpha}\frac{du}{dx}+u(x)=f(x),$$
$$0<x<1,\quad u(1)=0,\qquad u'(1)=0.$$

The exact solution is $u(x)=\sin((1-x)^3)$. We take α=1.5 and we apply the trigonometric splines. The plot of the error of the solution are given in Figure 11 and Figure 12. The numbers of the grid nodes are plotted along the axis x.

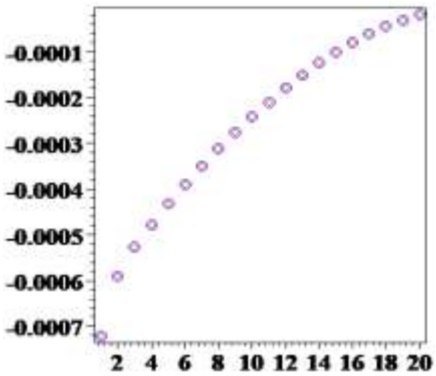

Fig. 11: The plot of the error of the solution

We obtained an approximate solution at the grid nodes (grid solution). Next, we can connect the points of the grid solution using the rule:

$$\widetilde{U}(x)=c_j\omega_j(x)+c_{j+1}\omega_{j+1}(x),x\in[x_j,x_{j+1}].$$

Here we used the trigonometric splines.
The plot of the error of the solution is shown in Figure 12.

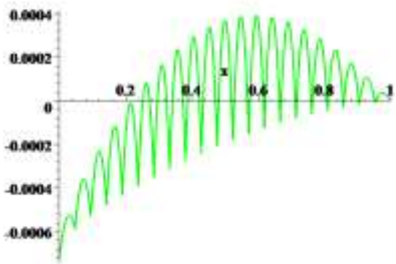

Fig. 12: The plot of the error of the solution

Example 4. We solve the boundary value problem

$$-\frac{d}{dx}(\sin x)^{\alpha}\frac{du}{dx}+u(x)=f(x),$$
$$0<x<1,\quad u(1)=0,\qquad u'(1)=0.$$

The exact solution is $u(x)=(1-x)^3$. We apply the trigonometric splines. The plot of the error of the solution is given in Figure 13. The numbers of the grid nodes are plotted along the axis x.

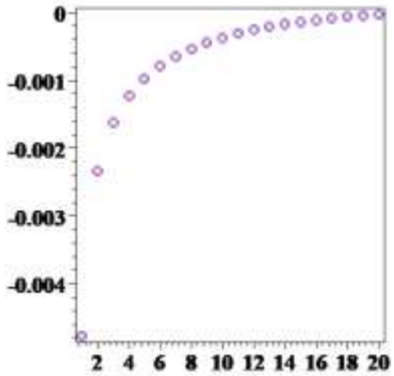

Fig. 13: The plot of the error of the solution

5 Conclusion

In the work, polynomial and trigonometric splines of the second order of approximation were used to solve a boundary value problem with a strong degeneracy. Experiments have shown the advantage of trigonometric splines if the equation has a trigonometric right part and trigonometrical coefficients. In the case of using splines of non-zero level, we simultaneously find a continuously differentiable solution and its derivative.

The next work will consider the use of spline approximation with wide support and a non-uniform grid for solving a boundary value problem with a strong degeneracy.

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- I. G.Burova developed algorithms and conducted numerical experiments,
- G. O. Alcybeev executed the numerical experiments.

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Conflict of Interest

The authors have no conflicts of interest to declare.