# About the Uniqueness of Approximate Numerical Solutions of Scalar Conservation Laws with a non Lipschitz Flux Function in an Infinite Space Domain

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*Abstract:* In this note, we investigate about discrete entropy solution of scalar conservation law. We establish uniqueness of finite volume approximate solution to scalar conservation laws with a non Lipschitz flux function in the whole space. Our arguments are based on properties of moduli of continuity of the components of the numerical flux.

*Key-Words:* Scalar conservation laws, Finite volume scheme, Modulus of continuity, Entropy formulation.

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## 1 Introduction

The aim of this paper is to propose a uniqueness result for approximate solution obtained by some finite volume schemes which approach the following nonlinear hyperbolic equation:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{\ell} \frac{\partial}{\partial x_k} f_k(u) = g \text{ in } Q = (0,T) \times \mathbb{R}^\ell \quad (1)$$

Moreover, (1) is supplemented with an initial condition

$$u(0,x) = u_0(x) \text{ in } \mathbb{R}^\ell. \quad (2)$$

The source term $g$ and initial data $u_0$ satisfy

$$g \in L^1(0,T; L^\infty_{loc}(\mathbb{R}^\ell));\ u_0 \in L^\infty(\mathbb{R}^\ell) \quad (3)$$

Here the convection flux $f = (f_1; \cdots, f_\ell) : \mathbb{R} \to \mathbb{R}^\ell$ is merely continuous not Lipschitz continuous. Problems like (1)-(2) that are the central point of our works occur in several applications, including porous media flow, sedimentation processes, road traffic, the dispersal of a single species of animals in a finite territory...For example, batch or continuous sedimentation processes are utilized in many industrial applications in which a solid-fluid suspension is separated into its solid and fluid components under the influence of gravity [1]. What we know in the study of hyperbolic problem is the lack of uniqueness of weak solution for general continuous flux functions $f_i$ even if the initial and source term are regulars. The global problem is : the infinite speed of propagation makes "infinity points" be "singular boundary points" for the equation. Indeed, with the method developed in the celebrate paper of [2], it is quite easy to show, even for general continuous flux field $f$, uniqueness of the so-called entropy solutions that are compactly supported in $\mathbb{R}$ (uniformly with respect to time); but, in the case of non locally Lipschitz flux functions, compactly supported data $f$ in (1) or $(u_0, g)$ in (1)-(2) do not yield in general compactly supported entropy solution. The authors in [3], under general anisotropic conditions on the modulus of continuity of the fluxes $f_i$ insure comparison principle for entropy solution and then prove uniqueness of entropy solution. For example in two space dimension, the propose a following family of flux

$$f_i(u) = \frac{|u|^{\alpha_i - 1} u}{\alpha_i}; 0 < \alpha_1 < \alpha_2 \quad (4)$$

In the numerical point of view, let us recall some non exhaustive results about (1). In recent paper, [4], employed implicit finite difference schemes for (1) and investigated about the monotonicity property of an implicit scheme. They construct a monotonicity notion that is based on a comparison of data sets using an induction principle to obtain a discrete comparison principle. In [1], the authors consider (1)-(2) with zero-flux boundary conditions imposed on the boundary of a rectangular multidimensional domain but the flux $f_i$ are Lipschitz-continuous contrarily to our case. They study monotone schemes applied to this problem and show that the approximate solutions produced by these schemes converge to the unique entropy solution in the sense of [5]. The authors in [6], showed that a front tracking method [7] converges to a weak solution of (1)-(2) in a bounded domain in one spatial dimension with zero-flux boundary condition. This weak solution is unique in the class of functions that can be constructed as the $L^1$ limit of front tracking approximations. Moreover, they present numerical results for the case of two spatial dimensions. However, for none of these cases they present a notion of entropy solution for which existence and uniqueness is proved. It should also be noted that many authors inspired by (1), generalize the study on the parabolic case which degenerates into hyperbolic according to values of the unknown $u$:[8], [9], [10], [11], [12], [13], [14], [15]. The outline of the paper is the following: in Section 2, we briefly review the main results contained in [3], in Section 3, we propose our finite volume scheme to approximates (1). Section 4 will be devoted to the proofs of uniqueness of approximates solution. In the last section, we discuss about convergence result.

## 2 Framework of Uniqueness Result for Continuous Problem

In this section, let us recall the main result obtained in [3]. We first give an entropy formulation for (1)-(2) then recall the Lemma 1.1 and Theorem 2.1 of [3].

**Definition 2.1** *we say that a bounded measurable function* $u \in L^\infty(\mathbb{R}^\ell)$ *is called an entropy solution of* (1)-(2) *if the following inequality for all non negative* $\xi \in C^\infty([0,T[\times\mathbb{R}^\ell)$, $k \in \mathbb{R}$ *holds:*

$$\int_0^T\int_{\mathbb{R}^\ell} |u(t,x)-k|\xi_t + \int_{\mathbb{R}^\ell}|u_0-k|\xi(0,x)dx + \int_0^T\int_{\mathbb{R}^\ell} sign(u-k)g\xi dxdt \geq 0. \tag{5}$$

Recall the following the main Lemma in [3]

**Lemma 2.2** *Let* $\lambda_1,\cdots,\lambda_\ell$ *be a positive finite functions on* $(0,+\infty]$ *and assume that for* $i=1,\cdots,\ell$

$$\begin{cases} \lambda_i(0) = \lim_{\epsilon\to 0}\lambda_i(\epsilon) \text{ exists in } (0,+\infty], \\ C = \liminf_{\epsilon\to 0}\epsilon\prod_{i=1}^{\ell}\lambda_i(\epsilon) < \infty. \end{cases}$$

*Let* $h \in L^1_{loc}(Q)$ *with* $h^+ = \max(0;h) \in L^1(Q)$, $w_0 \in L^1(\mathbb{R}^\ell)$ *and* $w \in L^1_{loc}(Q)$, $w \geq 0$ *with*

$$e^{-\delta|x|}w \in L^1(Q) \quad \textit{for any } \delta > 0. \tag{6}$$

*Assume that for some constant* $\lambda > 0$

$$\iint_Q u\xi_t + \iint_Q \sum_{\lambda_i(0)=\infty}(w+\epsilon)\lambda_i(\epsilon)|\xi_{x_i}| + \iint_Q \lambda w\sum_{\lambda_i(0)<\infty}|\xi_{x_i}| + \iint_Q h\xi \geq 0 \tag{7}$$

*for any* $\epsilon > 0$ *and* $\xi \in D(Q)$; $\xi \geq 0$ *and*

$$(w(t,.)-w_0)^+ \to 0 \text{ in } L^1(\mathbb{R}^\ell) \text{ as } t \to 0 \tag{8}$$

*Then*

$$\int_{\mathbb{R}^\ell} w(\tau,x)dx \leq \int_{\mathbb{R}^\ell} w_0 dx + \int_0^\tau\int_{\mathbb{R}^\ell} h(t,x)dtdx \tag{9}$$

*for a.e.* $\tau \in (0,T)$

Using the Lemma above in [3], the authors propose the following theorem

**Theorem 2.3** *Let* $\omega_{f_1},\cdots,\omega_{f_\ell}$ *be moduli of continuity of* $f_1;\cdots,f_\ell$ *that is sub-additive increasing continuous functions from* $[0;+\infty)$ *into* $(0,+\infty]$ *with*

$$\begin{cases} \omega_{f_1}(0) = \cdots = \omega_{f_\ell} = 0 \\ \liminf_{r\to 0} r^{1-\ell}\prod_{i=1}^{\ell}\omega_{f_i}(r) < \infty \end{cases}$$

*Let* $u$, $\hat{u}$ *entropy solution of* (1)-(2) *corresponding to data* $(u_0, g)$ *and* $(\hat{u}_0,\hat{g}_0)$ *in* $L^1(\mathbb{R}^\ell)\times L^1_{loc}(Q)$ *with* $w = (u+\hat{u})^+ \in L^1(Q)$ *satisfying* (6), $w_0 \in L^1(\mathbb{R}^\ell)$ *and* $w \in L^1_{loc}(Q)$ , $w \geq 0$. *Assume that for some constant* $i = 1,\cdots,\ell$

$$|f_i(u)-f_i(\hat{u})| \leq w_{f_i}(u-\hat{u}) \text{ a.e. on } \{u > \hat{u}\} \tag{10}$$

*If* $(u_0+\hat{u}_0)^+ \in L^1(\mathbb{R}^\ell)$ *and* $(g-\hat{g})^+Id_{\{u>\hat{u}\}} \in L^1(Q)$ *for* $i = 1,\cdots,\ell$. *Then,* $(u+\hat{u})^+ \in L^1(0,T;L^1(\mathbb{R}^\ell))$, $(g-\hat{g})Id_{\{u>\hat{u}\}} \in L^1(Q)$ *and*

$$\int_{\mathbb{R}^\ell}(u(\tau,x)-\hat{u}(\tau,x))^+ \leq \int_{\mathbb{R}^\ell}(u_0(x)-\hat{u}_0(x))^+ \tag{11}$$

$$+\iint_{Q_\tau\cap\{u>\hat{u}\}}(g-\hat{g}) + \iint_{Q_\tau\cap\{u=\hat{u}\}}(g-\hat{g})^+ \tag{12}$$

*for a.e.* $\tau \in (0,T)$ *in particular if* $u_0 \leq \hat{u}_0$ *on* $\mathbb{R}^\ell$ *and* $f \leq \hat{f}$ *a.e.* $\{u > \hat{u}\}$ *then* $u \leq \hat{u}$ *on* $Q$

An immediate consequence of Theorem 2.3 is a uniqueness result of entropy solution. The assumption on the moduli of continuity of the flux $f_i$ is in some sense sharp to obtain uniqueness of entropy solution.

## 3 Implicit Finite Volume Scheme to Approach (1)-(2)

Finite volume schemes are used to compute an approximation of the solution of a system of equations set on a certain domain. In this paper, we propose a finite volume scheme whose unknowns are the discrete values of the volume ratio on Cartesian meshes. The principle for example, to construct a finite volume scheme for a Partial Differential Equation (PDE) is to decompose the domain into small parts (the control volumes) and to integrate the equation on these volumes. The way is to decompose the time-space domain using small rectangles and integrating the PDE on each rectangle. Cartesian meshes are admissible meshes. The last step consists in using the fact that the approximate solution is bounded and using the nonlinear weak-star convergence to show that the sequence of approximate solutions converges towards the notion of the entropy process solution and show by the doubling of the Kruzkov variables that this notion of the entropy process solution coincides with the entropy solution the control volumes of which satisfying an orthogonality property between the "centers" of the control volumes and the edges.). The reader may consult: [8], [13], [16], [17], [18], [19], [20]. Introduce a constant (for simplicity) time step $\delta t > 0$, and for a control volume $K_i$ with center coordinate $x_{\vec{i}}$, $\vec{i} := (i_1, i_2, ..., i_\ell) \in \mathbb{Z}^\ell$ and $\vec{e}_k := \overbrace{(0,...,0,1,0,...,0)}^{1 \text{ at position } k}$, take $\delta x_k$ the space step in the direction $\vec{e}_k$. Let $u^n_{\vec{i}}$ and $g^n_{\vec{i}}$ denote the value of the numerical solution and the value of the source term at the point which is a center of volume $K_{\vec{i}}$ at the time level $n\delta t$. We define the numerical convection fluxes which approach the fluxes $f_k$ by $F_k : \mathbb{R}^2 \longrightarrow \mathbb{R}$ for $k = 1, ..., \ell$, $(a, b) \mapsto F_k(a, b)$. The numerical convection fluxes are monotone (non-decreasing with respect to the first variable and non-increasing with respect to the second variable) i.e: for all $k = 1, ..., \ell$,

$$\partial_b F_k(a, b) \leq 0 \leq \partial_a F_k(a, b). \tag{13}$$

The traditional Lipschitz continuity of $F_k$ makes no sense in our framework, because $f_k$ are non-Lipschitz; therefore, the classical Courant-Friedrichs-Levy (CFL) condition for explicit schemes is impossible to satisfy. We will formulate an implicit scheme, which does not require Lipschitz continuity of $F_k$ for the sake of stability and convergence analysis.

The discretization of (1) is performed with the classical upwind Finite Volume scheme for the convection term. Finite volume implicit scheme for (1) on uniform rectangular meshes is:

$$u^{n+1}_{\vec{i}} = u^n_{\vec{i}} - \delta t \sum_{k=1}^{\ell} \frac{\mathcal{F}(u^{n+1}_{\vec{i}-\vec{e}_k}, u^{n+1}_{\vec{i}}, u^{n+1}_{\vec{i}+\vec{e}_k})}{\delta x_k} + \delta t g^{n+1}_{\vec{i}} \tag{14}$$

where

$$\mathcal{F}(u^{n+1}_{\vec{i}-\vec{e}_k}, u^{n+1}_{\vec{i}}, u^{n+1}_{\vec{i}+\vec{e}_k}) := F_k\left(u^{n+1}_{\vec{i}}, u^{n+1}_{\vec{i}+\vec{e}_k}\right) - F_k\left(u^{n+1}_{\vec{i}-\vec{e}_k}, u^{n+1}_{\vec{i}}\right)$$

and this gives

$$u^{n+1}_{\vec{i}} = u^n_{\vec{i}} - \sum_{k=1}^{\ell} \lambda_k \mathcal{F}(u^{n+1}_{\vec{i}-\vec{e}_k}, u^{n+1}_{\vec{i}}, u^{n+1}_{\vec{i}+\vec{e}_k}) + \delta t g^{n+1}_{\vec{i}} \tag{15}$$

with $\lambda_k = \dfrac{\delta t}{\delta x_k}$.

To complete the discretization, we have to approximate initial datum. One can consider for example for given $u_0 \in L^1(\mathbb{R}^\ell)$, the approximation value

$$u^0_{\vec{i}} = \frac{1}{\prod_{k=1}^{\ell} \delta x_k} \int_{K_{\vec{i}}} u_0(x)dx \tag{16}$$

or for $u_0 \in L^\infty(\mathbb{R}^\ell)$, the approximation value

$$u^0_{\vec{i}} = u_0(x_{\vec{i}}). \tag{17}$$

Scheme (15) appears to be implicit, using the Godunov scheme for the convection term (which is the upstream weighting scheme in the present case where $f$ is non decreasing). It is then possible to show that the implicit scheme (15) has at least one solution, which allows to define the function by the value $u^{n+1}_{\vec{i}}$ for a.e. $x_{\vec{i}}$ center of control volume $K_{\vec{i}}$ and $t \in (n\delta t, (n+1)\delta t)$.

## 4 Uniqueness of Approximate Solution

Since the components $f_i$ of the flux $f$ are not Lipschitz continuous, the numerical flux is also not Lipschitz continuous. We suppose that $\omega_{F_k}$; for all $k = 1, \cdots, \ell$ are the modulus of continuity of $F_k(.,.)$ and

$$\begin{cases} |F_k(c,d) - F_k(\hat{c},\hat{d})| \leq \omega_{F_k}(|c-\hat{c}|) + \omega_{F_k}(|d-\hat{d}|), \\ \text{for } k = 1, ..., N \text{ and } (c,d), \, (\hat{c},\hat{d}) \in \mathbb{R}^2 \end{cases} \tag{H1}$$

Moreover, since $\omega_{F_k}$ is sub additive and increasing, for $a = bq + r$ with $q \in \mathbb{N}$ and for $0 \leq r < b$, $\omega_{F_k}(a) < (q+1)\omega_{F_k}(b)$.
We introduce now the following notation for all $(a,b) \in \mathbb{R}^2$:

$$a \top b = \max(a,b),\ a \bot b = \min(a,b)$$

and for $k, l \in \mathbb{N}$, $\vec{i} := (i_1, i_2, ..., i_\ell) \in \mathbb{Z}^\ell$, we let:

$$\begin{aligned}
Q_{\vec{i}+\frac{1}{2}\vec{e}_k} &:= F_k(\hat{u}^{n+1}_{\vec{i}} \top u^{n+1}_{\vec{i}}, \hat{u}^{n+1}_{\vec{i}+\vec{e}_k} \top u^{n+1}_{\vec{i}+\vec{e}_k}) \\
&\quad - F_k(\hat{u}^{n+1}_{\vec{i}} \bot u^{n+1}_{\vec{i}}, \hat{u}^{n+1}_{\vec{i}+\vec{e}_k} \bot u^{n+1}_{\vec{i}+\vec{e}_k}), \\
Q_{\vec{i}-\frac{1}{2}\vec{e}_k} &:= F_k(\hat{u}^{n+1}_{\vec{i}-\vec{e}_k} \bot u^{n+1}_{\vec{i}-\vec{e}_k}, \hat{u}^{n+1}_{\vec{i}} \bot u^{n+1}_{\vec{i}}) \\
&\quad - F_k(\hat{u}^{n+1}_{\vec{i}-\vec{e}_k} \top u^{n+1}_{\vec{i}-\vec{e}_k}, \hat{u}^{n+1}_{\vec{i}} \top u^{n+1}_{\vec{i}}), \\
\Delta^l_{\vec{i}} &= |u^l_{\vec{i}} - \hat{u}^l_{\vec{i}}|, \\
\mathcal{G}^l_{\vec{i}} &= sign(u^l_{\vec{i}} - \hat{u}^l_{\vec{i}})(g^l_{\vec{i}} - \hat{g}^l_{\vec{i}}).
\end{aligned}$$

**Lemma 4.1** *Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}; (a,b) \mapsto g(a,b)$ such that, g satisfies* (13). *Then, for all $(a,b,c,\hat{a},\hat{b},\hat{c}) \in \mathbb{R}^6$*

$$\begin{aligned}
&sign(a-\hat{a})\Big(g(a,b) - g(\hat{a},\hat{b})\Big) - \Big(g(c,a) - g(\hat{c},\hat{a})\Big) \\
&\geq \Big(g(\hat{a}\top a, \hat{b}\top b) - g(a\bot\hat{a}, b\bot\hat{b})\Big) \\
&- \Big(g(\hat{c}\top c, \hat{a}\top a) - g(\hat{c}\bot c, a\bot\hat{a})\Big). \qquad (18)
\end{aligned}$$

**Proof.** Let $A$ the left hand side term of inequality (18) and $B$, the right hand side term of inequality (18). Suppose $A = A_1 + A_2$ and $B = B_1 + B_2$ where

$$\begin{aligned}
A_1 &= sign(a-\hat{a})\Big(g(a,b) - g(\hat{a},\hat{b})\Big); \\
A_2 &= -sign(a-\hat{a})\Big(g(c,a) - g(\hat{c},\hat{a})\Big), \\
B_1 &= \Big(g(\hat{a}\top a, \hat{b}\top b) - g(a\bot\hat{a}, b\bot\hat{b})\Big); \\
B_2 &= -\Big(g(\hat{c}\top c, \hat{a}\top a) - g(\hat{c}\bot c, a\bot\hat{a})\Big).
\end{aligned}$$

In first time, we prove that $A_1 \geq B_1$ and after $A_2 \geq B_2$. We examine three situations.
Case 1: $a < \hat{a}$, then $A_1 = g(\hat{a},\hat{b}) - g(a,b)$ and $B_1 = g(\hat{a}, b\top\hat{b}) - g(a, b\bot\hat{b})$. As $b\top\hat{b} \geq \hat{b}$; $b\bot\hat{b} \leq b$ and $\partial_b g(a,b) \leq 0$, we get $A_1 \geq B_1$.
Case 2: $a > \hat{a}$, then $A_1 = g(a,b) - g(\hat{a},\hat{b})$ and $B_1 = g(a, b\top\hat{b}) - g(\hat{a}, b\bot\hat{b})$. As $b\top\hat{b} \geq b$; $b\bot\hat{b} \leq \hat{b}$ and $\partial_b g(a,b) \leq 0$, we get also $A_1 \geq B_1$.
Case 3: $a = \hat{a}$, then $A_1 = 0$ and $B_1 = g(a, b\top\hat{b}) - g(a, b\bot\hat{b})$. As $b\top\hat{b} \geq \hat{b}\bot b$; and $\partial_b g(a,b) \leq 0$, we get $B_1 \leq 0 = A_1$.
From now, the proof of the second inequality is similarly, because of $\partial_a g(a,b) \geq 0$.

**Lemma 4.2**
*If $(u^n_{\vec{i}})_{\vec{i}\in\mathbb{Z}^\ell, n\in\mathbb{N}}$ and $(\hat{u}^n_{\vec{i}})_{\vec{i}\in\mathbb{Z}^\ell, n\in\mathbb{N}}$ are two discrete solutions of* (1)-(2) *with initial data $u^0_{\vec{i}}$, $\hat{u}^0_{\vec{i}}$. Then, for all $\vec{i} \in \mathbb{Z}^\ell$,*

$$\Delta^{n+1}_{\vec{i}} + \sum_{k=1}^{\ell} \lambda_k \left(Q_{\vec{i}+\frac{1}{2}\vec{e}_k} - Q_{\vec{i}-\frac{1}{2}\vec{e}_k}\right) - \delta t \mathcal{G}^{n+1}_{\vec{i}} \leq \Delta^n_{\vec{i}}. \qquad (19)$$

The inequality (19) is called discrete entropy inequality.
**Proof.** Let $(\hat{u}_{\vec{i}})_{\vec{i}\in\mathbb{Z}^\ell}$ and $(u_{\vec{i}})_{\vec{i}\in\mathbb{Z}^\ell}$ two discrete solutions of (1)-(2). To simplify the notations, let $u^{n+1}_{\vec{j}} = u_{\vec{j}}, u^n_{\vec{j}} = s_{\vec{j}}$ and $\hat{u}^{n+1}_{\vec{j}} = \hat{u}_{\vec{j}}, \hat{u}^n_{\vec{j}} = \hat{s}_{\vec{j}}$. Then, they satisfy (15) and we have

$$\begin{aligned}
u_{\vec{i}} - \hat{u}_{\vec{i}} = &- \sum_{k=1}^{\ell} \lambda_k \overline{\mathcal{F}}(u_{\vec{i}}, u_{\vec{i}+\vec{e}_k}, \hat{u}_{\vec{i}}, \hat{u}_{\vec{i}+\vec{e}_k}) \\
&+ \sum_{k=1}^{N} \lambda_k \overline{\mathcal{F}}(u_{\vec{i}-\vec{e}_k}, u_{\vec{i}}, \hat{u}_{\vec{i}-\vec{e}_k}, \hat{u}_{\vec{i}})) \\
&+ (s_{\vec{i}} - \hat{s}_{\vec{i}}) - \delta t(g^{n+1}_{\vec{i}} - \hat{g}^{n+1}_{\vec{i}}). \qquad (20)
\end{aligned}$$

where

$$\overline{\mathcal{F}}(u_{\vec{i}}, u_{\vec{i}+\vec{e}_k}, \hat{u}_{\vec{i}}, \hat{u}_{\vec{i}+\vec{e}_k}) = F_k(u_{\vec{i}}, u_{\vec{i}+\vec{e}_k}) - F_k(\hat{u}_{\vec{i}}, \hat{u}_{\vec{i}+\vec{e}_k}),$$

$$\overline{\mathcal{F}}(u_{\vec{i}-\vec{e}_k}, u_{\vec{i}}, \hat{u}_{\vec{i}-\vec{e}_k}, \hat{u}_{\vec{i}}) = F_k(u_{\vec{i}-\vec{e}_k}, u_{\vec{i}}) - F_k(\hat{u}_{\vec{i}-\vec{e}_k}, \hat{u}_{\vec{i}})$$

Multiplying the relation (20) by $sign(u_{\vec{i}} - \hat{u}_{\vec{i}})$, we get

$$\begin{aligned}
\Delta^{n+1}_{\vec{i}} = &- \sum_{k=1}^{\ell} \lambda_k sign(u_{\vec{i}} - \hat{u}_{\vec{i}}) \mathcal{F}_k(u_{\vec{i}}, u_{\vec{i}+\vec{e}_k}, \hat{u}_{\vec{i}}, \hat{u}_{\vec{i}+\vec{e}_k}) \\
&+ \sum_{k=1}^{\ell} \lambda_k sign(u_{\vec{i}} - \hat{u}_{\vec{i}}) \mathcal{F}_k(u_{\vec{i}-\vec{e}_k}, u_{\vec{i}}, \hat{u}_{\vec{i}-\vec{e}_k}, \hat{u}_{\vec{i}}) \\
&+ sign(u_{\vec{i}} - \hat{u}_{\vec{i}})(s_{\vec{i}} - \hat{s}_{\vec{i}}) \\
&- sign(u_{\vec{i}} - \hat{u}_{\vec{i}})(g^{n+1}_{\vec{i}} - \hat{g}^{n+1}_{\vec{i}}).
\end{aligned}$$

Using (18), we get (19).

From now, we follow the techniques and approach of [3], to prove the main result (Theorem 4.3) which is a key of the uniqueness of discrete solution.

**Theorem 4.3**
*Let $(u_{\vec{i}})_{\vec{i}\in\mathbb{Z}^\ell}$ and $(\hat{u}_{\vec{i}})_{\vec{i}\in Z^\ell}$ be two discrete solutions of*

(1). *Let $\omega_{F_1}$, $\omega_{F_2}$,...,$\omega_{F_\ell}$ the modulus of continuity of components $(F_k)_{1\le k\le \ell}$ satisfy* (H1)*,* (13)*. Then*

$$\sum_{\vec{i}\in\mathbb{Z}^\ell} \Delta_{\vec{i}}^{n+1} \le \sum_{\vec{i}\in\mathbb{Z}^\ell} \Delta_{\vec{i}}^{0}. \tag{21}$$

**Proof.** From now, for $\epsilon > 0$ multiply the entropy inequality (19) by $p_{\vec{i}}(\epsilon)$ with positive sequence $(p_{\vec{i}}(\epsilon))_{\vec{i}\in\mathbb{Z}^\ell}$ such that $\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon) < \infty$ and sum on $\vec{i} \in \mathbb{Z}^\ell$, we have

$$\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n+1} + \sum_{k=1}^{N}\lambda_k \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\left(Q_{\vec{i}+\frac{1}{2}\vec{e}_k} - Q_{\vec{i}-\frac{1}{2}\vec{e}_k}\right) \le \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n} + \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\delta t\mathcal{G}_{\vec{i}}^{n+1}. \tag{22}$$

Thanks to the absolute convergence of the series $(p_{\vec{i}})_{\vec{i}\in\mathbb{Z}^\ell}$, and $(Q_{\vec{i}\pm\frac{1}{2}\vec{e}_k})_{\vec{i}\in\mathbb{Z}^N} \in l^\infty(\mathbb{Z}^\ell)$ one can apply Abel sum to obtain

$$\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\left(Q_{\vec{i}+\frac{1}{2}\vec{e}_k} - Q_{\vec{i}-\frac{1}{2}\vec{e}_k}\right) = -\sum_{\vec{i}\in\mathbb{Z}^\ell}\left(p_{\vec{i}+\vec{e}_k}(\epsilon) - p_{\vec{i}}(\epsilon)\right)Q_{\vec{i}+\frac{1}{2}\vec{e}_k}. \tag{23}$$

So, (22) becomes

$$\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n+1} - \sum_{k=1}^{\ell}\lambda_k \sum_{\vec{i}\in\mathbb{Z}^\ell}\left(p_{\vec{i}+\vec{e}_k}(\epsilon) - p_{\vec{i}}(\epsilon)\right)Q_{\vec{i}+\frac{1}{2}\vec{e}_k} \le \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n} + \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\delta t\mathcal{G}_{\vec{i}}^{n+1}. \tag{24}$$

Using the subadditivity property of the modulus of continuity of $F_k$ in the quantities $Q_{\vec{i}+\frac{1}{2}\vec{e}_k}$ and with help of (H1) we have

$$\begin{aligned}\left|Q_{\vec{i}+\frac{1}{2}\vec{e}_k}\right| &\le \omega_{F_k}\left(\left|u_{\vec{i}}\top\hat{u}_{\vec{i}} - u_{\vec{i}}\bot\hat{u}_{\vec{i}}\right|\right) + \\ &\quad \omega_{F_k}\left(\left|u_{\vec{i}+\vec{e}_k}\top\hat{u}_{\vec{i}+\vec{e}_k} - u_{\vec{i}+\vec{e}_k}\bot\hat{u}_{\vec{i}+\vec{e}_k}\right|\right) \\ &\le \omega_{F_k}\left(\Delta_{\vec{i}}^{n+1}\right) + \omega_{F_k}\left(\Delta_{\vec{i}+\vec{e}_k}^{n+1}\right)\end{aligned}$$

With the insertion of these inequalities in (24), we find

$$\begin{aligned}&-\sum_{k=1}^{\ell}|\lambda_k|\sum_{\vec{i}\in\mathbb{Z}^\ell}\left|p_{\vec{i}+\vec{e}_k}(\epsilon) - p_{\vec{i}}(\epsilon)\right|\omega_{F_k}\left(\Delta_{\vec{i}}^{n+1}\right) + \\ &-\sum_{k=1}^{\ell}|\lambda_k|\sum_{\vec{i}\in\mathbb{Z}^\ell}\left|p_{\vec{i}+\vec{e}_k}(\epsilon) - p_{\vec{i}}(\epsilon)\right|\omega_{F_k}\left(\Delta_{\vec{i}+\vec{e}_k}^{n+1}\right) \\ &+\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n+1} \\ &\le \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n} + \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\delta t\mathcal{G}_{\vec{i}}^{n+1}. \end{aligned} \tag{25}$$

In the sequel, the decisive step is to construct an appropriate discrete test function. So, for $k = 1, ..., \ell$ and $\vec{i} = (i_1, ..., i_\ell) \in \mathbb{Z}^\ell$, we pose:

$$p_{\vec{i}}(\epsilon) = \prod_{k=1}^{\ell}\exp\left(-\epsilon^{\theta_k}|i_k|\right) \tag{26}$$

([3], for the continuous case).

With this choice, we have $(p_{\vec{i}}(\epsilon))_{\vec{i}\in\mathbb{Z}^\ell} \in l^1(\mathbb{Z}^\ell)$ since

$$\begin{aligned}\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon) &= \sum_{\vec{i}\in\mathbb{Z}^\ell}\left(\prod_{k=1}^{\ell}\exp\left(-\epsilon^{\theta_k}|i_k|\right)\right) \\ &= \prod_{k=1}^{\ell} 2\sum_{\vec{i}\in N^\ell}\left(\exp\left(-\epsilon^{\theta_k}|i_k|\right)\right) \\ &= \prod_{k=1}^{\ell}\frac{2\left(1-\left(\exp\left(-\epsilon^{\theta_k}|i_k|\right)\right)^\gamma\right)}{1-\exp\left(-\epsilon^{\theta_k}|i_k|\right)}; \gamma\to\infty \\ &= \prod_{k=1}^{\ell}\frac{2}{1-(1-\epsilon^{\theta_k}+\bar{\bar{o}}(1))} \\ &\le \prod_{k=1}^{\ell}\frac{2}{\epsilon^{\theta_k}} = \frac{2^\ell}{\epsilon^{\sum_{k=1}^{\ell}\theta_k}} = \frac{2^\ell}{\epsilon^{\theta_1+\cdots+\theta_\ell}}.\end{aligned}$$

So, for fixed $\epsilon$, $\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon) < \infty$.

For $\epsilon$ small enough, we see that on the one hand

$$\begin{aligned}\left|p_{\vec{i}+\vec{e}_k}(\epsilon) - p_{\vec{i}}(\epsilon)\right| &= p_{\vec{i}}(\epsilon)\left|1-\exp\left(-\epsilon^{\theta_k}\right)\right| \\ &\le 2p_{\vec{i}}(\epsilon)\left(1-(1-\epsilon^{\theta_k}+\bar{\bar{o}}(1))\right) \\ &\le 2p_{\vec{i}}(\epsilon)\epsilon^{\theta_k} \\ \text{and } \left|p_{\vec{i}+\vec{e}_k}(\epsilon) - p_{\vec{i}}(\epsilon)\right| &= p_{\vec{i}+\vec{e}_k}(\epsilon)\left|1-\exp\left(\epsilon^{\theta_k}\right)\right| \\ &\le 2p_{\vec{i}+\vec{e}_k}(\epsilon)\epsilon^{\theta_k};\end{aligned} \tag{27}$$

Euclidean division of $\Delta_{\vec{i}}^{n+1}$ by $\epsilon$:

$$\Delta_{\vec{i}}^{n+1} = \epsilon\left[\frac{\Delta_{\vec{i}}^{n+1}}{\epsilon}\right] + r \text{ with } 0 \le r < \epsilon$$

and

$$\omega_{F_k}(\Delta_{\vec{i}}^{n+1}) \leq \left(\frac{\Delta_{\vec{i}}^{n+1}}{\epsilon} + 1\right)\omega_{F_k}(\epsilon) \qquad (28)$$

Returning to (25) with (28) and inequalities (27) we get

$$\begin{aligned}
&-\sum_{k=1}^{\ell} 2\,|\lambda_k|\,\epsilon^{\theta_k} \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\left(\frac{\Delta_{\vec{i}}^{n+1}}{\epsilon}+1\right)\omega_{F_k}(\epsilon)\\
&-\sum_{k=1}^{\ell} 2\,|\lambda_k|\,\epsilon^{\theta_k} \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}+\vec{e}_k}\left(\frac{\Delta_{\vec{i}+\vec{e}_k}^{n+1}}{\epsilon}+1\right)\omega_{F_k}(\epsilon)\\
&+\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n+1}\\
&\leq \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n} + \delta t \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\mathcal{G}_{\vec{i}}^{n+1} \qquad (29)
\end{aligned}$$

from which we say that

$$\begin{aligned}
&\left[1-\sum_{k=1}^{\ell}\frac{\epsilon^{\theta_k}}{\epsilon}\left(4\,|\lambda_k|\,\omega_{F_k}(\epsilon)\right)\right]\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n+1}\\
&\leq \sum_{k=1}^{\ell}\epsilon^{\theta_k}\left(4\,|\lambda_k|\,\omega_{F_k}(\epsilon)\right)\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon) + \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n}\\
&\quad + \delta t \sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\mathcal{G}_{\vec{i}}^{n+1}.
\end{aligned}$$

$$\begin{aligned}
&(1-\Lambda)\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n+1} \leq \epsilon\Lambda\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\\
&+\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\Delta_{\vec{i}}^{n} + \delta t\sum_{\vec{i}\in\mathbb{Z}^\ell} p_{\vec{i}}(\epsilon)\mathcal{G}_{\vec{i}}^{n+1}.
\end{aligned}$$

where $\Lambda = \sum_{k=1}^{\ell}\epsilon^{\theta_k-1}\left(4\,|\lambda_k|\,\omega_{F_k}(\epsilon)\right)$. It remains to see easily that $\Lambda$ goes to zero when $\epsilon \to 0$. For example if $\omega_{F_k}(a) = a^{\alpha_k}$, just choose $\alpha_k + \theta_k > 1$. Now consider the same source term $g = \hat{g}$ for $\epsilon \to 0$, we obtain (21).

## 5 Discussion on Convergence Result

The question of the convergence of numerical schemas has always been at the center of the concerns of numerical analysis. The proof of convergence can be sketched as follows. First of all, the existence of the discrete solution. Knowing that we are in infinite dimension, we have to be careful. We can use a topological fixed point argument in infinite dimension, [17]. Then, the result of the uniqueness of a discrete solution extends to a result of discrete contraction in $L^1$ (either directly within the proof; or, by approximation of the $f$ flux by regular fluxes $f_n$; note that for a regular flux, the contraction $L^1$ is demonstrated in a "classical" way). Thanks to this result of discrete contraction, and to the invariance of the scheme by translation, we can affirm that there is a "modulus of uniform continuity in space", as in Kruzkov's founding paper. This comes from the combination of the two facts:

1. For the discretized initial data $(u_{\vec{i}}^0)_{\vec{i}\in\mathbb{Z}^\ell}$, the continuity modulus in space is uniform because $(u_{\vec{i}}^0)_{\vec{i}\in\mathbb{Z}^\ell}$ converges in $L^1$ to the continuous initial data $u_0$.

2. The discrete contraction then ensures that for any $t$, $u_{\vec{i}}(t, .)$ has the same continuity module.

It remains, as in Kruzkov's founding paper, to deduce "the modulus of uniform continuity in time" from that in space and from the itself. In a slightly different context, this is done in the paper, [21], and in more detail, in the appendix of the paper, [22].
Once the compactness in space-time is obtained, the passage to the limit in the formulation of the scheme with a test function does not require much. This is very standard.
The last step consists in using the fact that the approximate solution is bounded and using the nonlinear weak-star convergence to show that the sequence of approximate solutions converges towards the notion of entropy process solution and show by the doubling of the Kruzkov variables that this notion of entropy process solution coincides with the entropy solution.

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