Riemann-Liouville Generalized Fractional Integral Inequalities

SÜMEYYE ERMEYDAN ÇİRİŞ, HÜSEYİN YILDIRIM
Department of Mathematics,
University of Kahramanmaraş Sütçü İmam,
Kahramanmaraş, 46100,
TURKEY

Abstract: In this paper, we define Riemann-Liouville generalized fractional integral. Moreover, we obtained some significant inequalities for Riemann-Liouville generalized fractional integrals.

Key-Words: Fractional integrals; Generalized fractional integrals; Inequalities; Riemann-liouville fractional integrals; Chebyshev function; Integral inequalities [1]

Received: June 19, 2024. Revised: November 2, 2024. Accepted: November 25, 2024. Published: December 30, 2024.

1 Introduction

Fractional integrals are a generalization of the concept of integration to non-integer orders. In particular, fractional integrals extend the idea of differentiation to non-integer orders, which means they provide a way to define integrals of functions to fractional powers, [1], [2], [3].

Let's start with the Riemann-Liouville fractional integral . Given a function $f(x)$ defined on an interval $[a, b]$, and a real number $\alpha > 0$, the Riemann-Liouville fractional integral of order α of $f(x)$, denoted by $I^{\alpha} f(x)$,is defined as:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt. \tag{1}$$

Where $\Gamma(\alpha)$ is the gamma function. This definition can be extended to other types of integrals, such as the Caputo fractional integral, which is often used in fractional calculus.

Definition 1. [*4*, *5*] *Let $h(\tau)$ be an increasing and positive monotone function on $[0, \infty)$. Furthermore, we'll consider h as a monotonically increasing and positive function defined on the interval $[0, \infty)$, with its derivative h' being continuous and $\gamma(0) = 0$. The space $X_h^d(0, \infty)$ is the following form for $(1 \leq d < \infty)$,*

$$\|f\|_{X_h^d} = \left(\int_0^{\infty} |f(\theta)|^d h'(\tau)\, d\theta \right)^{\frac{1}{d}} < \infty \tag{2}$$

and if we choose $d = \infty$,

$$\|f\|_{X_h^{\infty}} = ess \sup_{1 \leq \theta < \infty} \left[f(\theta) h'(\tau) \right]. \tag{3}$$

Additionally, if we take $h(\tau) = \tau$ $(1 \leq d < \infty)$ the space $X_h^d(0, \infty)$, then we have the $L_d[0, \infty)-$space. Moreover, if we take $h(\tau) = \frac{\tau^{k+1}}{k+1}$ $(1 \leq d < \infty,\ k \geq 0)$ the space $X_h^d(0, \infty)$, then we have the $L_{d,k}[0, \infty)-$space [6].

Senouci and Khirani obtained newly the following definition of fractional integral [7].

Definition 2. *Let $f \in L_1([a, b])$, $a < b$, $\alpha > 0$, $k > 0$. Then, we have*

$$I_{0,k}^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t-x)^{\frac{\alpha}{k}-1} f(x)\, dx. \tag{4}$$

Where

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} \exp\left(-\frac{t^k}{k}\right) dt,\ k > 0. \tag{5}$$

Furthermore, we generalized this definition obtained by Abdelkader and Mohammed as the following

Definition 3. *Let $f \in L_1([a, b])$, $a < b$, $\alpha > 0$, $k > 0$. Suppose that $h(x)$ be an increasing and positive monotone function on $[0, \infty)$. Furthermore, we'll consider h as a monotonically increasing and positive function defined on the interval $[0, \infty)$, with its derivative h' being continuous and $\gamma(0) = 0$. Then,*

$$I_{k,h}^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (h(t) - h(x))^{\frac{\alpha}{k}-1} f(x) h'(x)\, dx. \tag{6}$$

Where

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} \exp\left(-\frac{t^k}{k}\right) dt,\ k > 0. \tag{7}$$

The Chebyshev fractional for two integrable functions f ve g which are synchronous (i,e

$(f(x)-f(y))(g(x)-g(y)) \geq 0$) for any $x, y \in [a, b]$, is defined as follows

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \frac{1}{b-a}\int_a^b g(x)\,dx. \tag{8}$$

Chebyshev fractional holds a significant position not only in mathematics but also in statistics, finding wide-ranging applications across various disciplines. Thereby, there are a lot of investigation on its. (See [8], [9] and [10]). Let give some definitions associated with the fractional integration in the sense of Riemann-Liouville.

2 Main Results

Lemma 1. *Let $f \in L_1([0,\infty))$ and $t > 0$, $\alpha > 0$, $k > 0$. Suppose that $h(x)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, we'll consider h as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative h' being continuous and $\gamma(0) = 0$. Then*

$$I_{k,h}^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)}\Gamma\left(\frac{\alpha}{k}\right) I^{\frac{\alpha}{k}} f(t). \tag{9}$$

Proof. For all $f \in L_1([0,\infty))$ and $t > 0$, $\alpha > 0$, $k > 0$, we have

$$I^{\frac{\alpha}{k}} f(t) = \frac{1}{\Gamma\left(\frac{\alpha}{k}\right)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) f(x)\,dx, \tag{10}$$

and

$$I_{k,h}^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) f(x)\,dx. \tag{11}$$

Then

$$\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) f(x)\,dx = \Gamma\left(\frac{\alpha}{k}\right) I^{\frac{\alpha}{k}} f(t), \tag{12}$$

finally

$$I_{k,h}^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)}\Gamma\left(\frac{\alpha}{k}\right) I^{\frac{\alpha}{k}} f(t). \tag{13}$$

The proof is done. □

Theorem 1. *Let the functions f and g be two synchronous functions on $[0,\infty[$. Suppose that $h(x)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, we'll consider h as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative h' being continuous and $\gamma(0) = 0$. Then for all $t > 0$, $\alpha > 0$, $k > 0$*

$$I_{k,h}^{\alpha}(fg)(t) \geq \frac{1}{I_{k,h}^{\alpha}(1)}.I_{k,h}^{\alpha} f(t).I_{k,h}^{\alpha} g(t) \tag{14}$$

Proof. The functions f and g are synchronous functions on then for all $x \geq 0$, $b \geq 0$, then

$$(f(x)-f(b))(g(x)-g(b)) \geq 0 \tag{15}$$

and

$$f(x)g(x) + f(b)g(b) \geq f(x)g(b) + f(b)g(x). \tag{16}$$

We have (16). Multiplying both hand sides of (16) by $\frac{(h(t)-h(x))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(x)$, $x \in (0,t)$,

$$\begin{aligned}&\frac{(h(t)-h(x))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(x)f(x)g(x) + \frac{(h(t)-h(x))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(x)f(b)g(b)\\ &\geq \frac{(h(t)-h(x))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(x)f(x)g(b) + \frac{(h(t)-h(x))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(x)f(b)g(x).\end{aligned} \tag{17}$$

By integrating (17) from 0 to t, we have

$$\begin{aligned}&\frac{1}{k\Gamma_k(\alpha)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) f(x) g(x)\,dx + \frac{1}{k\Gamma_k(\alpha)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) f(b) g(b)\,dx\\ &\geq \frac{1}{k\Gamma_k(\alpha)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) f(x) g(b)\,dx + \frac{1}{k\Gamma_k(\alpha)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) f(b) g(x)\,dx.\end{aligned} \tag{18}$$

In here, we can write

$$\begin{aligned}&I_{k,h}^{\alpha}(fg)(t) + f(b)g(b)\frac{1}{k\Gamma_k(\alpha)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x)\,dx\\ &\geq g(b)\frac{1}{k\Gamma_k(\alpha)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) f(x)\,dx + f(b)\frac{1}{k\Gamma_k(\alpha)}\int_0^t (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(x) g(x)\,dx.\end{aligned} \tag{19}$$

Finally, we get

$$I_{k,h}^{\alpha}(fg)(t) + f(b)g(b)I_{k,h}^{\alpha}(1) \geq g(b)I_{k,h}^{\alpha} f(t) + f(b)I_{k,h}^{\alpha} g(t). \tag{20}$$

Now, multiplying both hand sides of (20) by $\frac{(h(t)-h(b))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(b)$, $b \in (0,t)$,

$$\begin{aligned}&\frac{(h(t)-h(b))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(b)I_{k,h}^{\alpha}(fg)(t) + \frac{(h(t)-h(b))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(b)f(b)g(b)I_{k,h}^{\alpha}(1)\\ &\geq \frac{(h(t)-h(b))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(b)g(b)I_{k,h}^{\alpha} f(t) + \frac{(h(t)-h(b))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}h'(b)f(b)I_{k,h}^{\alpha} g(t).\end{aligned} \tag{21}$$

In here, by integrating (21) from 0 to t,

$$\begin{aligned}&I_{k,h}^{\alpha}(fg)(t)\,\tfrac{1}{k\Gamma_k(\alpha)}\\&\left[\times\int_0^t(h(t)-h(b))^{\frac{\alpha}{k}-1}h'(b)\,db\right]\\&+I_{k,h}^{\alpha}(1)\,\tfrac{1}{k\Gamma_k(\alpha)}\\&\left[\times\int_0^t(h(t)-h(b))^{\frac{\alpha}{k}-1}h'(b)\,f(b)\,g(b)\,db\right]\\&\geq I_{k,h}^{\alpha}f(t)\,\tfrac{1}{k\Gamma_k(\alpha)}\\&\left[\times\int_0^t(h(t)-h(b))^{\frac{\alpha}{k}-1}h'(b)\,g(b)\,db\right]\\&+I_{k,h}^{\alpha}g(t)\,\tfrac{1}{k\Gamma_k(\alpha)}\\&\left[\times\int_0^t(h(t)-h(b))^{\frac{\alpha}{k}-1}h'(b)\,f(b)\,db.\right]\end{aligned}\tag{22}$$

We can write that

$$I_{k,h}^{\alpha}(fg)(t)\geq\frac{1}{I_{k,h}^{\alpha}(1)}I_{k,h}^{\alpha}f(t)\,I_{k,h}^{\alpha}g(t).\tag{23}$$

The proof is done. □

Corollary 1. *If the functions f and g are asynchronous (i.e $(f(x)-f(y))(g(x)-g(y))\leq 0$, for any $x,y\in[a,b]$), then*

$$I_{k,h}^{\alpha}(fg)(t)\leq\tfrac{1}{I_{k,h}^{\alpha}(1)}I_{k,h}^{\alpha}f(t)\,I_{k,h}^{\alpha}g(t).\tag{24}$$

Theorem 2. *Let the functions f and g be two synchronous functions on $[0,\infty[$. Suppose that $h(x)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, we'll consider h as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative h' being continuous and $\gamma(0)=0$. Then for all $t>0$, $\alpha>0$, $k>0$, $\beta>0$, the following inequality, we have*

$$\begin{aligned}&I_{k,h}^{\alpha}(fg)(t)\,I_{k,h}^{\beta}(1)\\&+I_{k,h}^{\alpha}(1)\,I_{k,h}^{\beta}(fg)(t)\\&\geq I_{k,h}^{\alpha}(f)(t)\,I_{k,h}^{\beta}(g)(t)\\&+I_{k,h}^{\alpha}(g)(t)\,I_{k,h}^{\beta}(f)(t).\end{aligned}\tag{25}$$

Proof. By utilizing the proof of $Theorem\ 1$, we can write

$$\begin{aligned}&\tfrac{(h(t)-h(y))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h'(y)\,I_{k,h}^{\alpha}(fg)(t)\\&+\tfrac{(h(t)-h(y))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h'(y)\,f(y)\,g(y)\,I_{k,h}^{\alpha}(1)\\&\geq\tfrac{(h(t)-h(y))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h'(y)\,g(y)\,I_{k,h}^{\alpha}f(t)\\&+\tfrac{(h(t)-h(y))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h'(y)\,f(y)\,I_{k,h}^{\alpha}g(t).\end{aligned}\tag{26}$$

By integrating (26) from 0 to 1, we get

$$\begin{aligned}&\tfrac{I_{k,h}^{\alpha}(fg)(t)}{k\Gamma_k(\beta)}\\&\left[\times\int_0^t(h(t)-h(y))^{\frac{\beta}{k}-1}h'(y)\,dy\right]\\&+\tfrac{I_{k,h}^{\alpha}(1)}{k\Gamma_k(\beta)}\\&\left[\times\int_0^t(h(t)-h(y))^{\frac{\beta}{k}-1}h'(y)\,f(y)\,g(y)\,dy\right]\\&\geq\tfrac{I_{k,h}^{\alpha}f(t)}{k\Gamma_k(\beta)}\\&\left[\times\int_0^t(h(t)-h(y))^{\frac{\beta}{k}-1}h'(y)\,g(y)\,dy\right]\\&+\tfrac{I_{k,h}^{\alpha}g(t)}{k\Gamma_k(\beta)}\\&\left[\times\int_0^t(h(t)-h(y))^{\frac{\beta}{k}-1}h'(y)\,f(y)\,dy\right].\end{aligned}\tag{27}$$

Then,

$$\begin{aligned}&I_{k,h}^{\alpha}(fg)(t)\,I_{k,h}^{\beta}(1)\\&+I_{k,h}^{\alpha}(1)\,I_{k,h}^{\beta}(fg)(t)\\&\geq I_{k,h}^{\alpha}(f)(t)\,I_{k,h}^{\beta}(g)(t)\\&+I_{k,h}^{\alpha}(g)(t)\,I_{k,h}^{\beta}(f)(t).\end{aligned}\tag{28}$$

The proof is done. □

Corollary 2. *If the functions f and g are asynchronous, then inequality (28) holds in the reversed direction.*

Remark 1. *If we choose $\alpha=\beta$ in $Theorem\ 2$, then we obtain inequality of $Theorem\ 1$.*

Theorem 3. *Let $(f_i)_{i=1,\ldots,n}$ be n positive increasing functions on $[0,\infty[$. Suppose that $h(x)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, we'll consider h as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative h' being continuous and $\gamma(0)=0$. Then for any $t>0$, $\alpha>0$, $k>0$, we have*

$$\begin{aligned}&I_{k,h}^{\alpha}(\pi_{i=1}^{n}f_i)(t)\\&\geq\left(I_{k,h}^{\alpha}(1)\right)^{1-n}\left(\pi_{i=1}^{n}I_{k,h}^{\alpha}f_i(t)\right).\end{aligned}\tag{29}$$

Proof. By utilizing inequality in $Theorem\ 1$ for $n=2$, we have for $\alpha>0$ and $k>0$

$$\begin{aligned}&I_{k,h}^{\alpha}(f_1f_2)(t)\\&\geq\left(I_{k,h}^{\alpha}(1)\right)^{-1}I_{k,h}^{\alpha}f_1(t)\,I_{k,h}^{\alpha}f_2(t).\end{aligned}\tag{30}$$

In here, we can write as the following inequality for $t>0$

$$\begin{aligned}&I_{k,h}^{\alpha}(\pi_{i=1}^{n}f_i)(t)\\&\geq\left(I_{k,h}^{\alpha}(1)\right)^{2-n}\left(\pi_{i=1}^{n-1}I_{k,h}^{\alpha}f_i(t)\right).\end{aligned}\tag{31}$$

If $(f_i)_{i=1,2,\ldots,n}$ are positive increasing functions, then $\left(\pi_{i=1}^{n-1}f_i\right)(t)$ is an increasing function.Moreover, we can apply $Theorem$ 1 to the functions $\pi_{i=1}^{n-1} f_i$ and $f_n = f$. Then,

$$\begin{aligned} & I_{k,h}^{\alpha} \left(\pi_{i=1}^{n} f_i\right)(t) \\ & = I_{k,h}^{\alpha} (fg)(t) \\ & \geq \left(I_{k,h}^{\alpha}(1)\right)^{-1} I_{k,h}^{\alpha} \left(\pi_{i=1}^{n-1} f_i\right)(t)\, I_{k,h}^{\alpha} f_n(t), \end{aligned} \tag{32}$$

by using inequality in (31), we get

$$\begin{aligned} & I_{k,h}^{\alpha} \left(\pi_{i=1}^{n} f_i\right)(t) \\ & \geq \left(I_{k,h}^{\alpha}(1)\right)^{-1} \left(I_{k,h}^{\alpha}(1)\right)^{2-n} I_{k,h}^{\alpha} \left(\pi_{i=1}^{n-1} f_i\right)(t)\, I_{k,h}^{\alpha} f_n(t) \\ & \geq \left(I_{k,h}^{\alpha}(1)\right)^{1-n} \left(\pi_{i=1}^{n} I_{k,h}^{\alpha} f_i(t)\right). \end{aligned} \tag{33}$$

The proof is done. □

Theorem 4. *Let f and g be two functions defined on $[0,\infty[$, such that f is increasing and g is differentiable and there is a real number $m = \inf_{t>0} g'(t)$. Suppose that $h(x)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, we'll consider h as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative h' being continuous and $\gamma(0) = 0$. Then we have as the following inequality for $t > 0$, $\alpha > 0$ and $k > 0$,*

$$\begin{aligned} & I_{k,h}^{\alpha} (fg)(t) \\ & \geq \left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} g(t) \\ & - I_{k,h}^{\alpha} f(t) \frac{m(kh(t)+\alpha h(0))}{(\alpha+k)} + m I_{k,h}^{\alpha}(hf)(t). \end{aligned} \tag{34}$$

Proof. Let $H(t) := g(t) - mh(t)$. It is clear that H is differentiable and increasing on $[0,\infty[$. Additionally, Let $h(t)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, if we consider $h^{'}(t)$ is continuous on $[0,\infty)$ and $h(0) = 0$. Then by means of $Theorem$ 1, we have

$$\begin{aligned} & I_{k,h}^{\alpha} \left((g(t) - mh(t))(f(t))\right) \\ & \geq \left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} (g(t) - mh(t)) \\ & \geq \left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} g(t) \\ & - m\left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} h(t) \\ & \geq \left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} g(t) \\ & - m\left[I^{\frac{\alpha}{k}}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I^{\frac{\alpha}{k}} h(t) \\ & = \left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} g(t) \\ & - m\left[I_{k,h}^{\frac{\alpha}{k}}(1)\right]^{-1} I_{k,h}^{\alpha} f(t) \frac{(h(t)-h(0))^{\frac{\alpha}{k}}(kh(t)+\alpha h(0))}{\Gamma\left(\frac{\alpha}{k}+1\right)(\alpha+k)} \\ & = \left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} g(t) \\ & - m\frac{\Gamma\left(\frac{\alpha}{k}+1\right)}{(h(t)-h(0))^{\frac{\alpha}{k}}} I_{k,h}^{\alpha} f(t) \frac{(h(t)-h(0))^{\frac{\alpha}{k}}(kh(t)+\alpha h(0))}{\Gamma\left(\frac{\alpha}{k}+1\right)(\alpha+k)} \\ & = \left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} g(t) \\ & - I_{k,h}^{\alpha} f(t) \frac{m(kh(t)+\alpha h(0))}{(\alpha+k)}. \end{aligned}$$

Where

$$\begin{aligned} & I^{\frac{\alpha}{k}} h(t) \\ & = \frac{1}{\Gamma\left(\frac{\alpha}{k}\right)} \int_0^t (h(t) - h(0))^{\frac{\alpha}{k}-1} h(x)\, h'(x)\, dx \\ & = \frac{1}{\Gamma\left(\frac{\alpha}{k}+1\right)} \frac{(h(t)-h(0))^{\frac{\alpha}{k}}(kh(t)+\alpha h(0))}{(\alpha+k)} \end{aligned} \tag{35}$$

and

$$\left[I^{\frac{\alpha}{k}}(1)\right]^{-1} = \frac{\Gamma\left(\frac{\alpha}{k}+1\right)}{(h(t)-h(0))^{\frac{\alpha}{k}}}. \tag{36}$$

The proof is done. □

Corollary 3. *Let f and g be two functions defined on $[0,\infty[$. Suppose that $h(x)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, we'll consider h as a monotonically increasing and positive function defined on the interval $[0,\infty)$, with its derivative h' being continuous and $\gamma(0) = 0$.*

1. While f is decreasing, g is differentiable and there is a real number $M := \sup_{t\geq 0} g'(t)$, then for all $t > 0$, $\alpha > 0$, $k > 0$, we acquire

$$\begin{aligned} & I_{k,h}^{\alpha} (fg)(t) \\ & \geq \left[I_{k,h}^{\alpha}(1)\right]^{-1} I_{k,h}^{\alpha} f(t)\, I_{k,h}^{\alpha} g(t) \\ & - I_{k,h}^{\alpha} f(t) \frac{M(kh(t)+\alpha h(0))}{(\alpha+k)} + M I_{k,h}^{\alpha}(hf)(t). \end{aligned} \tag{37}$$

2. If f and g are differentiable and we assume that $m_1 := \inf_{t\geq 0} f'(t)$ and $m_2 := \inf g'(t)$, then

we obtain

$$\begin{aligned}&I^{\alpha}_{k,h}(fg)(t)-m_1I^{\alpha}_{k,h}(gh)(t)\\&-m_2I^{\alpha}_{k,h}(fh)(t)+m_1m_2I^{\alpha}_{k,h}(hh)(t)\\&\geq\left[I^{\alpha}_{k,h}(1)\right]^{-1}\\&\times\Big[I^{\alpha}_{k,h}f(t)\,I^{\alpha}_{k,h}g(t)-m_1I^{\alpha}_{k,h}g(t)\,I^{\alpha}_{k,h}h(t)\\&-m_2I^{\alpha}_{k,h}f(t)\,I^{\alpha}_{k,h}h(t)+m_1m_2I^{\alpha}_{k,h}h(t)\,I^{\alpha}_{k,h}h(t)\Big].\end{aligned}\tag{38}$$

3. If f and g are differentiable and we assume that $M_1:=\sup f'(t)$ and $M_2:=\sup_{t\geq 0}g'(t)$, then we obtain

$$\begin{aligned}&I^{\alpha}_{k,h}(fg)(t)-M_1I^{\alpha}_{k,h}(gh)(t)\\&-M_2I^{\alpha}_{k,h}(fh)(t)+M_1M_2I^{\alpha}_{k,h}(hh)(t)\\&\geq\left[I^{\alpha}_{k,h}(1)\right]^{-1}\\&\times\Big[I^{\alpha}_{k,h}f(t)\,I^{\alpha}_{k,h}g(t)-M_1I^{\alpha}_{k,h}g(t)\,I^{\alpha}_{k,h}h(t)\\&-M_2I^{\alpha}_{k,h}f(t)\,I^{\alpha}_{k,h}h(t)+M_1M_2I^{\alpha}_{k,h}h(t)\,I^{\alpha}_{k,h}h(t)\Big].\end{aligned}\tag{39}$$

Proof. 1. If we take $G(t):=g(t)-Mh(t)$, then we obtain (38) by utilizing (14) to the decreasing functions f and G.

2. If we take $F(t):=f(t)-m_1h(t)$ and $G(t):=g(t)-m_2h(t)$, then we obtain (39) by utilizing (14) to the increasing functions F and G as the following

$$\begin{aligned}&I^{\alpha}_{k,h}\left((f(t)-m_1h(t))\,(g(t)-m_2h(t))\right)\\&\geq\left[I^{\alpha}_{k,h}(1)\right]^{-1}\\&\times\left[\left(I^{\alpha}_{k,h}f(t)-m_1I^{\alpha}_{k,h}h(t)\right)\left(I^{\alpha}_{k,h}g(t)-m_2I^{\alpha}_{k,h}h(t)\right)\right]\\&\geq\left[I^{\alpha}_{k,h}(1)\right]^{-1}\\&\times\Big[I^{\alpha}_{k,h}f(t)\,I^{\alpha}_{k,h}g(t)-m_1I^{\alpha}_{k,h}g(t)\,I^{\alpha}_{k,h}h(t)\\&-m_2I^{\alpha}_{k,h}f(t)\,I^{\alpha}_{k,h}h(t)+m_1m_2I^{\alpha}_{k,h}h(t)\,I^{\alpha}_{k,h}h(t)\Big].\end{aligned}\tag{40}$$

which

$$\begin{aligned}&I^{\alpha}_{k,h}\left((f(t)-m_1h(t))\,(g(t)-m_2h(t))\right)\\&=I^{\alpha}_{k,h}(fg)(t)-m_1I^{\alpha}_{k,h}(gh)(t)\\&-m_2I^{\alpha}_{k,h}(fh)(t)+m_1m_2I^{\alpha}_{k,h}(hh)(t).\end{aligned}\tag{41}$$

3. If we take $F(t):=f(t)-M_1h(t)$ and $G(t):=g(t)-M_2h(t)$, then we obtain (40) by utilizing (14) to the decreasing functions F and G as the following

$$\begin{aligned}&I^{\alpha}_{k,h}\left((f(t)-M_1h(t))\,(g(t)-M_2h(t))\right)\\&\geq\left[I^{\alpha}_{k,h}(1)\right]^{-1}\\&\times\left[\left(I^{\alpha}_{k,h}f(t)-M_1I^{\alpha}_{k,h}h(t)\right)\left(I^{\alpha}_{k,h}g(t)-M_2I^{\alpha}_{k,h}h(t)\right)\right]\\&\geq\left[I^{\alpha}_{k,h}(1)\right]^{-1}\\&\times\Big[I^{\alpha}_{k,h}f(t)\,I^{\alpha}_{k,h}g(t)-M_1I^{\alpha}_{k,h}g(t)\,I^{\alpha}_{k,h}h(t)\\&-M_2I^{\alpha}_{k,h}f(t)\,I^{\alpha}_{k,h}h(t)+M_1M_2I^{\alpha}_{k,h}h(t)\,I^{\alpha}_{k,h}h(t)\Big].\end{aligned}\tag{42}$$

which

$$\begin{aligned}&I^{\alpha}_{k,h}\left((f(t)-M_1h(t))\,(g(t)-M_2h(t))\right)\\&=I^{\alpha}_{k,h}(fg)(t)-M_1I^{\alpha}_{k,h}(gh)(t)\\&-M_2I^{\alpha}_{k,h}(fh)(t)+M_1M_2I^{\alpha}_{k,h}(hh)(t).\end{aligned}\tag{43}$$

□

Remark 2. *If we choose $h(t)=t$ and $k=1$ in Theorems and Corollaries presented in this article, we acquire the consequences equivalent to those found in [6] Theorems and Corollaries. Similarly, if we take $h(t)=t$ in Theorems and Corollaries presented in this article, we obtain results of Theorems and Corollaries in [7].*

3 Conclusion

In this paper, we introduce the Riemann-Liouville generalized fractional integral and derive several important inequalities associated with it. Additionally, we establish key properties and bounds for Riemann-Liouville generalized fractional integrals.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.