Mittag-Leffler Functions and the Sawi Transform: A New Approach to Fractional Calculus

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Abstract: - This study uses the Sawi transform and Mittag-Leffler function to introduce a novel method of fractional calculus. The basic characteristics of the Sawi transform and its connection to other transforms such as the Laplace transform are explored. We demonstrate the application of the Sawi transform in deriving novel solutions to fractional differential equations and elucidate the importance of the Mittag-Leffler function in these scenarios. We show the efficiency of the presented method by presenting some nonhomogeneous equations and initial value problems. This study proves the efficacy of the Sawi transform by handling some problems in the sense of Caputo fractional calculus.

Key-Words: - Integral transform, Sawi transform, Mittag-Leffler function, Caputo fractional derivative, Fractional differential equations, Nonhomogeneous problems.

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1 Introduction

The study of fractional calculus involves derivatives and integrals of arbitrary orders, has gained a lot of attention due to some applications in various fields including engineering and physics. This branch of mathematics extends the concepts of classical calculus by introducing new models for explaining real-world phenomena, especially those involving memory effects or diffusion. Modeling complicated systems and resolving fractional differential equations require the use of the Mittag-Leffler function, that generalizes the exponential function and a concept in fractional calculus, [1], [2]. It is known as the "Queen Function" in fractional calculus, it is important in dealing with several fields, including bioengineering, [3], luminescence degradation [4], and others [5]. Multi-index Mittag-Leffler functions were introduced to handle some kinds of fractional-order equations in applied mathematics, [6], [7].

The Sawi transform (SWT), a new integral transform, that is related to the Laplace transform. It has shown promise applications in managing the difficulty in solving fractional differential equations, [8], [9].

The SWT offers special characteristics including convolution, scaling, and linearity that make solving differential equations much easier. It shows efficacy in solving many kinds of problems, such as linear ordinary differential equations [10], fractional partial differential equations [11], and nonlinear fractional differential equations [12], exhibiting advantages in accuracy, simplicity, and fast convergence, [11]. The SWT has been used to develop hybrid methods for solving difficult problems [11], [12], by combining with numerical approaches such as the homotopy perturbation and decomposed methods. Researchers have also investigated its relation to other integral transforms [13], and the application of solving equations involving Hilfer-Prabhakar fractional derivatives [14], and how it can be applied to Hyers-Ulam stability analysis [15]. The SWT shows that it is a valuable tool for handling mathematical analysis and applications in various disciplines.

In order to solve fractional differential equations, this study presents a unique method that uses the SWT to provide simpler solutions. The main goal is to find the SWT and its relation to the Mittag-Leffler function. This paper concentrates on the SWT and Mittag-Leffler function, it investigate novel approaches to solve fractional differential equations.

Power series solutions for linear and nonlinear fractional differential equations are provided by Mittag-Leffler functions [16], [17]. Fractional calculus is used to produce new connections between the parameters in Mittag-Leffler functions [18], whereas generalized multivariable Mittag-Leffler functions are solved using the Laplace transform approach, [19], [20]. These methods provide efficient, accurate, and broadly applicable solutions to complex fractional differential equations [21], [22], [23], improving the understanding of fractional calculus and its various applications, [24], [25], [26].

We begin this study by describing the definitions and basic features [27], [28] related to SWT. We then examine the uses and importance of the Mittag-Leffler function in this article. We demonstrate the efficiency of SWT in solving fractional differential equations by solving some examples including the Mittag-Leffler function, [29], [30].

The rest of this paper is organized as follows: In Section 2, we present some basic definitions and theorems, in Section 3, we in discuss the application of SWT on the fractional operators, and obtain some related results concerning Mittag-Leffler function. We solve four interesting results using the obtained theorems in Section 4, finally we get the conclusion.

2 Basic Definitions and Properties

In this section, we introduce some basic definitions and theorems that are related to our work.

Definition 1. [8] The SWT of the function $w(t)$, defined on $[0,\infty)$, is denoted by $S[w(t)]$ and given by:

$$S[w(\mathrm{t})] = R(v) = \frac{1}{v^2}\int_0^\infty w(t)e^{-\frac{t}{v}}dt. \qquad (1)$$

If $S[w(t)] = R(v)$, then $w(t)$, is referred to as the inverse SWT of $R(v)$, and is denoted by $S^{-1}[R(v)] = w(t)$, that is

$$S^{-1}[R(v)] = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R(v)\, e^{\frac{t}{v}}\, dv. \qquad (2)$$

Theorem 1. [8] Let $w(t)$ be a continuous function defined for $t > 0$ and has exponential order α property; $|w(t))| \leq \mu e^{\alpha t}$ where $\mu > 0$. Then, the SWT $S[w(t)]$ exists for $Re\left(\frac{1}{v}\right) > \alpha$.

Theorem 2. [8] Let $R(v)$ be SWT of $w(t)$. Then

$$(i) \quad S[w'(t)] = \frac{R(v)}{v} - \frac{w(0)}{v^2}. \qquad (3)$$

$$(ii) \; S[w''(t)] = \frac{R(v)}{v^2} - \frac{w(0)}{v^3} - \frac{w'(0)}{v^2}. \qquad (4)$$

$$(iii) \; S\left[w^{(n)}(t)\right] = \frac{R(v)}{v^n} - \sum_{k=0}^{n-1}\frac{w^{(k)}(0)}{v^{n-k+1}}. \qquad (5)$$

Remark 1. The relation between SWT and Laplace transform is given by: If $S[w(t)] = R(v)$ and $L[w(t)] = W(v)$, is the Laplace transform of $w(t)$, then $R(v) = \frac{1}{v^2}W\left(\frac{1}{v}\right)$.

Proof. The Laplace transform of a function $w(t)$, is given by:

$$L[w(t)] = W(v) = \int_0^\infty \mathrm{w}(t)e^{-vt}dt.$$

Now, $R(v) = \frac{1}{v^2}\int_0^\infty w(t)\, e^{\frac{-t}{v}}\, dt$, which implies:

$$R(v) = \frac{1}{v^2}W\left(\frac{1}{v}\right). \qquad (6)$$

In the following we present some properties of SWT, assuming that $S[w(t)] = R(v)$ and, $S[u(t)] = U(v)$:

$$S[\, a\, w(t) + b\, u(t)] = a\, S[w(t)] + b\, S[u(t)] = aR(v) + bU(v),$$

where a & b are arbitrary constants. Moreover, the inverse of SWT is linear,

$$S^{-1}[aR(v) + bU(v)] = a\, w(t) + b\, u(t).$$

$S[\, w(at)] = a\, R(av\,)$, where $a \neq 0$.

$$S[e^{at}w(t)] = \frac{1}{(1-av)^2}R\left(\frac{v}{1-av}\right), \text{where } av \neq 1.$$

$$S\,[w(t) * u(t)] = v^2\, R(v)\, U(v).$$

In Table 1, we state the formula of SWT of some functions.

Table 1. SWT of some elementary functions

$w(t)$	$S[w(t)]$
1	$\frac{1}{v}$
t	1
$t^n, n\epsilon\mathbb{N}$	$n!\,v^{n-1}$
$t^\alpha, \alpha\epsilon\mathbb{R}^+$	$\Gamma(\alpha+1)\,v^{\alpha-1}$
e^{at}	$\frac{1}{v(1-av)}$
$\sin at$	$\frac{a}{1+a^2v^2}$
$\cos at$	$\frac{1}{v(1+a^2v^2)}$
$\sinh at$	$\frac{a}{1-a^2v^2}$
$\cosh at$	$\frac{1}{v(1-a^2v^2)}$

3 The SWT of Mittag-Leffler Function and Riemann- Liouville

The Riemann-Liouville integral is motivated from Cauchy formula for repeated integration and the Mittag-Leffler function is one of the important special functions, which is considered as a generalization of the exponential function, and it is frequently used in the solutions of fractional differential equations and systems of fractional differential equations.

Definition 2. [20] The Riemann–Liouville fractional integral of a function $w(t)$ of order $\alpha > 0$ is defined by:

$$I^\alpha w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\, w(\tau)d\tau. \tag{7}$$

Definition 3. [20] The Caputo fractional derivative of a function $w(t)$ of order $\alpha > 0$, is defined by:

$$D^\alpha w(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{w^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau, & m-1<\alpha<m, \\ w^{(m)}(t), & \alpha = m \in N. \end{cases} \tag{8}$$

Definition 4. [20] The Mittag-Leffler function is defined by:

$$E_{\alpha,\beta}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j+\beta)}, \tag{9}$$
$$t, \alpha, \beta \in \mathbb{C} \text{ and } Re(\alpha) > 0.$$

Theorem 3. If $R(v)$ is the SWT of $w(t)$, then SWT of Riemann-Liouville fractional integral is given by:

$$S[I^\alpha w(t)] = v^\alpha R(v). \tag{10}$$

Proof. From the definition of Riemann-Liouville integral, we have:

$$I^\alpha w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} w(\tau)d\tau,$$
$$I^\alpha w(t) = \frac{1}{\Gamma(\alpha)}(t^{\alpha-1} * w(t)). \tag{11}$$

Taking SWT to both sides of (11), we obtain:

$$S[I^\alpha w(t)] = \frac{1}{\Gamma(\alpha)}\, S[t^{\alpha-1} * w(t)].$$

By using convolution property of SWT, we obtain:

$$\begin{aligned} S[I^\alpha w(t)] &= \frac{v^2}{\Gamma(\alpha)} S[t^{\alpha-1}]\, S[w(t)] \\ &= \frac{v^2}{\Gamma(\alpha)} \Gamma(\alpha) v^{\alpha-2}\, R(v) \\ &= v^\alpha R(v). \end{aligned}$$

Theorem 4. If $R(v)$ is SWT of the function $w(t)$, then SWT of Caputo functional derivative of a function $w(t)$, is given by

$$S[D^\alpha w(t)] = \frac{1}{v^\alpha}\, R(v) - \sum_{k=0}^{m-1}\left(\frac{1}{v}\right)^{m-(k-1)} w^{(k)}(0), \tag{12}$$

where, $m-1 < \alpha \le m, m \in N$.

Proof. The definition of Caputo derivative of a function $w(t)$ is:

$$\begin{aligned} D^\alpha w(t) &= \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{w^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau \\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} w^{(m)}(\tau)d\tau, \end{aligned}$$

where $m-1 < \alpha < m$. Thus, we can write

$$D^\alpha w(t) = \frac{1}{\Gamma(m-\alpha)}(t^{m-\alpha-1} * w^{(m)}(t)). \tag{13}$$

Taking SWT to both sides of Eq (13), we obtain:

$$S[D^\alpha w(t)] = \frac{1}{\Gamma(m-\alpha)} S[t^{m-\alpha-1} * w^{(m)}(t)].$$

By using convolution property of SWT,

$$S[D^\alpha w(t)] = \frac{v^2}{\Gamma(m-\alpha)} S[t^{m-\alpha-1}]\, S\left[w^{(m)}(t)\right]$$
$$= \frac{v^2}{\Gamma(m-\alpha)} \Gamma(m-\alpha) v^{m-\alpha-2} \left(\frac{1}{v^m} R(v) - \sum_{k=0}^{m-1} \left(\frac{1}{v}\right)^{m-(k-1)} w^{(k)}(0)\right)$$
$$= v^{m-\alpha} \left(\frac{1}{v^m} R(v) - \sum_{k=0}^{m-1} \left(\frac{1}{v}\right)^{m-(k-1)} w^{(k)}(0)\right)$$
$$= \frac{1}{v^\alpha} R(v) - \sum_{k=0}^{m-1} \left(\frac{1}{v}\right)^{m-(k-1)} w^{(k)}(0).$$

Theorem 5. The SWT of the Mittag-Leffler function is given by

$$S[\, E_{\alpha,\beta}(t)] = \sum_{j=0}^{\infty} \frac{\Gamma(j+1)}{\Gamma(\alpha j+\beta)} v^{j-1}. \tag{14}$$

Proof. The Mittag-Leffler function is defined by,

$$E_{\alpha,\beta}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j+\beta)}, \tag{15}$$
$$t, \alpha, \beta \in \mathbb{C} \text{ and } Re(\alpha) > 0.$$

Applying SWT for Mittag-Leffler function in Eq (15), to get:

$$S[\, E_{\alpha,\beta}(t)] = S\left[\sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j+\beta)}\right]$$
$$= \sum_{j=0}^{\infty} \frac{1}{\Gamma(\alpha j+\beta)} S\left[t^j\right]$$
$$= \sum_{j=0}^{\infty} \frac{1}{\Gamma(\alpha j+\beta)} v^{j-1}\, \Gamma(j+1).$$

Thus,

$$S[\, E_{\alpha,\beta}(t)] = \sum_{j=0}^{\infty} \frac{\Gamma(j+1)}{\Gamma(\alpha j+\beta)} v^{j-1}.$$

Theorem 6. For $\alpha > 0$, $a \in \mathbb{R}$ and $|av^\alpha| < 1$, we have the following formulas of inverse SWT as

$$(i)\quad S^{-1}\left[\frac{1}{v(1+av^\alpha)}\right] = E_\alpha(-at^\alpha). \tag{16}$$

$$(ii)\quad S^{-1}\left[\frac{av^\alpha}{v(1+av^\alpha)}\right] = t^\alpha\, E_\alpha(-at^\alpha). \tag{17}$$

Proof. (i) The formula of Mittag-Leffler function is,

$$E_\alpha(-at^\alpha) = \sum_{j=0}^{\infty} \frac{(-at^\alpha)^j}{\Gamma(\alpha j+1)}. \tag{18}$$

By taking SWT of Eq (18), we get:

$$S[E_\alpha(-at^\alpha)] = S\left[\sum_{j=0}^{\infty} \frac{(-at^\alpha)^j}{\Gamma(\alpha j+1)}\right]$$
$$= S\left[\sum_{j=0}^{\infty} \frac{(-a)^j}{\Gamma(\alpha j+1)} \left(t^{\alpha j}\right)\right]$$
$$= \sum_{j=0}^{\infty} \frac{(-a)^j\, S\left[t^{\alpha j}\right]}{\Gamma(\alpha j+1)}$$
$$= \frac{1}{v}\left(\frac{1}{1+av^\alpha}\right).$$

Thus,

$$S[E_\alpha(-at^\alpha)] = \frac{1}{v(1+av^\alpha)}. \tag{19}$$

By taking inverse SWT of Eq (19), we get:

$$S^{-1}\left[\frac{1}{v(1+av^\alpha)}\right] = E_\alpha(-at^\alpha).$$

Proof. (ii) The formula of Mittag-Leffler function is:

$$t^\alpha E_\alpha(-at^\alpha) = \sum_{n=0}^{\infty} \frac{(-at^\alpha)^j (\, t^\alpha)}{\Gamma(\alpha j+1)}. \tag{20}$$

By taking SWT of Eq (20),

$$S[t^\alpha\, E_\alpha(-at^\alpha)] = S\left[\sum_{j=0}^{\infty} \frac{(-at^\alpha)^j (\, t^\alpha)}{\Gamma(\alpha j+1)}\right]$$
$$= \sum_{j=0}^{\infty} \frac{(-a)^j\;\, S\left[t^{\alpha j}\, t^\alpha\right]}{\Gamma(\alpha j+\beta)}$$
$$= \sum_{j=0}^{\infty} \frac{(-a)^j\, v^{\alpha(j+1)-1}\, \Gamma(\alpha j+1)}{\Gamma(\alpha j+1)}$$
$$= \frac{1}{v^{1-\alpha}}\left(\frac{1}{(1+av^\alpha)}\right).$$

Thus,

$$S[t^{\alpha}\,E_{\alpha}(-at^{\alpha})]=\frac{v^{\alpha-1}}{(1+av^{\alpha})}. \tag{21}$$

By taking SWT inverse of Eq (21), we get:

$$S^{-1}\left[\frac{v^{\alpha-1}}{(1+av^{\alpha})}\right]=\,t^{\alpha}\,E_{\alpha}(-at^{\alpha}).$$

Corollary 1. For $\alpha>0,\ a\in\mathbb{R}\ \&\ |av^{\alpha}|<1$, we have the following formula of inverse SWT as:

$$S^{-1}\left[\frac{v^{\alpha-2}}{(1+av^{\alpha})}\right]=\,t^{\alpha-1}\,E_{\alpha}(-at^{\alpha}).$$

The formula of Mittag-Leffler function is:

$$t^{\alpha-1}\,E_{\alpha}(-at^{\alpha})=\sum_{j=0}^{\infty}\frac{(-at^{\alpha})^{j}(\,t^{\alpha-1})}{\Gamma(\alpha j+1)}. \tag{22}$$

By taking SWT of Eq (22), we obtain:

$$\begin{aligned}S[t^{\alpha-1}\,E_{\alpha}(-at^{\alpha})]&=S\left[\sum_{j=0}^{\infty}\frac{(-at^{\alpha})^{j}(\,t^{\alpha-1})}{\Gamma(\alpha j+1)}\right]\\&=\sum_{j=0}^{\infty}\frac{(-a)^{j}\ S[t^{\alpha j}\,t^{\alpha-1}]}{\Gamma(\alpha j+1)}\\&=\sum_{j=0}^{\infty}\frac{(-a)^{j}\,S\left[t^{\alpha(j+1)-1}\right]}{\Gamma(\alpha j+1)}\\&=\sum_{j=0}^{\infty}\frac{(-a)^{j}\,v^{\alpha(j+1)-2}\,(\Gamma(\alpha j+1)}{\Gamma(\alpha j+1)}\\&=\frac{1}{v^{2-\alpha}}\left(\frac{1}{(1+av^{\alpha})}\right).\end{aligned}$$

Thus,

$$S[t^{\alpha-1}\,E_{\alpha}(-at^{\alpha})]=\frac{v^{\alpha-2}}{(1+av^{\alpha})}. \tag{23}$$

By taking SWT inverse of Eq (23), we get:

$$S^{-1}\left[\frac{v^{\alpha-2}}{(1+av^{\alpha})}\right]=\,t^{\alpha-1}\,E_{\alpha}(-at^{\alpha}).$$

4 Applications on Fractional Differential Equations

In this section, we present four interesting examples on fractional differential equations, that can be solved using the obtained results.

Example 4.1. At the first example, we consider the following initial value problem in the case of nonhomogeneous Bagley-Torvik equation.

$$D^{2}w(t)+D^{\frac{3}{2}}w(t)+w(t)=1+t, \tag{24}$$

with the initial conditions:

$$w(0)=w'(0)=1. \tag{25}$$

Solution: By applying SWT on Eq (24), we get:

$$\begin{aligned}S[D^{2}w(t)]+\mathrm{S}\left[D^{\frac{3}{2}}w(t)\right]+\mathrm{S}[w(t)]\\=S[1+t].\end{aligned} \tag{26}$$

Running SWT for Eq (26), and using Theorem 2 for $\alpha=\frac{3}{2}$ and $m=2$, we have:

$$\begin{aligned}&\frac{R(v)}{v^{2}}-\frac{w(0)}{v^{3}}-\frac{w'(0)}{v^{2}}+v^{-\frac{3}{2}}R(v)\\&\quad-\sum_{k=0}^{1}\left(\frac{1}{v}\right)^{\frac{3}{2}-(k-1)}w^{(k)}(0)\\&\quad+R(v)=\frac{1}{v}+1,\end{aligned} \tag{27}$$

Substituting the initial conditions in Eq (25), we obtain:

$$\frac{R(v)}{v^{2}}-\frac{1}{v^{3}}-\frac{1}{v^{2}}+v^{-\frac{3}{2}}R(v)-\left(\frac{1}{v}\right)^{\frac{5}{2}}-\left(\frac{1}{v}\right)^{\frac{3}{2}}+R(v)=\frac{1}{v}+1.$$

Now,

$$\begin{aligned}&R(v)\left(\frac{1}{v^{2}}+\frac{1}{v^{\frac{3}{2}}}+1\right)\\&\qquad=1+\frac{1}{v}+\frac{1}{v^{2}}+\frac{1}{v^{3}}+\frac{1}{v^{\frac{3}{2}}}+\frac{1}{v^{\frac{5}{2}}},\\&R(v)\left(\frac{1}{v^{2}}+\frac{1}{v^{\frac{3}{2}}}+1\right)=\left(1+\frac{1}{v}\right)\left(\frac{1}{v^{2}}+\frac{1}{v^{\frac{3}{2}}}+1\right).\end{aligned}$$

Thus,

$$R(v)=1+\frac{1}{v}. \tag{28}$$

Now, take inverse SWT for both sides of Eq (28), which implies:

$$w(t)=t+1.$$

Example 4.2. The second example discusses the idea of a nonhomogeneous linear equation:

$$\begin{aligned}D^{\alpha}w(t)+w(t)=\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\\+t^{2}-t,\end{aligned} \tag{29}$$

with the initial condition:

$$w(0)=0\ \ ,\ 0<\alpha\leqslant 1. \tag{30}$$

Solution: Applying SWT to Equation (29) yields

$$S[D^{\alpha}w(t)] + S[w(t)] = \frac{2}{\Gamma(3-\alpha)}S[t^{2-\alpha}] - \frac{1}{\Gamma(2-\alpha)}S[t^{1-\alpha}] + S[t^2 - t]. \quad (31)$$

Running SWT for Eq (31), and using Theorem 2 for $0 < \alpha \leq 1$ and $m = 1$, we have:

$$\left(\frac{R(v)}{v^{\alpha}} - \left(\frac{1}{v}\right)^{\alpha+1} w(0)\right) + R(v) = \frac{2v^{2-\alpha-1}\Gamma(3-\alpha)}{\Gamma(3-\alpha)} - \frac{v^{1-\alpha-1}\Gamma(2-\alpha)}{\Gamma(2-\alpha)} + 2v - 1\,. \quad (32)$$

Substituting the initial condition in Eq (32), we obtain:

$$\frac{R(v)}{v^{\alpha}} + R(v) = 2v^{1-\alpha} - v^{-\alpha} + 2v - 1.$$

Now,

$$R(v)\left(\frac{1}{v^{\alpha}} + 1\right) = \frac{1}{v^{\alpha}}\left(\frac{2}{v^{-1}} - 1\right) + 2v - 1,$$

$$R(v)\left(\frac{1}{v^{\alpha}} + 1\right) = \frac{1}{v^{\alpha}}(2v - 1) + 2v - 1.$$

$$R(v) = 2v - 1. \quad (33)$$

Taking inverse SWT for both sides of Eq (33), we get:

$$S^{-1}[R(v)] = S^{-1}(2v - 1),$$
$$w(t) = t^2 - t.$$

Example 4.3. Evaluate the following linear initial value problem:

$$D^{\alpha}w(t) + w(t) = 0, \quad (34)$$

with the initial conditions,

$$w(0) = 1,\ w'(0) = 0. \quad (35)$$

The second initial condition is for $\alpha > 1$ only, thus, we have two cases of α and $D^{\alpha}w(t)$ considered as $0 < \alpha < 1$ *and* $1 < \alpha < 2$.

Solution.
(i) For $0 < \alpha < 1$.

By applying SWT for Eq (34), we get

$$S[D^{\alpha}w(t)] + S[w(t)] = 0. \quad (36)$$

Running SWT for Eq (39) and using Theorem 2 for $0 < \alpha < 1$ and $m = 1$, we have

$$\frac{R(v)}{v^{\alpha}} - \left(\frac{1}{v}\right)^{\alpha+1} w(0) + R(v) = 0. \quad (37)$$

By substituting the beginning conditions of Equation (37), we derive

$$\frac{R(v)}{v^{\alpha}} - \left(\frac{1}{v}\right)^{\alpha+1} + R(v) = 0. \quad (38)$$

Simplifying Eq (38), we get

$$R(v) = \frac{1}{v(v^{\alpha+1})} = \frac{1}{v}\left(\frac{1}{v^{\alpha+1}}\right).$$

(ii) For $1 < \alpha < 2$. By applying SWT for Eq (34), we get

$$S[D^{\alpha}w(t)] + S[w(t)] = 0. \quad (39)$$

Running SWT for Eq (39) and using Theorem 2, for $1 < \alpha < 2$, and $m = 2$.

$$\frac{R(v)}{v^{\alpha}} - \left(\left(\frac{1}{v}\right)^{\alpha+1} w(0) - \left(\frac{1}{v}\right)^{\alpha} w'(0)\right) + R(v) = 0. \quad (40)$$

We obtain the expression by changing the beginning conditions in Eq. (40)

$$\frac{R(v)}{v^{\alpha}} - \left(\frac{1}{v}\right)^{\alpha+1} + R(v) = 0,$$

$$R(v) = \frac{1}{v(v^{\alpha+1})} = \frac{1}{v}\left(\frac{1}{v^{\alpha+1}}\right),$$

which gives the same results in both cases. Now by applying inverse SWT, we have

$$S^{-1}[R(v)] = S^{-1}\left[\frac{1}{v}\left(\frac{1}{v^{\alpha+1}}\right)\right].$$

Thus,

$$w(t) = E_{\alpha}(-t^{\alpha}). \quad (41)$$

Example 4.4. Consider the following linear initial value problem

$$D^{\alpha}w(t) = w(t) + 1, \quad (42)$$

with the initial condition,

$$w(0) = 0 \quad , \quad 0 < \alpha \leqslant 1. \quad (43)$$

Solution. By applying SWT for Eq (42),

$$S[D^{\alpha}w(t)] = S[w(t) + 1]. \quad (44)$$

Running SWT for Eq (44) and using Theorem 2, for $0 < \alpha \leqslant 1$ and $m = 1$.

$$\frac{R(v)}{v^{\alpha}} - \left(\left(\frac{1}{v}\right)^{\alpha+1} \mathrm{w}(0)\right) = R(v) + \frac{1}{v}. \qquad (45)$$

Substituting the initial terms into equation (45), we conclude:

$$\frac{R(v)}{v^{\alpha}} = R(v) + \frac{1}{v},$$
$$R(v) = \frac{v^{\alpha-1}}{(1-v^{\alpha})}.$$

Now by applying inverse SWT, we have:

$$S^{-1}[R(v)] = S^{-1}\left[\frac{v^{\alpha-1}}{(1-v^{\alpha})}\right].$$

Thus,

$$w(t) = t^{\alpha} E_{\alpha}(-t^{\alpha}). \qquad (46)$$

5 Conclusion

This paper investigates a unique method for fractional calculus that combines the SWT with the Mittag-Leffler function. We demonstrated how the SWT's basic features may simplify and improve the solution of fractional differential equations, a foundation for dealing with complicated mathematical problems. Solving fractional differential equations in this study, is based on the properties of SWT such as linearity, scaling, and convolution. Hiring the Mittag-Leffler function, we presented new solutions that demonstrate the effectiveness and adaptability of SWT in solving fractional differential equations. This article helps us to get better understanding for fractional calculus and presents new techniques for researchers.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work the authors used Grammarly for language editing. After using this service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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