On Some Characteristics of Generalized γ-Closure Spaces

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Abstract: In this paper, we study that a pointwise symmetric γ-isotonic (γIso), γ-closure (γCl) mapping is uniquely specified by the pairs of sets it separates. Then, we demonstrate that when the γCl mapping of the domain is -γIso and the γCl mapping of the co-domain is γIso and pointwise γ-symmetric (γsym), mappings that only separate already separated pairs of sets are γ -continuous.

Key-words: - γCl separated; γCl mapping; γ-continuous mappings.

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1 Introduction

The Nobel Prize 2016 in Physics was jointly bestowed upon three researchers in recognition of their work on phase transitions and topological phases of matter. As this occurrence, people are more aware of the necessity of gaining further topological understanding.
General topological spaces are becoming more and more important in several fields of use, for example data mining, [1]. Information systems are the essential tools used in any real-world field to learn from data. Mathematical models can be made for both quantitative and qualitative data based on topological structures that describe how data is collected. So far, there have been a lot of attempts by topologists to use the idea of closure spaces to investigate different topological problems, [2, 3, 4]. Within this context, the symbols $(U_1;\tau)$ and $(U_2;\sigma)$ or just $U_1 and U_2$ stand for topological spaces. Let $K_1 \subseteq U_1$ is called a ($\gamma - open$, [5]) or ($b - open$, [6]) or (sp-open, [7]), if $K_1 \subseteq (Cl(Int(K_1)))\bigcup(Int(Cl(K_1)))$. The complement of a $\gamma - open$ set is called $\gamma - closed$. The intersection of all $\gamma - closed$ sets containing a set K_1 is called the $\gamma Cl(K_1)$.
We may not have at the moment an application of theoretical mathematics that is being formulated, but someone will find an application for it and who will be fluent in using it, and that is why theoretical mathematics is very important. Recently, a lot of research has been published in Gamma, Nano Topology and Soft, [8]. Previously, we have used topology to study the similarity of DNA sequences and to identify mutations in genes, chromosomes, [9]. Many topologists studied topological models in medicine, [10, 11]. We also used topology to study the recombination of DNA and to form a mathematical model for the recombination process. El-Sharkasy[12] used τ to study the recombination of DNA and it was better as it gives more than one space. In biomathematics, topological concepts can be used to build flexible mathematical models.

2 Generalized γCl structures

Throughout this section, we will proceed under the assumption that set W is a nonempty finite universal set and that set 2^W is its power set.

Definition 2.1. A closure structure ([13], [14], [15]) is a pair (W, Cl) and $Cl : 2^W \to 2^W$ is a mapping associating with $\forall U_1 \subseteq W$ and $Cl(U_1) \subseteq W$, called the closure of U_1, where:

(1) $Cl(\phi) = \phi$,

(2) $U_1 \subseteq Cl(U_1)$,

(3) $Cl(U_1 \bigcup U_2) = Cl(U_1) \bigcup Cl(U_2)$,

(4) The interior Int(U_1) of U_1 is $(Cl(U_1^c))^c$,

(5) The set U_1 is a neighborhood of an element $u_1 \in W$ if $u_1 \in Int(U_1)$,

(6) U_1 is closed if U_1=$Cl(U_1)$,

(7) U_1 is open if U_1=$Int(U_1)$.

Because the subsequent lemmas' proofs are clear, we omit them.

Lemma 2.2. *In a closure structure (W, Cl), the following hold:*

(1) U_1 is open if and only if U_1^c is closed, for any $U_1 \subseteq W$.

(2) If $U_1 \subseteq U_2$, then $Cl(U_1) \subseteq Cl(U_2)$ and consequently $Int(U_1) \subseteq Int(U_2)$.

(3) $Cl(\bigcap_{i \in I} U_{1i}) \subseteq \bigcap_{i \in I} Cl(U_{1i})$.

Lemma 2.3. *The open sets of a closure structure (W, Cl), satisfy*

(1) ϕ and W are open,

(2) If U_1 , U_2 is open, therefore $(U_1 \bigcap U_2)$ is open,

(3) If U_{1i} is open, $\forall i \in I$, then $\bigcup_{i \in I} U_{1i}$ is open.

Definition 2.4. (1) The generalized γCl structure is a pair $(U_1, \gamma Cl)$ consisting of a set U_1 , a γCl mapping (γCl), a mapping from the power set of U_1 to itself.

(2) γCl of $K_1 \subseteq U_1$, denoted $\gamma Cl(K_1)$ is the image of K_1 under γCl.

(3) The γ-Exterior (γExt) of K_1 is $\gamma Ext(K_1) = U_1 \backslash \gamma Cl(K_1)$ and γ-Interior (γInt) of K_1 is $\gamma Int(K_1) = U_1 \backslash \{\gamma Cl(U_1 \backslash K_1)\}$.

(4) K_1 is γ-closed when $K_1 = \gamma Cl(K_1)$, K_1 is γ-open when $K_1 = \gamma Int(K_1)$ and M is a γ-neighborhood (γnbd) of u_1 if $u_1 \in \gamma Int(M)$.

Definition 2.5. A γCl mapping defined on U_1 is called:

(1) γ-grounded if $\gamma Cl(\phi) = \phi$.

(2) γIso if $\gamma Cl(K_1) \subseteq \gamma Cl(K_2)$ for $K_1 \subseteq K_2$.

(3) γ-enlarging if $K_1 \subseteq \gamma Cl(K_1), \forall K_1 \subseteq U_1$.

(4) γ-idempotent if $\gamma Cl(K_1) = \gamma Cl(\gamma Cl(K_1))$, $\forall K_1 \subseteq U_1$.

(5) γ-sublinear if $\gamma Cl(K_1 \bigcup K_2) \subseteq \gamma Cl(K_1) \bigcup \gamma Cl(K_2), \forall K_1, K_2 \subseteq U_1$.

(6) γ-additive if $\bigcup_{i \in U_1} \gamma(K_{1i}) = \gamma Cl(\bigcup_{i \in I} K_{1i})$ for $K_{1i} \subseteq U_1$.

Definition 2.6. (1) $A, B \subseteq U_1$ are called γCl-separated(γCl- sep) in a generalized γCl structure $(U_1, \gamma Cl)$ if $K_1 \bigcap \gamma Cl(K_2) = \phi$ and $\gamma Cl(K_1) \bigcap K_2 = \phi$, or a similar expression, if $K_1 \subseteq \gamma Ext(K_2)$ and $K_2 \subseteq \gamma Ext(K_1)$.

(2) γExt points are called γCl- sep in a generalized γCl structure $(U_1, \gamma Cl)$ if $\forall K_1 \subseteq U_1$ and $\forall u_1 \in \gamma Ext(K_1)$, K_1 and $\{u_1\}$ are $\gamma Cl - sep$.

3 Some Basic Properties

Theorem 3.1. *Let $(U_1, \gamma Cl)$ be a generalized γ-closure structure in which γExt points are γCl-sep and let K_1 be the pairs of γCl-sep sets in U_1. Therefore, $\forall K_1 \subseteq U_1$, the γCl of K_1 is $\gamma Cl(K_1) = \{u_1 \in U_1 : (\{u_1\}, K_1) \notin K_1\}$.*

Proof. . For any generalized γCl structure $\gamma Cl(K_1) \subseteq \{u_1 \in U_1 : (\{u_1\}, K_1) \notin K\}$. Indeed, Suppose that $u_2 \notin \{u_1 \in U_1 : (\{u_1\}, K_1) \notin K\}$ which is, $(\{u_2\}, K_1) \in K$, therefore $\{u_2\} \bigcap \gamma Cl(K_1) = \phi$, then $u_2 \notin \gamma Cl(K_1)$. Let $u_2 \notin \gamma Cl(K_1)$. According to the hypothesis, $(\{u_2\}, K_1) \in K$. Hence $u_2 \notin \{u_1 \in U_1 : (\{u_1\}, K_1) \notin K\}$. □

Definition 3.2. (1) A γCl mapping defined on a set U_1 is called pointwise γsym when, for any u_1,$u_2 \in U_1$, if $u_1 \in \gamma Cl(\{u_2\})$, then $u_2 \in \gamma Cl(\{u_1\})$.

(2) A generalized γCl structure $(U_1, \gamma Cl)$ is said to be $R_0\gamma$ when, for any $u_1, u_2 \in U_1$, if u_1 is in every γnbd of u_2, then u_2 is in every γnbd of u_1.

Corollary 3.3. *Let $(U_1, \gamma Cl)$ be a generalized γCl structure in which γExt points are γCl-sep. Then γCl is pointwise γsym and $(U_1, \gamma Cl)$ is $R_0\gamma$.*

Proof. . Assume that γExt points be γCl-sep in $(U_1, \gamma Cl)$. If $u_1 \in \gamma Cl(\{u_2\})$, then $\{u_1\}$ and $\{u_2\}$ are not γCl-separated. This means that $u_2 \in \gamma Cl(\{u_1\})$. Then, γCl is pointwise γsym. Assume that u_1 belongs to every γnbd of u_2, that is, $U_1 \in M$ whenever $y \in \gamma Int(M)$. Letting $K_1 = \{U_1 \backslash M\}$ and rewriting in the other direction, $u_2 \in \gamma Cl(K_1)$ whenever $u_1 \in K_1$. Suppose that $u_1 \in \gamma Int(M)$, $u_1 \notin \gamma Cl(U_1 \backslash M)$. So u_1 is γCl-sep from $\{U_1 \backslash M\}$, and hence, $\gamma Cl(\{u_1\}) \subseteq M$, $u_1 \in (\{u_1\})$, so $u_2 \in \gamma Cl(\{u_1\}) \subseteq M$. Hence $(U_1, \gamma Cl)$ is $R_0\gamma$. □

Note that these three axioms are not equal with one another in general; nevertheless, they are equivalent with one another when γCl mapping is γIso

Theorem 3.4. *If $(U_1, \gamma Cl)$ is a generalized γCl structure with γCl γIso, then the next statement are equivalent:*

(1) γExt points are γCl-sep.

(2) (γCl) is pointwise γsym.

(3) $(U_1, \gamma Cl)$ is $R_0\gamma$.

Proof. . Let (2) be true. Assume that $K_1 \subseteq U_1$, and $u_1 \in \gamma Ext(K_1)$. Then, as γCl is γIso, $\forall u_2 \in K_1$, $u_1 \notin \gamma Cl(\{u_2\})$, and hence, $u_2 \notin \gamma Cl(\{u_1\})$. Thus $K_1 \bigcap \gamma Cl\{u_1\} = \phi$. Then $(2) \rightarrow (1)$, and by Corollary 3.1,$(1) \rightarrow (2)$. Let (2) be true and suppose

that $u_1, u_2 \in U_1$ since u_1 is in each γnbd of u_2, i. e., $u_1 \in M$ when $u_2 \in \gamma Int(M)$. Therefore $u_2 \in \gamma Cl(K_1)$ when $u_1 \in K_1$, and especially since $u_1 \in \{u_1\}$, $u_2 \in \gamma Cl(\{u_1\})$.As a consequence, $u_1 \in \gamma Cl(\{u_2\})$. Hence if $u_2 \in K_2$, therefore $u_1 \in \gamma Cl(\{u_2\}) \subseteq \gamma Cl(K_2)$, as γCl is γIso. Then, if $u_1 \in \gamma Cl(K_3)$, then $u_2 \in K_3$, that is, u_2 is in each $\gamma nbd of u_1$. Then, (2) implies (3).
Now, assume that $(U_1, \gamma Cl)$ is $R_0\gamma$ and $u_1 \in \gamma Cl(\{u_2\})$ such that γCl is γIso, $u_1 \in \gamma Cl(K_2)$ whenever $u_2 \in K_2$ or, u_2 is in each γnbd of u_1 where $(U_1, \gamma Cl)$ is $R_0\gamma$, $u_1 \in \gamma Int(M)$. Then, $u_2 \in \gamma Cl(K_1)$ whenever $u_1 \in K_1$, and especially since $u_1 \in \{u_1\}$, $u_2 \in \gamma Cl(\{u_1\})$. Hence, (3) $\Longrightarrow$ (2). □

Theorem 3.5. *Let W be a set of unordered pairs of subsets of a set of U_1 . Then:*

(1) If $K_1 \subseteq K_2$ and $(K_2, K_3) \in W$, then $(K_1, K_3) \in W$, $\forall K_1, K_2, K_3 \subseteq U_1$;

(2) If $(\{u_1\}, K_2) \in W$, $\forall u_1 \in K_1$ and $(\{u_2\}, K_1) \in W, \forall u_2 \in K_2$, then $(K_1, K_2) \in W$, $\forall K_1, K_2, K_3 \subseteq U_1$. Then there is a unique pointwise γsym, γIso and γCl mappings γCl on U_1 which γCl-sep the elements of W.

Proof. . Define γCl by $\gamma Cl(K_1) = \{u_1 \in U_1 : (\{u_2\}, K_1) \notin W\}$ for each $K_1 \subseteq U_1$. If $K_1 \subseteq K_2 \subseteq U_1$ and $u_1 \in \gamma Cl(K_1)$, then $(\{u_2\}, K_1) \notin W$, thus $(\{u_1\}, K_2) \notin W$, meaning that, $u_1 \in \gamma Cl(K_2)$. Then, $\gamma Cl(\{u_1\})$ is γIso, Moreover, $u_1 \in \gamma Cl(\{u_2\})$ iff $(\{u_1\}, \{u_2\}) \notin W$ iff $u_2 \in \gamma Cl(\{u_1\})$. Hence γCl is pointwise γsym. Let $(K_1, K_2) \in W$. Therefore $K_1 \bigcap \{\gamma Cl(K_2)\} = K_1 \bigcap \{u_1 \in U_1 : (\{u_1\}, K_2) \notin W\} = \{u_1 \in K_1 : (\{u_1\}, K_1) \notin W\} = \phi$. Similarly, $\gamma Cl(K_1) \bigcap K_2 = \phi$. Then, if $(K_1, K_2) \in W$, then K_1, and K_2 are γCl-sep.
Now, Let K_1 and K_2 be γCl-sep. Then $\{u_1 \in K_1 : (\{u_1\}, K_2) \notin W\}$=$K_1 \bigcap \gamma Cl(K_2) = \phi$ and $\{u_1 \in K_2 : (\{u_2\}, K_1) \notin W\} = \gamma Cl\{K_1 \bigcap K_2 = \phi$. Thus, $(\{u_1\}, K_2) \in W, \forall u_1 \in K_1$ and $(\{u_2\}, K_1) \in W, \forall u_2 \in K_2$. Then, $(K_1, K_2) \in W$. □

Many features of γCl mappings can be stated wise the sets they separate, as shown below:

Theorem 3.6. *If W is the pairs of γCl-sep sets of a generalized γCl structure $(U_1, \gamma Cl)$ in which γExt points are γCl-sep, then γCl is*

(1) γ-grounded iff $\forall u_1 \in U_1, (\{u_1\}, \phi) \in W$.

(2) γ-enlarging iff $\forall (K_1, K_2) \in W$, $K_1 \cap K_2 = \phi$.

(3) γ-sub linear iff $(K_1, K_2 \bigcup K_3) \in W$ whenever $(K_1, K_2) \in W$ and $(K_1, K_3) \in W$.
In addition, if γCl is γ-enlarging and for $K_1, K_2 \subseteq U_1$. $(\{u_1\}, K_1) \notin W$ whenever $(\{u_1\}, K_2) \notin W\}$ and $(\{u_2\}, K_1) \notin W, \forall u_2 \in K_2$, then γCl is γ-idempotent. Also,if γCl-Iso and γ-idepotent, then $(\{u_1\}, K_1) \notin W$ whenever $(\{u_1\}, K_1) \notin W$ and $(\{u_2\}, K_1) \notin W, \forall u_2 \in K_2$.

Proof. . By Theorem 3.1, $\gamma Cl(K_1) = \{u_1 \in U_1 : (\{u_1\}, K_1) \notin W\}$ for each $K_1 \subseteq U_1$. Let $\forall u_1 \in U_1$, $(\{u_1\}, \phi) \in W$. Therefore $\gamma Cl(\phi) = \{u_1 \in U_1, (\{u_1\}, \phi) \notin W\} = \phi$. Then γCl is γ-grounded. Conversely, if $\phi = \gamma Cl(\phi) = \{u_1 \in U_1, (\{u_1\}, \phi) \notin W\}$, and hence $(\{u_1\}, \phi) \in W$, for every $u_1 \in U_1$. Let for each $(K_1, K_2) \in W$, $K_1 \cap K_2 = \phi$. Since $(\{a\}, K_1) \notin W$ if $a \in K_1$, $K_1 \subseteq \gamma Cl(K_1), \forall K_1 \subseteq U_1$. Then γCl is γ-enlarging. Conversely, suppose that γCl is γ-enlarging and $(K_1, K_2) \in W$. Therefore $K_1 \bigcap K_2 \subseteq \gamma Cl(K_1) \bigcap K_2 = \phi$.
Let $(K_1, K_2 \bigcup K_3) \in W$ whenever $(K_1, K_2) \in W$ and $(K_1, K_3) \in W$, and Let $u_1 \in U_1$ and $K_2, K_3 \subseteq U_1$ where $(\{u_1\}, K_2 \bigcup K_3) \notin W$. Therefore $(\{u_1\}, K_2) \notin W$ or $(\{u_1\}, K_3) \notin W$. Thus $\gamma Cl(K_2 \bigcup K_3) \subseteq \gamma Cl(K_2) \bigcup \gamma Cl(K_3)$. Hence, γCl is γ-sublinear.
Conversely, let γCl be γ-sublinear, and $(K_1, K_2), (K_1, K_3) \in W$. Therefore $\gamma Cl(K_2 \bigcup K_3) \bigcap K_1 \subseteq (\gamma Cl(K_2) \bigcup \{\gamma Cl(K_3)) \bigcap K_1 = (\gamma Cl(K_2) \bigcap K_1) \bigcup (\gamma Cl(K_3) \bigcap K_1) = \phi$ and $(K_2 \bigcup K_3) \bigcap \gamma Cl(K_1) = (K_2 \bigcap \gamma Cl(K_1)) \bigcup (K_3 \bigcap \gamma Cl(K_1)) = \phi$. Assume that γCl is γ-enlarging and let $(\{u_1\}, K_1) \notin W$ whenever $(\{u_2\}, K_2) \notin W$ and $(\{u_2\}, K_1) \notin W, \forall u_2 \in K_2$, then $\gamma Cl(\gamma Cl(K_1)) \subseteq \gamma Cl(K_1)$. If $u_1 \in \{\gamma Cl\{\gamma Cl(K_1)\}\}$, hence $(\{u_1\}, \gamma Cl(K_1)) \notin W$, $(\{u_2\}, K_1) \notin W, \forall u_2 \in \gamma Cl(K_1)$, and hence $(\{u_1\}, K_1) \notin W$. Where γCl is γ-enlarging, therefore $\gamma Cl(K_1) \subseteq \gamma Cl(\gamma Cl(K_1))$. Then, $\gamma Cl(\gamma Cl(K_1)) = \gamma Cl(K_1)$, $\forall K_1 \subseteq U_1$. Lastly, let's say that γCl be γIso and γ-idempotent. Suppose that $u_1 \in U_1$ and $K_1, K_2 \subseteq U_1$ since $(\{u_1\}, K_2) \notin W$ and $\forall u_2 \in K_2$, $(\{u_2\}, K_1) \notin W$, then $u_1 \in \gamma Cl(K_2)$ and $\forall u_2 \in K_2$, $u_2 \in \gamma Cl(K_1)$, (i. e., $K_2 \subseteq \gamma Cl(K_1)$. Then, $u_1 \in \gamma Cl(K_2) \subseteq \gamma Cl(\gamma Cl(K_1)) = \gamma Cl(K_1)$. □

Definition 3.7. If $(U_1, (\gamma Cl)_{U_1})$ and $(U_2, (\gamma Cl)_{U_2})$ are generalized γCl structures, then a mapping $T : U_1 \rightarrow U_2$ is called:

(1) γCl preserving if $T((\gamma Cl)_{U_1}(K_1)) \subseteq (\gamma Cl)_{U_2}(T(K_1)), \forall K_1 \subseteq U_1$.

(2) γ-continuous if $(\gamma Cl)_{U_1}(T^{-1}(K_2)) \subseteq T^{-1}((\gamma Cl)_{U_2}(K_2)), \forall K_2 \subseteq U_2$.

Theorem 3.8. *Let $(U_1, (\gamma Cl)_{U_1})$ and $(U_2, (\gamma Cl)_{U_2})$ be generalized γCl structures and $T : U_1 \to U_2$ be a mapping:*

(1) If T is γCl preserving and $(\gamma Cl)_{U_2}$ is γIso, then T is $\gamma - continuous$.

(2) If T is γ-continuous and $(\gamma Cl)_{U_1}$ is γIso, then T is $\gamma Cl preserving$.

Proof. . Assume that T is γCl preserving and, $(\gamma Cl)_{U_2}$ is γIso. Let $K_2 \subseteq U_2$, therefore $T(\gamma Cl)_{U_1}((T^{-1}(K_2))) \subseteq (\gamma Cl)_{U_2}((T(T^{-1}(K_2))) \subseteq (\gamma Cl)_{U_2}(K_2)$ hence, $(\gamma Cl)_{U_1}((T^{-1}T(K_2))) \subseteq T^{-1}((T(\gamma Cl)_{U_1}T^{-1}(K_2))) \subseteq T^{-1}((\gamma Cl)_{U_2}(K_2))$. Let T be γ-continuous and $(\gamma Cl)_{U_1}$ is γIso. Suppose that $K_1 \subseteq U_1$. Then $(\gamma Cl)_{U_1}(K_1) \subseteq (\gamma Cl)_{U_1}(T^{-1}(T(K_1))) \subseteq T^{-1}((\gamma Cl)_{U_2}(T^{-1}(K_1)))$. Then, $T(\gamma Cl)_{U_1}(K_1) \subseteq T(T^{-1}(\gamma Cl)_{U_2}T(K_1)) \subseteq (\gamma Cl)_{U_2}(T(K_1)$. □

Definition 3.9. Let $(U_1, (\gamma Cl)_{U_1})$ and $(U_2, (\gamma Cl)_{U_2})$ be generalized γCl structures and $T : U_1 \to U_2$ be a mapping, if $\forall K_1, K_2 \subseteq U_1$, $T(K_1)$ and $T(K_2)$ are not $(\gamma Cl)_{U_2}$-sep whenever K_1 and k_2 are not $(\gamma Cl)_{U_1}$-sep, Then, we say, T is non γ-sep.
Notice that T is non γ-sep iff K_1 and K_2 are $(\gamma Cl)_{U_1}$-sep, whenever $T(K_1)$ and $T(K_2)$ are $(\gamma Cl)_{U_2}$-sep.

Theorem 3.10. *Suppose that $(U_1, (\gamma Cl)_{U_1})$ and $(U_2, (\gamma Cl)_{U_2})$ are a generalized γCl structures and $T : U_1 \to U_2$ is a mapping:*

(1) If $(\gamma Cl)_{U_2}$ is γIso and T is non γ-sep, then $T^{-1}(C)$ and $f^{-1}(D)$ are $((\gamma Cl)_{U_1})$-sep for every C and D are $((\gamma Cl)_{U_2})$-sep.

(2) If $((\gamma Cl)_{U_1})$ is γIso and $T^{-1}(C)$ and $T^{-1}(D)$ are $(\gamma Cl)_{U_1}$-sep for every C, D is $(\gamma Cl)_{U_2}$-sep, then T is non $\gamma - sep$.

Proof. . Let C and D be $(\gamma Cl)_{U_2}$-sep subsets, such that $(\gamma Cl)_{U_2}$ is γIso. Suppose that $K_1 = T^{-1}(C)$, $K_2 = T^{-1}(D)$ then $T(K_1) \subseteq C$, $T(K_2) \subseteq D$ and $(\gamma Cl)_{U_2}$ is γIso, $T(K_1)$ and $T(K_2)$ are also $(\gamma Cl)_{U_2}$-sep, as a result of this, K_1 and K_2 are $(\gamma Cl)_{U_2}$-sep in U_1. Assume that $(\gamma Cl)_{U_1}$ is γIso and let $K_1, K_2 \subseteq U_1$ where $C = T(K_1)$ and $D = T(K_2)$ are $(\gamma Cl)_{U_1}$-sep, therefore $T^{-1}(C)$ and $T^{-1}(D)$ are $(\gamma Cl)_{U_1}$-sep and since $(\gamma Cl)_{U_1}\gamma Iso$, $K_1 \subseteq T^{-1}(T(K_1)) = T^{-1}(C)$ and $K_2 \subseteq T^{-1}(T(K_2)) = T^{-1}(D)$ are $(\gamma Cl)_{U_1}$-sep as well. □

Theorem 3.11. *Let $(U_1, (\gamma Cl)_{U_1})$ and $((U_2, \gamma Cl)_{U_2})$ be generalized γCl structures and Assume that $T : U_1 \to U_2$ be a mapping. If T is γCl preserving, then T is non γ-sep.*

Proof. . Let T be γCl preserving and $K_1, K_2 \subseteq U_1$ be not $(\gamma Cl)_{U_1}$-sep. Assume that $(\gamma Cl)_{U_1}(K_1) \bigcap K_2 \neq \phi$. Therefore $\phi \neq T((\gamma Cl)_{U_1}(K_1)) \bigcap K_2)) \subseteq T((\gamma Cl)_{U_1}(K_1)) \bigcap T(K_2) \subseteq (\gamma Cl)_{U_2}(T(K_1)) \bigcap T(K_2)$. Similarly $K_1 \bigcap (\gamma Cl)_{U_1}(K_2) \neq \phi$, hence $T(K_1) \bigcap ((\gamma Cl)_{U_2}(T(K_2))) \neq \phi$. Then $T(K_1)$ and $T(K_2)$ are not $(\gamma Cl)_{U_2}$-sep. □

Theorem 3.12. *Let $(U_1, (\gamma Cl)_{U_1})$ and $(U_2, (\gamma Cl)_{U_2})$ be generalized γCl structures which γExt points $(\gamma Cl)_{U_2}$-sep in U_2 and let $T : U_1 \to U_2$ be a mapping. Then T is γCl preserving iff T is non γ-sep.*

Proof. . If T is γCl preserving, then T is non γ-sep. Let T be non γ-sep and $K_1 \subseteq U_1$. If $(\gamma Cl)_{U_1}(K_1) = \phi$, therefore $T(\gamma Cl)_{U_1}(K_1)) = \phi \subseteq (\gamma Cl)_{U_2}(T(K_1))$. Assume $(\gamma Cl)_{U_1}(K_1) \neq \phi$. Let W_{U_1} and W_{U_2} be denote pairs of $(\gamma Cl)_{U_1} - sep \subseteq U_1$ and the pairs of $(\gamma Cl)_{U_2}$-sep subsets of U_2, respectively. Suppose $u_2 \in T((\gamma Cl)_{U_1}(K_1))$, $u_1 \in (\gamma Cl)_{U_1}(K_1) \bigcap T^{-1}(u_2)$. Where $u_1 \in \{(\gamma Cl)_{U_1}(K_1),(\{u_2\}, K_1)\} \notin W_{U_1}$ and since T is non γ-sep, $\{\{u_2\}, T(K_1)\} \notin W_{U_2}$. Where γExt points are $(\gamma Cl)_{U_2}$-sep, $u_2 \in (\gamma Cl)_{U_2}(T(K_2))$. Thus $T(\gamma Cl)_{U_1}(K_1)) \subseteq (\gamma Cl)_{U_2}(T(K_1)), \forall K_1 \subseteq U_1$. □

Corollary 3.13. *Let $(U_1, (\gamma Cl)_{U_1}))$ and $(U_2, (\gamma Cl)_{U_2}))$ be generalized γCl structures with $(\gamma Cl)_{U_1}$ gamma-Iso and assume that $T : U_1 \to U_2$ be a mapping, if T is γ-continuous. Then T is non γ-sep.*

Proof. . If T is γ-continuous and $(\gamma Cl)_{U_1}\gamma Iso$, then by (Theorem 3.4) if T is γCl preserving. by (Theorem 3.7), T is non γ-sep. □

Corollary 3.14. *Let $(U_1, (\gamma Cl)_{U_1})$ and $(U_2, (\gamma Cl)_{U_2})$ be a generalized γCl structures with γIso closure mappings and with $(\gamma Cl)_{U_2}$-pointwise Syγ and let $f : U_1 \to U_2$ be a mapping. Then f is γ-continuous iff f non-γ-sep.*

Proof. . Since $(\gamma Cl)_{U_2}$ is γIso and pointwise Syγ, Ext γ points are γCl sep in $(U_2, (\gamma Cl)_{U_2})$ (Theorem 3.1). Since both γCl mappings are γIso, f is γCl preserving (Theorem 3.4) iff f is γ-continuous. Thus, we can apply the (Theorem 3.7). □

4 γ-Connected Generalized γCl structures

Definition 4.1. Let $(U_1, \gamma Cl)$ be a generalized γCl structure. U_1 is called γ-connected if U_1 is not a union of disjoint nontrivial γCl-sep pair of sets.

Theorem 4.2. *If $(U_1, \gamma Cl)$ be a generalized γCl structure with γ-grounded γIso and γ-enlarging γCl, then the next are equivalent:*

(1) $(U_1, \gamma Cl)$ is γ-connected,

(2) U_1 can not be a union of nonempty disjoint γ-open sets.

Proof. . **(1)⇒(2):** Assume that $K_1, K_2 be \gamma$-open sets. Then $U_1 = K_2 \cup K_2$ and $K_1 \cap K_2 = \phi$. This implies $K_2 = U_1\ K_1$ and K_1 is a $\gamma - open$ set. Thus, K_2 is $\gamma - closed$, hence $K_1 \bigcap \gamma Cl(K_2) = K_1 \bigcap K_2 = \phi$. By using the same method, we get $\gamma Cl(K_1) \bigcap K_2 = \phi$. Hence, K_1 and K_2 are γCl- sep and hence U_1 is not γ-connected. A contradiction.

(2)⇒(1): Assume that U_1 is not γ-connected. Then $U_1 = K_1 \bigcup K_2$, since $K_1 \cap K_2 = \phi$, γCl- sep sets, i.e, $K_1 \bigcap \gamma Cl(K_2) = \gamma Cl(K_1) \bigcap K_2 = \phi$. We have $\gamma Cl(K_2) \subset U_1 - K_1 \subset K_2$ such that γCl is γ-enlarging, we get $\gamma Cl(K_2) = K_2$, then, K_2 is γ-closed. By using $\gamma Cl(K_1) \bigcap K_2 = \phi$. Similarly, it is clear that K_1 is γ-closed. Inconsistency. □

Definition 4.3. Let $(U_1, \gamma Cl)$ be a generalized γCl structure with γ-grounded $\gamma Iso \gamma Cl$. Then, $(U_1, \gamma Cl)$ is called a $T_1 - \gamma$-grounded γIso structure if $\gamma Cl(u_1) \subset \{u_1\}, \forall u_1 \in U_1$.

Theorem 4.4. *If $(U_1, \gamma Cl)$ is a generalized γCl structure with γ-grounded γIso γCl,then the next statement are equivalent:*

(1) $(U_1, \gamma Cl)$ is γ-connected,

(2) Every γ-continuous mapping $T : U_1 \to U_2$ is constant for all $T_1 - \gamma$-grounded γIso structure $U_2 = \{0, 1\}$.

Proof. . **(1)⇒(2):** Assume that U_1 is γ-connected and $T : U_1 \to U_2$ is γ-continuous and it isn't constant. Therefore there is a set $U_{11} \subseteq U_1$ where $U_{11} = \{T^{-1}\{0\}\}$ and $U_1 \backslash U_{11} = T^{-1}(\{1\})$. such that T is γ-continuous and U_2 is $T_1 - \gamma$-grounded γIso structure, then $\gamma Cl(U_{11}) = \gamma Cl(T^{-1}\{0\}) \subset T^{-1}(\gamma Cl(\{0\})) \subset T^{-1}(\{0\}) = U_{11}$ and hence $\gamma Cl(U_{11}) \bigcap \{U_1 \backslash U_{11}\} = \phi$. Similarly, we have $U_{11} \bigcap \gamma Cl(U_1 \backslash U_{11}) = \phi$. Contradiction, then, T is constant.

(2)⇒(1): Let U_1 is not γ-connected. Therefore there exists γCl sep sets U_{11} and U_{12} where $U_{11} \bigcup U_{12} = U_1$. We have $\gamma Cl(U_{11}) \subset U_{11}$ and $\gamma Cl(U_{12}) \subset U_{12}$ and $U_1\ U_{11} \subset U_{12}$. Such that γCl is γIso and U_{11}, U_{12} are γCl- sep, hence $\{\gamma Cl(U_1 \backslash U_{11}) \subset \gamma Cl(U_{12}) \subset \{U_1 \backslash U_{11}\}$. Let the structure $(U_2, \gamma Cl)$ by $U_2 = \{0, 1\}$, $\gamma Cl(\phi) = \phi$, $\gamma Cl(\{0\}) = \{0\}$, $\gamma Cl(\{1\}) = \{1\}$, and $\gamma Cl(U_2) = U_2$, therefore the structure $(U_2, \gamma Cl)$ is a $T_1 - \gamma$-grounded γ-Iso structure, we define the mapping $T : U_1 \to U_2$ as $T(U_{11}) = \{0\}$ and $T(U_1 \backslash U_{11}) = \{1\}$. Assume that $K_1 \neq \phi$ and $K_1 \subset U_2$. If $K_1 = U_2$, and hence $T^{-1}(K_1) = U_1$, $\gamma Cl(\{U_1\}) = \gamma Cl(T^{-1}(K_1)) \subset U_1 = T^{-1}(K_1) = T^{-1}(\gamma Cl(K_1))$. If $K_1 = \{0\}$, therefore $T^{-1}(K_1) = U_{11}$, then $\gamma Cl(U_{11}) = \gamma Cl(T^{-1}(K_1)) \subset U_{11} = T^{-1}(K_1) = T^{-1}(\gamma Cl(K_1))$. If $K_1 = \{1\}$, then $T^{-1}(K_1) = \{U_1 \backslash U_{11}\}$, hence $\gamma Cl(U_1 \backslash U_{11}) = \gamma Cl(T^{-1}(K_1)) \subset U_1 \backslash U_{11} = T^{-1}(K_1) = T^{-1}(\gamma Cl(K_1))$. Thus, T is γ-continuous such that T is not constant. Inconsistency. □

Theorem 4.5. *If $T : (U_1, \gamma Cl) \to (U_2, \gamma Cl)$ and $g : (U_2, \gamma Cl) \to (U_3, \gamma Cl)$ are γ-continuous maps, then, $g \circ T : U_1 \to U_3$ is γ-continuous.*

Proof. . Let T and g be γ-continuous. For every $K_1 \subset U_3$ we get $\gamma Cl((g \circ T)^{-1}(K_1)) = \gamma Cl(T^{-1}(g^{-1}(K_1))) \subset T^{-1}(\gamma Cl(g^{-1}(K_1))) \subset T^{-1}(g^{-1}(\gamma Cl(K_1))) = (g \circ T)^{-1}(\gamma Cl(K_1))$. Then, $g \circ T : U_1 \to U_3$ is γ-continuous. □

Theorem 4.6. *Let $(U_1, \gamma Cl)$, $(U_2, \gamma Cl)$ be generalized γCl structures with γ-grounded $\gamma Iso \gamma Cl$ and $T : (U_1, \gamma Cl) \to (U_2, \gamma Cl)$ be a γ-continuous mapping from U_1 onto U_2. If U_1 is γ-connected, then U_2 is γ-connected, $g \circ T : U_1 \to \{0, 1\}$ is constant and then "g" is constant mapping. By (Theorem 4.2), U_2 is γ-connected.*

Proof. . Let $\{0, 1\}$ be a generalized γCl structures with γ-grounded γIso γCl and $g : U_2 \to \{0, 1\}$ is a γ-continuous mapping. Since T is γcontinuous, by (Theorem 3.3), $g \circ T : U_1 \to \{0, 1\}$ is γ-continuous. where U_1 is γ-connected, $g \circ T$ is constant, then g is constant. Consequently, U_2 is γ-connected. □

Definition 4.7. Let $(U_2, \gamma Cl)$ be a generalized γCl structure with γ-grounded γIso γCl, and more than one element. A generalized γCl structure $(U_1, \gamma Cl)$ with γ-grounded γIso γCl is called U_2-γ-connected if any γ-continuous mapping $T : U_1 \to U_2$ is constant.

Theorem 4.8. *Let $(U_2, \gamma Cl)$ be a generalized γCl structure with γ-grounded γIso, γ-enlarging γCl, and more than one element. Then every U_2-γ-connected generalized γCl structure with γ-grounded γIso is γ-connected.*

Proof. . Assume that $(U_1, \gamma Cl)$ be a U_2-γ-connected generalized γCl structure with γ-grounded $\gamma Iso\, \gamma Cl$. Let $T : U_1 \to \{0, 1\}$ is a γ-continuous mapping, since $\{0, 1\}$ is a $T_1\gamma$-grounded γIso structure. Where U_2 is a generalized γCl structure with γ-grounded γIso, γ-enlarging γCl and more than one element, therefore there is a γ-continuous injection $g : \{0, 1\} \to U_2$. By (Theorem 4.3), $g \circ T : U_1 \to U_2$ is γ-continuous. Such that U_1 is U_2-γ-connected, then $g\circ T$ is constant. Then, T is constant hence, by (Theorem 4.2), U_1 is γ-connected. □

Theorem 4.9. *Let $(U_1, \gamma Cl)$ and $(U_2, \gamma Cl)$ be a generalized γCl structures with γ-grounded $\gamma Iso\ \gamma Cl$ and $T : (U_1, \gamma Cl) \to (U_2, \gamma Cl)$ be a γ-continuous mapping onto U_2. If U_1 is Z-γ-connected, then U_2 is Z-$\gamma-$connected.*

Proof. . Let $g : U_2 \to Z$ is a γ-continuous mapping. Then $g \circ T : U_1 \to Z$ is γ-continuous. Since U_1 is Z-γ-connected, therefore $g \circ T$ is constant. Therefor "g' is constant. Then U_2 is Z-γ-connected. □

Definition 4.10. the generalized γCl structure $(U_1, \gamma Cl)$ is strongly γ-connected if there is no a countable collection of pairwise γCl- sep sets $\{K_n\}$ where $U_1 = \bigcup\{K_n\}$.

Theorem 4.11. *Every strongly γ-connected generalized γCl structure with γ-grounded γIso γCl is γ-connected.*

Theorem 4.12. *Let $(U_1, \gamma Cl)$ and $(U_2, \gamma Cl)$ be a generalized γCl structure with γ-grounded $\gamma Iso\, \gamma Cl$ and $T : (U_1, \gamma Cl) \to (U_2, \gamma Cl)$ be a γ-continuous mapping onto U_2. If U_1 is strongly γ-connected, then U_2 is strongly γ-connected.*

Proof. . let U_2 is not strongly γ-connected. Then, there exists a countable collection of pairwise γCl sep sets $\{K_n\}$ such that $U_2 = \bigcup\{K_n\}$.
Since $T^{-1}(K_n) \bigcap \gamma Cl(T^{-1}(K_m)) \subset (T^{-1}(K_n)) \bigcap (T^{-1}(\gamma Cl(K_m))) = \phi$ for all $n \neq m$, then the collection $\{T^{-1}(K_n)\}$ is pairwise γCl-sep. This is a contradiction. Thus, U_2 is strongly γ-connected. □

Theorem 4.13. *Let $(U_1, \gamma Cl)_{U_1}$, $(U_2, \gamma Cl)_{U_2}$ be a generalized γCl structures. Then the subsequent are equivalent for a mapping $T : U_1 \to U_2$*

(1) T is γ-continuous,

(2) $T^{-1}(\gamma Int(K_2)) \subseteq \gamma Int(T^{-1}(K_2)), \forall K_2 \subseteq U_2$.

Theorem 4.14. *If $(U_1, \gamma Cl)$ is a generalized γCl structure with γ-grounded γIso γCl, then $(U_1, \gamma Cl)$ is strongly γ-connected iff $(U_1, \gamma Cl)$ is U_2-γ-connected for any countable $T_1 - \gamma$-grounded γIso structure $(U_2, \gamma Cl)$.*

Proof. . (Necessity): Assume that $(U_1, \gamma Cl)$ is strongly connected and $(U_1, \gamma \mathrm{Cl})$ is not $U_2 - \gamma$-connected for some countable $T_1 - \gamma$-grounded. There is a γ-continuous mapping $T : U_1 \to U_2$ this isn't constant and as a result $H = T(U_1)$ is a type of countable set that contains several elements. for every $a_n \in H$, there exists $K_n \subset U_1$ since $K_n = T^{-1}(a_n)$ hence $U_2 = \bigcup K_n$. Such that T is γ-continuous and U_2 is γ-grounded, then $\forall n \neq m$, $\{K_n\} \bigcap \gamma Cl K_m)\} = T^{-1}(a_n) \bigcap \gamma Cl(T^{-1}(a_m)) \subset T^{-1}\{a_n\} \bigcap T^{-1}(\gamma Cl\{a_m\}) \subset T^{-1}\{a_n\} \bigcap T^{-1}\{a_m\} = \phi$. Contradicts, for strong γ-connectedness of U_1. Hence, U_1 is U_2-γ-connected.
(Sufficiency): Assume that $U_1$is $U_2 - \gamma$-connected for any countable $T_1 - \gamma$-grounded γIso structure $(U_2, \gamma Cl)$. Let U_1 b not strongly γ-connected. There is a countable collection of pairwise γCl sep sets $\{K_n\}$ where $U_1 = \bigcup K_n$. We take the structure $(Z, \gamma Cl)$ such that Z is the set of integers and $\gamma Cl : P(Z) \to P(Z)$ is defined as $\gamma Cl(H) = H, \forall H \subset Z$. Obviously $(Z, \gamma Cl)$ is a countable T_1-γ-grounded γIso structure. Let $K_k \in \{K_n\}$. We define a mapping $T : U_1 \to Z$ by $T(K_k) = \{u_1\}$ and $T(U_1\backslash K_k) = \{u_2\}$ since $u_1, u_2 \in Z$ and $u_1 \neq u_2$. Where $\gamma Cl(K_k) \bigcap\{K_n\} = \phi$, for each $n \neq k$, therefore $\gamma Cl(K_k) \bigcap(\bigcup\{K_n\}) = \phi$, $n \neq k$ hence, $\gamma Cl(K_k) \subset \{K_k\}$. Put $\phi \neq H \subset Z$. If $u_1, u_2 \in H$ then $T^{-1}(H) = U_1$ and $\gamma Cl(T^{-1}(H)) = \gamma Cl(U_1) \subset U_1 = T^{-1}(H) = T^{-1}(\gamma Cl(H))$. If $u_1 \in H$ and $u_2 \notin H$, then $T^{-1}(H) = K_k$ and $\gamma Cl(T^{-1}(H)) = \gamma Cl(K_k) \subset K_k = T^{-1}(H) = T^{-1}(\gamma Cl(H))$. If $u_2 \in H$ and $u_1 \notin H$ then $T^{-1}(H) = \{U_1\backslash K_k\}$. Since $\gamma Cl(H) = H, \forall H \subset Z$, then $\gamma Int(H) = H$, $\forall H \subset Z$. Also, $(U_1\backslash K_k) \subset K_{n\neq k}\{U_k\} \subset \{U_1\backslash \gamma Cl(K_k)\} = \gamma Int(U_1\backslash K_k)$. Then, $T^{-1}(\gamma Int(H)) = U_1\backslash K_k = T^{-1}(H) \subset \gamma Int(U_1\backslash K_k) = \gamma Int(T^{-1}(H))$. Alternatively, $\forall n \neq k$, $K_k \cap \gamma Cl(k_n) = \emptyset$, hence $K_k \cap \bigcup\{\gamma Cl(K_n), n \neq k\} = \emptyset$. As a result, it can be concluded that $K_k \cap \gamma Cl(K_n), n \neq k) = \emptyset$. Thus, $\gamma Cl(U_1\backslash K_k) \subset U_1\backslash K_k$. Such that $\gamma Cl(H) = H, \forall H = Z$, we get $\gamma Cl(T^{-1}(H)) = \gamma Cl(U_1\backslash K_k) \subset U_1\backslash K_k$. Hence, T is γ-continuous. Since T is not constant, this goes against what was stated in Z-γ-connectedness of U_1. Hence, U_1 is strongy γ-connected. □

5 Conclusion

This research explores closure structures in point-set topology, with emphasis on γ-closure operators.

We investigate the characteristics of γ-isotonic and γ-closure mappings and provide useful definitions, lemma, and propositions related to the γ-closure structure.
Closure structure in point-set topology gives novel topological qualities (such as separation axioms, connectedness, and continuity) that are useful in studying digital topology, [16]. Thus, we may emphasize γ closure operators as a branch of them and their application in quantum physics, [17], and computer graphics, [18]. In the future, we can use these results to study the processes of nucleic acid ”mutation, recombination, and crossover.”

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Conflict of Interest

The authors declare no conflict of interest.