(q,γ)-Bernstein Basis Functions on the Interval $[a;b]$

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Abstract: -In this paper, we present a novel set of (q,γ)-Bernstein basis functions that are parameterized by q and γ and defined over an interval. We give detailed proofs for several key properties of these functions, including the partition of unity, recurrence relations, degree elevation, and the Marsden identity. Additionally, we introduce and validate the (q,γ)-De Casteljau algorithm, providing comprehensive examples to illustrate its implementation. These results are analyzed to highlight the theoretical and practical implications of these (q,γ)-Bernstein basis functions in various fields, such as polynomial approximation, numerical methods, and Computer Aided Geometric Design (CAGD). Furthermore, we discuss potential extensions and applications of these functions, considering their impact on future research and developments in the domain. By exploring these aspects, we aim to offer a robust framework for understanding and utilizing (q,γ)-Bernstein basis functions.

Key-Words: - (q,γ)-Bernstein polynomials, (q,γ)-Bernstein basis, Marsden's identity, (q,γ)-Bezier curves, De Casteljau, interval.

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1 Introduction

Bernstein's basic polynomials, developed by the Russian mathematician Sergei Natanovich Bernstein in the early 20th century, represent a class of polynomials that play a fundamental role in various fields of applied mathematics, from numerical analysis to computer-aided geometric design (CAGD). These polynomials form the basis of the representation of Bézier curves and surfaces.

The foundations of Bézier theory, developed by French engineers Pierre Bézier and Paul de Casteljau in the 1960s, have found widespread application in fields such as computer-aided geometric design (CAGD), surface modelling and graphic animation.

Recently, basic q-Bernstein polynomials have been develeped in [1], [2], [3] and inspired by classical Bernstein polynomials on the segment, have added an extra dimension thanks to the parameter q, thus opening up new perspectives for the more precise approximation of functions on the interval $[a;b]$. More recently, a special case of q-Bernstein polynomials on the triangle has been developed by [4], inspired by classical Bernstein polynomials on triangular [5]. Thus, the introduction of the q parameter offers greater flexibility and more refined modelling possibilities for q-Bézier curves and surfaces [4], [6], [7]. Very recently, the theory of h-Bézier curves was developed by [8] and that of h-Bézier surfaces by [9].

In this article, we add a new real parameter γ to the parameter q parameter in order to introduce new polynomial functions with base (q,γ)-Bernstein and (q,γ)-Bézier curves on an interval $[a;b]$. We define these new (q,γ)-Bernstein basis polynomials and (q,γ)-Bézier curves, then formulate and prove several properties of these polynomials and curves, including recurrence relations, partition of unity, degree elevation, linear independence (polynomial basis), De Casteljau's algorithm and the (q,γ)-Marsden identity. Finally we conclude and provide some perspectives.

2 Bernstein polynomials

Bernstein's bases functions currently come in three elementary forms

n is a natural integer.

• **on the segment** $[0;1]$**:** [6]

1. the classic form:
$B_k^n(t) = \binom{n}{k} t^k(1-t)^{n-k},\ k=0,\cdots,n,$
2. the h-form:
$$B_k^n(t;h) = \binom{n}{k}\frac{\prod_{i=0}^{k-1}(t+ih)\prod_{i=0}^{n-k-1}(1-t+ih)}{\prod_{i=0}^{n-1}(1+ih)},$$
$k=0,\cdots,n,$
3. the q-form:
$$B_k^n(t;q) = \begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{i=0}^{n-k-1}(1-tq^i),$$
$k=0,\cdots,n,$

• **on an interval** $[a;b]$**:** [6]

1. the classic form on $[a;b]$:
$$B_k^n(t;[a;b]) = \binom{n}{k}\frac{(t-a)^k(b-t)^{n-k}}{(b-a)^n},$$
$k=0,\cdots,n,$
2. the h-form on $[a;b]$:
$$B_k^n(t;[a;b];h) = \binom{n}{k}\frac{\prod_{i=0}^{k-1}(t-a+ih)\prod_{i=0}^{n-k-1}(b-t+ih)}{\prod_{i=0}^{n-1}(b-a+ih)},$$
$k=0,\cdots,n,$
3. the q-form on $[a;b]$:
$$B_k^n(t;[a;b];q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(t-aq^i)\prod_{i=0}^{n-k-1}(b-tq^i)}{\prod_{i=0}^{n-1}(b-aq^i)},$$
$k=0,\cdots,n,$

where [10]: $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$

$$[k]_q! = [k]_q[k-1]_q\cdots[1]_q,$$

$$[0]_q! = 1$$

$$[k]_q = 1+q+\cdots+q^{k-1} = \frac{1-q^k}{1-q},\ \text{with}\quad q\neq 1.$$

3 $(q,\gamma)-$Bernstein basis functions on an interval $[a;b]$

Let γ be a real number.

Let $\tau=b-a;\ \tau_1=t-a;\ \tau_2=b-t;$

$\tilde{\gamma}=\gamma+t-a=\gamma+\tau_1.$

On the interval $[a;b]$, we define the (q,γ)-Bernstein polynomials of degree n by:

$$B_k^n(t;[a;b];q,\gamma) = \begin{cases} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau+\gamma(1-q^i)]}, & \text{for } n\geq 1 \\ 1, & \text{for } n=0 \end{cases} \quad (1)$$

We have:

• $\lim_{q\mapsto 1} B_k^n(t;[a;b];q,\gamma) = B_k^n(t;[a,b]);$ where
$$B_k^n(t;[a,b]) = \binom{n}{k}\frac{(t-a)^k(b-t)^{n-k}}{(b-a)^n}$$

• $\lim_{\gamma\mapsto a} B_k^n(t;[a;b];q,\gamma) = B_k^n(t;[a,b];q);$ where
$$B_k^n(t;[a;b];q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(t-aq^i)\prod_{i=0}^{n-k-1}(b-tq^i)}{\prod_{i=0}^{n-1}(b-aq^i)}$$

• Note that $B_k^n(t;[a;b];q,\gamma)\geq 0,$ for $-1<q\leq 1$ and $\gamma\geq 0$

• The effect of the γ parameter can be seen in

Figure 1, Figure 2, Figure 3 and Figure 4, in Appendix, followed by that of the q parameter Figure 5, Figure 6, Figure 7 and Figure 8, in Appendix.

4 Recurrence relation for the (q,γ)-Bernstein polynomials

For $k = 0, \cdots, n$. we have [10]:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{2}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{3}$$

then the (q,γ)-Bernstein polynomials verify the following recurrence relations:

$$\begin{aligned} B_k^n(t;[a;b];q,\gamma) = \\ \left(\frac{\tau_2 + \tilde{\gamma} q^{n-k-1}}{\tau + \gamma q^{n-1}}\right) B_k^{n-1}(t;[a;b];q,\gamma) \\ + q^{n-k}\left(\frac{\tau_1 + \gamma q^{k-1}}{\tau + \gamma q^{n-1}}\right) B_{k-1}^{n-1}(t;[a;b];q,\gamma) \end{aligned} \tag{4}$$

and

$$\begin{aligned} B_k^n(t;[a;b];q,\gamma) = \\ q^k\left(\frac{\tau_2 + \tilde{\gamma} q^{n-k-1}}{\tau + \gamma q^{n-1}}\right) B_k^{n-1}(t;[a;b];q) \\ + \left(\frac{\tau_1 + \gamma q^{k-1}}{\tau + \gamma q^{n-1}}\right) B_{k-1}^{n-1}(t;[a;b];q) \end{aligned} \tag{5}$$

Proof:

We fix $\begin{bmatrix} n \\ -1 \end{bmatrix}_q = 0$ et $\begin{bmatrix} n \\ n+1 \end{bmatrix}_q = 0$
By using (2) we have:

for $k = 0, \cdots, n$.

$$\begin{aligned} &B_k^n(t;[a;b];q,\gamma) \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}[\tau_1 + \gamma(1-q^i)] \prod_{i=0}^{n-k-1}[\tau_2 + \tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau + \gamma(1-q^i)]} \\ &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{\left(\tau_2 + \tilde{\gamma}(1-q^{n-k-1})\right)}{\left(\tau + \gamma(1-q^{n-1})\right)\prod_{i=0}^{n-2}[\tau + \gamma(1-q^i)]} \times \\ &\prod_{i=0}^{k-1}[\tau_1 + \gamma(1-q^i)] \prod_{i=0}^{n-k-2}[\tau_2 + \tilde{\gamma}(1-q^i)] \\ &+ q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{\left(\tau_1 + \gamma(1-q^{k-1})\right)}{\left(\tau + \gamma(1-q^{n-1})\right)} \times \\ &\frac{\prod_{i=0}^{k-2}[\tau_1 + \gamma(1-q^i)] \prod_{i=0}^{n-k-1}[\tau_2 + \tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-2}[\tau + \gamma(1-q^i)]} \\ &= \left(\frac{\tau_2 + \tilde{\gamma}(1-q^{n-k-1})}{\tau + \gamma(1-q^{n-1})}\right) B_k^{n-1}(t;[a;b];q,\gamma) + \\ &q^{n-k}\left(\frac{\tau_1 + \gamma(1-q^{k-1})}{\tau + \gamma(1-q^{n-1})}\right) B_{k-1}^{n-1}(t;[a;b];q,\gamma) \end{aligned}$$

□

Similarly, by using (3) we show that:

$$\begin{aligned} &B_k^n(t;[a;b];q,\gamma) = \\ &q^k\left(\frac{\tau_2 + \tilde{\gamma}(1-q^{n-k-1})}{\tau + \gamma(1-q^{n-1})}\right) B_k^{n-1}(t;[a;b];q) + \\ &\left(\frac{\tau_1 + \gamma(1-q^{k-1})}{\tau + \gamma(1-q^{n-1})}\right) B_{k-1}^{n-1}(t;[a;b];q) \end{aligned}$$

5 Partition of unity

The (q,γ)-Bernstein polynomials verify the partition of unity property:

$$\sum_{k=0}^{n} B_k^n(t;[a;b];q,\gamma) = 1 \tag{6}$$

Proof: to prove (6), we proceed by recurrence respect $n \geq 0$

1. for $n = 0$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$, replacing $B_k^n(t;[a;b];q,\gamma)$ by its expression given in (1) gives:
$B_0^0(t;[a;b];q,\gamma) = 1$. Therefore relation (6) in true for $n = 0$

2. suppose there exists $n \geq 0$, such that (6) is true and show that (6) is true at rank $n+1$.
Using (4) we have:

$$B_k^{n+1}(t;[a;b];q,\gamma) = \left(\frac{\tau_2 + \tilde{\gamma}(1-q^{n-k})}{\tau + \gamma(1-q^n)}\right) B_k^n(t;[a;b];q,\gamma) + q^{n+1-k}\left(\frac{\tau_1 + \gamma(1-q^{k-1})}{\tau + \gamma(1-q^n)}\right) B_{k-1}^n(t;[a;b];q,\gamma)$$

hence

$$\sum_{k=0}^{n+1} B_k^{n+1}(t;[a;b];q,\gamma) = \sum_{k=0}^{n+1}\Bigg[\left(\frac{\tau_2 + \tilde{\gamma}(1-q^{n-k})}{\tau + \gamma(1-q^n)}\right) B_k^n(t;[a;b];q,\gamma) + q^{n+1-k}\left(\frac{\tau_1 + \gamma(1-q^{k-1})}{\tau + \gamma(1-q^n)}\right) B_{k-1}^n(t;[a;b];q,\gamma)\Bigg]$$

with $B_{n+1}^n = 0$, $B_{-1}^n = 0$ and $\tilde{\gamma} = \gamma + \tau_1$.

This gives

$$\begin{aligned}
&\sum_{k=0}^{n+1} B_k^{n+1}(t;[a;b];q,\gamma) \\
&= \sum_{k=0}^{n}\left(\frac{\tau_2 + \tilde{\gamma}(1-q^{n-k})}{\tau + \gamma(1-q^n)}\right) B_k^n(t;[a;b];q,\gamma) + \sum_{k=0}^{n} q^{n-k}\left(\frac{\tau_1 + \gamma(1-q^{k})}{\tau + \gamma(1-q^n)}\right) B_k^n(t;[a;b];q,\gamma) \\
&= \sum_{k=0}^{n}\Bigg[\left(\frac{\tau_2 + \tilde{\gamma}(1-q^{n-k})}{\tau + \gamma(1-q^n)}\right) + q^{n-k}\left(\frac{\tau_1 + \gamma(1-q^{k})}{\tau + \gamma(1-q^n)}\right)\Bigg] B_k^n(t;[a;b];q,\gamma) \\
&= \sum_{k=0}^{n}\Bigg[\left(\frac{\tau_2 + (\gamma+\tau_1)(1-q^{n-k})}{\tau + \gamma(1-q^n)}\right) + q^{n-k}\left(\frac{\tau_1 + \gamma(1-q^{k})}{\tau + \gamma(1-q^n)}\right)\Bigg] B_k^n(t;[a;b];q,\gamma) \\
&= \sum_{k=0}^{n}\Bigg[\frac{\tau_2 + (\gamma+\tau_1)(1-q^{n-k})}{\tau + \gamma(1-q^n)} + \frac{q^{n-k}\left(\tau_1 + \gamma(1-q^{k})\right)}{\tau + \gamma(1-q^n)}\Bigg] B_k^n(t;[a;b];q,\gamma) \\
&= \sum_{k=0}^{n}\left(\frac{\tau + \gamma(1-q^n)}{\tau + \gamma(1-q^n)}\right) B_k^n(t;[a;b];q,\gamma) \\
&= \sum_{k=0}^{n} B_k^n(t;[a;b];q,\gamma) \\
&= 1
\end{aligned}$$

Therefore relation (6) is true for all $n \geq 0$.

□

6 Degree elevation for (q,γ)-Bernstein polynomials

The (q,γ)-Bernstein polynomials of degree n can be written as (q,γ)-Bernstein polynomials of degree $n+1$ as follows:

$$B_k^n(t;[a;b];q,\gamma) = q^{n-k}\frac{[k+1]_q}{[n+1]_q} B_{k+1}^{n+1}(t;[a;b];q,\gamma) + \frac{[n+1-k]_q}{[n+1]_q} B_k^{n+1}(t;[a;b];q,\gamma) \tag{7}$$

Proof

By writing $B_{k+1}^{n+1}(t;[a;b];q,\gamma)$ and $B_k^{n+1}(t;[a;b];q,\gamma)$ with (1) we have:

$$\begin{aligned}
&q^{n-k}\frac{[k+1]_q}{[n+1]_q} B_{k+1}^{n+1}(t;[a;b];q,\gamma) + \frac{[n+1-k]_q}{[n+1]_q} B_k^{n+1}(t;[a;b];q,\gamma) \\
&= q^{n-k}\frac{[k+1]_q}{[n+1]_q}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \times \frac{\prod_{i=0}^{k}[\tau_1 + \gamma(1-q^i)] \prod_{i=0}^{n-k-1}[\tau_2 + \tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n}[\tau + \gamma(1-q^i)]} +
\end{aligned}$$

$$\frac{[n+1-k]_q}{[n+1]_q}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q \times \frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n}[\tau+\gamma(1-q^i)]}$$

$$=q^{n-k}\left(\frac{\tau_1+\gamma(1-q^k)}{\tau+\gamma(1-q^n)}\right)\left[\begin{array}{c} n \\ k \end{array}\right]_q \times \frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau+\gamma(1-q^i)]}$$

$$+\left(\frac{\tau_2+\tilde{\gamma}(1-q^{n-k})}{\tau+\gamma(1-q^n)}\right)\left[\begin{array}{c} n \\ k \end{array}\right]_q \times \frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau_1+\gamma(1-q^i)]}$$

$$=\underbrace{\left[q^{n-k}\left(\frac{\tau_1+\gamma(1-q^k)}{\tau+\gamma(1-q^n)}\right)+\left(\frac{\tau_2+\tilde{\gamma}(1-q^{n-k})}{\tau+\gamma(1-q^n)}\right)\right]}_{1}\times$$

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau+\gamma(1-q^i)]}$$

$$=\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau+\gamma(1-q^i)]}$$

$$=B_k^n(t;[a;b];q,\gamma)$$

□

7 Polynomial basis

The (q,γ)-Bernstein polynomials form a basis for $\mathbb{P}_n([a;b])$. Where $\mathbb{P}_n([a;b])$ is the space of polynomials of degree at most n on $[a;b]$

Proof

It suffices to show that $\forall n \geq 0$ there exist coefficients $\{C_{n,i,k}\}_{k=0}^{n}$ such that:

$$t^i = \sum_{k=0}^{n} C_{n,i,k}B_k^n(t;[a;b];q,\gamma)\ , \qquad i=0,\cdots,n. \tag{8}$$

We reason by recurrence

1. For $n=0$ we have $B_0^0(t;[a;b];q,\gamma)=1$, $C_{0,0,0}=1$

2. Suppose there exists $n \geq 0$ such that $\forall\ 0 \leq i \leq n$, (8) is true and using (7)
We have:

$$\begin{aligned} t^i &= \sum_{k=0}^{n} C_{n,i,k}B_k^n(t;[a;b];q,\gamma) \\ &= \sum_{k=0}^{n} C_{n,i,k}\Big\{B_k^n(t;[a;b];q,\gamma)\Big\} \\ &= \sum_{k=0}^{n} C_{n,i,k}\Big\{q^{n-k}\frac{[k+1]_q}{[n+1]_q}B_{k+1}^{n+1}(t;[a;b];q,\gamma)+ \\ &\quad \frac{[n+1-k]_q}{[n+1]_q}B_k^{n+1}(t;[a;b];q,\gamma)\Big\} \\ &= \sum_{k=0}^{n+1} C_{n+1,i,k}B_k^{n+1}(t;[a;b];q,\gamma) \end{aligned}$$

with

$$\begin{aligned} C_{n+1,i,k} =& C_{n,i,k-1}q^{n-k-1}\frac{[k]_q}{[n+1]_q}+ \\ & C_{n,i,k}\frac{[n+1-k]_q}{[n+1]_q} \end{aligned} \tag{9}$$

3.

$$\begin{aligned} t^{n+1} = t.t^n &= \sum_{k=0}^{n} C_{n,n,k}tB_k^n(t;[a;b];q,\gamma) \\ &= \sum_{k=0}^{n} C_{n,n,k}\Big(\tilde{D}_{n,k}B_{k+1}^{n+1}(t;[a;b];q,\gamma)+ \\ &\quad \tilde{E}_{n,k}B_k^{n+1}(t;[a;b];q,\gamma)\Big), \end{aligned} \tag{10}$$

This last line of (10) is due to the degree elevation for (q,γ)-Bernstein polynomials (7). We now determine $\tilde{D}_{n,k}$ and $\tilde{E}_{n,k}$, in such a way t^{n+1} is a linear combination of $B_k^{n+1}(t;[a;b];q,\gamma)$.

By applying (1) to this last equality, we obtain:

$$
\begin{aligned}
&B_{k+1}^{n+1}(t;[a;b];q,\gamma)\\
&=\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \times \\
&\frac{\prod_{i=0}^{k}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n}[\tau+\gamma(1-q^i)]}\\
&=\frac{[n+1]_q!}{[k+1]_q![n-k]_q!}\times\frac{\tau_1+\gamma(1-q^k)}{\tau+\gamma(1-q^n)}\times\\
&\quad\frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau+\gamma(1-q^i)]}\\
&=\frac{[n+1]_q}{[k+1]_q}\frac{\tau_1+\gamma(1-q^k)}{\tau+\gamma(1-q^n)}\times\frac{[n]_q!}{[k]_q![n-k]_q!}\\
&\quad\frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau+\gamma(1-q^i)]}\\
&=\frac{[n+1]_q}{[k+1]_q}\left(\frac{\tau_1+\gamma(1-q^k)}{\tau+\gamma(1-q^n)}\right)B_k^n(t;[a;b];q,\gamma)\\
&=\frac{[n+1]_q}{[k+1]_q}\left(\frac{t-a+\gamma(1-q^k)}{\tau+\gamma(1-q^n)}\right)B_k^n(t;[a;b];q,\gamma)
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
&B_{k}^{n+1}(t;[a;b];q,\gamma)\\
&=\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \times \\
&\quad\frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n}[\tau+\gamma(1-q^i)]}\\
&=\frac{[n+1]_q!}{[n+1-k]_q![k]_q!}\times\frac{\tau_2+\tilde{\gamma}(1-q^{n-k})}{\tau+\gamma(1-q^n)}\times\\
&\quad\frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau+\gamma(1-q^i)]}\\
&=\frac{[n+1]_q}{[n+1-k]_q}\frac{\tau_2+\tilde{\gamma}(1-q^{n-k})}{\tau+\gamma(1-q^n)}\frac{[n]_q!}{[n-k]_q![k]_q!}\times\\
&\quad\frac{\prod_{i=0}^{k-1}[\tau_1+\gamma(1-q^i)]\prod_{i=0}^{n-k-1}[\tau_2+\tilde{\gamma}(1-q^i)]}{\prod_{i=0}^{n-1}[\tau+\gamma(1-q^i)]}\\
&=\frac{[n+1]_q}{[n+1-k]_q}\left(\frac{\tau_2+\tilde{\gamma}(1-q^{n-k})}{\tau+\gamma(1-q^n)}\right)\\
&\quad B_k^n(t;[a;b];q,\gamma)
\end{aligned}
$$

$$
\begin{aligned}
&B_k^{n+1}(t;[a;b];q,\gamma)=\frac{[n+1]_q}{[n+1-k]_q}\times\\
&\left(\frac{b-t+(\gamma+t-a)(1-q^{n-k})}{\tau+\gamma(1-q^n)}\right)\times\\
&B_k^n(t;[a;b];q,\gamma)
\end{aligned}
\tag{12}
$$

from (10) , (11) and (12); $\tilde{D}_{n,k}$ and $\tilde{E}_{n,k}$ verify the equation:

$$
\begin{aligned}
&\tilde{D}_{n,k}\frac{[n+1]_q}{[k+1]_q}\left(\frac{t-a+\gamma(1-q^k)}{\tau+\gamma(1-q^n)}\right)+\\
&\quad\tilde{E}_{n,k}\frac{[n+1]_q}{[n+1-k]_q}\times\\
&\left(\frac{b-t+(\gamma+t-a)(1-q^{n-k})}{\tau+\gamma(1-q^n)}\right)=t
\end{aligned}
\tag{13}
$$

with

$$
\begin{aligned}
&b-t+(\gamma+t-a)(1-q^{n-k})\\
&=b-t+t-tq^{n-k}+(\gamma-a)(1-q^{n-k})\\
&=-tq^{n-k}+b+(\gamma-a)(1-q^{n-k})
\end{aligned}
$$

by identification (13) is equivalent to:
●for the coefficient of t:

$$
\left(\tilde{D}_{n,k}\frac{[n+1]_q}{[k+1]_q}-\tilde{E}_{n,k}\frac{[n+1]_q}{[n+1-k]_q}q^{n-k}\right)=\tau+\gamma(1-q^n)
\tag{14}
$$

•for constant term:

$$\Big(\tilde{D}_{n,k}\frac{[n+1]_q}{[k+1]_q}\big(-a+\gamma(1-q^k)\big)+ \tilde{E}_{n,k}\frac{[n+1]_q}{[n+1-k]_q}\big[b+(\gamma-a)(1-q^{n-k})\big]\Big)=0$$
(15)

Multiplying (14) by $-\big(-a+\gamma(1-q^k)\big)$ and adding the result to (15) gives:

$$\tilde{E}_{n,k}\frac{[n+1]_q}{[n+1-k]_q}\times \Big(q^{n-k}\big[-a+\gamma(1-q^k)\big]+ \big[b+(\gamma-a)(1-q^{n-k})\big]\Big) = -\big[-a+\gamma(1-q^k)\big]\big[\tau+\gamma(1-q^n)\big],$$
with $\tau=b-a$

hence

$$\tilde{E}_{n,k}=\frac{[n+1-k]_q}{[n+1]_q}\times \frac{-\Big[-a+\gamma(1-q^k)\Big]\Big[b-a+\gamma(1-q^n)\Big]}{A};$$

with

$$\begin{aligned} A=&q^{n-k}\Big[-a+\gamma(1-q^k)\Big]+ \Big[b+(\gamma-a)(1-q^{n-k})\Big]\\ =&-aq^{n-k}+\gamma q^{n-k}-\gamma q^n+ b+\gamma-a-\gamma q^{n-k}+aq^{n-k}\\ =&-\gamma q^n+b+\gamma-a\\ A=&\big[b-a+\gamma(1-q^n)\big]; \end{aligned}$$

with the last equality of A above, we have :

$$\tilde{E}_{n,k}=\frac{[n+1-k]_q}{[n+1]_q}\times \frac{-\Big[-a+\gamma(1-q^k)\Big]\Big[b-a+\gamma(1-q^n)\Big]}{\Big[b-a+\gamma(1-q^n)\Big]}$$

thus

$$\tilde{E}_{n,k}=\frac{[n+1-k]_q}{[n+1]_q}\big[a-\gamma(1-q^k)\big] \quad (16)$$

From (14) and (16) we deduce that:

$$\begin{aligned} \tilde{D}_{n,k}=&\frac{[k+1]_q}{[n+1]_q}\Big(\tau+\gamma(1-q^n)+ \tilde{E}_{n,k}\frac{[n+1]_q}{[n+1-k]_q}q^{n-k}\Big)\\ =&\frac{[k+1]_q}{[n+1]_q}\Big(\tau+\gamma(1-q^n)+ q^{n-k}\big[a-\gamma(1-q^k)\big]\Big)\\ =&\frac{[k+1]_q}{[n+1]_q}\Big(\tau+\gamma-\gamma q^n+aq^{n-k}- \gamma q^{n-k}+\gamma q^n\Big)\\ =&\frac{[k+1]_q}{[n+1]_q}\Big(\tau+\gamma+aq^{n-k}-\gamma q^{n-k}\Big),\\ &\text{avec } \tau=b-a\\ =&\frac{[k+1]_q}{[n+1]_q}\Big(b-a+\gamma+aq^{n-k}-\gamma q^{n-k}\Big) \end{aligned}$$

hence

$$\tilde{D}_{n,k}=\frac{[k+1]_q}{[n+1]_q}\Big(b+(\gamma-a)(1-q^{n-k})\Big) \quad (17)$$

Therefore $\Big\{B_k^{n+1}(t;[a;b];q,\gamma)\Big\}_{k=0}^{n+1}$ is a basis of the space of $\mathbb{P}_{n+1}([a;b])$
Which completes the demonstration.

8 The De Casteljau algorithm for (q,γ)-Bezier curves

Let be a polynomial curve of the form

$$P(t)=\sum_{k=0}^{n}P_kB_k^n(t;[a;b];q,\gamma) \quad (18)$$

where the coefficients $\{P_k\}_{k=0}^n$ are polynomials called control points.
The (18) form of P is a (q,γ)-Bezier curve of degree n on the interval $[a;b]$.

$$P(t)=\sum_{k=0}^{n-r}P_k^rB_k^{n-r}(t;[a;b];q,\gamma) \quad (19)$$

$P_k^r=P_k^r(t)$, $k=0,\cdots,n-r$, are polynomials of degree r. By the recurrence relation (4) (with

$B_{-1}^{n-r-1} = B_{n-r}^{n-r-1} = 0$), (19) becomes

$$\begin{aligned}
&P(t)\\
&=\sum_{k=0}^{n-r} P_k^r \Bigg[\left(\frac{\tau_2 + \tilde{\gamma} q^{n-r-k-1}}{\tau + \gamma q^{n-r-1}}\right) B_k^{n-r-1}(t;[a;b];q,\gamma) +\\
&\quad q^{n-r-k}\left(\frac{\tau_1 + \gamma q^{k-1}}{\tau + \gamma q^{n-r-1}}\right) B_{k-1}^{n-r-1}(t;[a;b];q,\gamma)\Bigg]\\
&= \sum_{k=0}^{n-r-1} B_k^{n-r-1}(t;[a;b];q,\gamma) \Bigg[\left(\frac{\tau_2 + \tilde{\gamma} q^{n-r-k-1}}{\tau + \gamma q^{n-r-1}}\right) \times\\
&\qquad P_k^r + q^{n-r-k-1}\left(\frac{\tau_1 + \gamma q^{k}}{\tau + \gamma q^{n-r-1}}\right) P_{k+1}^r \Bigg]\\
&= \sum_{k=0}^{n-r-1} P_k^{r+1} B_k^{n-r-1}(t;[a;b];q,\gamma)
\end{aligned}$$

with (at rank $r+1$)

$$P_k^{r+1} = \left(\frac{\tau_2 + \tilde{\gamma} q^{n-r-k-1}}{\tau + \gamma q^{n-r-1}}\right) P_k^r + q^{n-r-k-1}\left(\frac{\tau_1 + \gamma q^{k}}{\tau + \gamma q^{n-r-1}}\right) P_{k+1}^r \quad (20)$$

At rank n of this algorithm we obtain a single point P_0^n which gives the value of the Bézier curve at t, i.e. $P_0^n = P(t)$.

Similarly, using the recurrence relation (5), we obtain

$$P_k^{r+1} = q^k \left(\frac{\tau_2 + \tilde{\gamma} q^{n-r-k-1}}{\tau + \gamma q^{n-r-1}}\right) P_k^r + \left(\frac{\tau_1 + \gamma q^{k}}{\tau + \gamma q^{n-r-1}}\right) P_{k+1}^r, \quad (21)$$

where $k = 0, 1, \cdots, n-r-1;\ r = 0, \cdots, n-1$, and, at the last level n we get $P_0^n = P(t)$.

9 The (q,γ)-Marsden identity

Property 1. *Let $t \in [a;b]$ and $n \geq 1$.*
The (q,γ)-Bernstein polynomials on the interval $[a;b]$ satisfy for any real number:

$$\prod_{i=0}^{n-1}\left(x - t + \tilde{\gamma}(1-q^i)\right) = \sum_{k=0}^{n}\prod_{j=k}^{n-1}\left(x - a + \gamma(1-q^j)\right)\times \prod_{j=0}^{k-1}\left(x - bq^j + (\gamma - a)(1-q^j)\right) B_k^n(t;[a;b];q,\gamma). \quad (22)$$

Proof: we proceed by recurrence on n

1. for $n = 1$.
   Using (1) we have:

$$\begin{aligned}
B_0^0(t;[a;b];q,\gamma) &= 1,\\
B_0^1(t;[a;b];q,\gamma) &= \frac{b-t}{b-a},\\
B_1^1(t;[a;b];q,\gamma) &= \frac{t-a}{b-a}.
\end{aligned} \quad (23)$$

   So the left-hand side of (22) is $(x-t)$ and the right-hand side is

$$(x-a)B_0^1 + (x-b)B_1^1 = (x-a)\frac{b-t}{b-a} + (x-b)\frac{t-a}{b-a} = (x-t)$$

   So (22) is true for $n = 1$.

2. Assume that (22) is true for some $n \geq 1$ and show that (22) is true at rank $n+1$.
   Let's set

$$P_{n,k}(x) = \prod_{j=k}^{n-1}\left(x - a + \gamma(1-q^i)\right)\times \prod_{j=0}^{k-1}\left(x - bq^i + (\gamma - a)(1-q^i)\right), \quad k = 0,\cdots,n \quad (24)$$

   then (22) is written as

$$\prod_{i=0}^{n-1}\left(x - t + \tilde{\gamma}(1-q^i)\right) = \sum_{k=0}^{n} P_{n,k}(x) B_k^n(t;[a;b];q,\gamma). \quad (25)$$

   Let's prove that (25) is true for $n+1$. For $n+1$, the left-hand side of (25) is

$$\prod_{i=0}^{n}\Big(x-t+\tilde{\gamma}(1-q^i)\Big)$$
$$=\Big(x-t+\tilde{\gamma}(1-q^n)\Big)\times \prod_{i=0}^{n-1}\Big(x-t+\tilde{\gamma}(1-q^i)\Big)$$
$$=\Big(x-t+\tilde{\gamma}(1-q^n)\Big)\times \sum_{k=0}^{n}P_{n,k}(x)B_k^n(t;[a;b];q,\gamma)$$
$$=\sum_{k=0}^{n}P_{n,k}(x)\Big(x-t+\tilde{\gamma}(1-q^n)\Big)\times B_k^n(t;[a;b];q,\gamma)$$
$$=\sum_{k=0}^{n}P_{n,k}(x)\Big(f_{n,k}B_k^{n+1}(t;[a;b];q,\gamma)+ g_{n,k}B_{k+1}^{n+1}(t;[a;b];q,\gamma)\Big)$$
$$=\sum_{k=0}^{n+1}\Big(P_{n,k}(x)f_{n,k}+P_{n,k-1}(x)g_{n,k-1}\Big)\times B_k^{n+1}(t;[a;b];q,\gamma) \quad (26)$$

with $P_{n,n-1}=0$ et $P_{n,-1}=0$.

Let's determine coefficients $f_{n,k}=f_{n,k}(x)$ and $g_{n,k}=g_{n,k}(x)$ such that

$$\Big(x-t+\tilde{\gamma}(1-q^n)\Big)B_k^n(t;[a;b];q,\gamma)= f_{n,k}B_k^{n+1}(t;[a;b];q,\gamma)+g_{n,k}B_{k+1}^{n+1}(t;[a;b];q,\gamma). \quad (27)$$

Using (27) , (12) et (11), we obtain the equation:

$$\bullet\ \Big(x-t+\tilde{\gamma}(1-q^n)\Big)= \tilde{f}_{n,k}\Big(b-t+(\gamma+t-a)(1-q^{n-k})\Big)+ \tilde{g}_{n,k}\Big(t-a+\gamma(1-q^k)\Big).$$

$$\bullet\ \Big(x-tq^n+(\gamma-a)(1-q^n)\Big)= \tilde{f}_{n,k}\Big(-tq^{n-k}+b+(\gamma-a)(1-q^{n-k})\Big)+ \tilde{g}_{n,k}\Big(t-a+\gamma(1-q^k)\Big) \quad (28)$$

with

$$\tilde{f}_{n,k}=\frac{[n+1]_q}{[n+1-k]_q}\frac{f_{n,k}}{\Big(\tau+\gamma(1-q^n)\Big)}$$
$$\text{and} \quad (29)$$
$$\tilde{g}_{n,k}=\frac{[n+1]_q}{[k+1]_q}\frac{g_{n,k}}{\Big(\tau+\gamma(1-q^n)\Big)}$$

Using (28) and equalizing the constant terms and the terms in variable t we obtain the following (30), (31) system:

$$-q^n=-q^{n-k}\tilde{f}_{n,k}+\tilde{g}_{n,k}, \quad (30)$$

$$x+(\gamma-a)(1-q^n)= \Big(b+(\gamma-a)(1-q^{n-k})\Big)\tilde{f}_{n,k}+ \Big(-a+\gamma(1-q^k)\Big)\tilde{g}_{n,k}, \quad (31)$$

Multiplying equation (30) by $-\Big(-a+\gamma(1-q^k)\Big)$ and adding it to equation (31) gives:

$$\tilde{f}_{n,k}=\frac{x+(\gamma-a)(1-q^n)+q^n\big(-a+\gamma(1-q^k)\big)}{\big[b+(\gamma-a)(1-q^{n-k})\big]+q^{n-k}\big[-a+\gamma(1-q^k)\big]}$$
$$=\frac{x-a+\gamma q^n(1-q^k)+\gamma(1-q^n)}{\Big(\tau+\gamma(1-q^n)\Big)}$$
$$=\frac{x-a+\gamma(1-q^{n+k})}{\Big(\tau+\gamma(1-q^n)\Big)} \quad (32)$$

using (30), we obtain:

$$\begin{aligned}
\tilde{g}_{n,k} &= q^{n-k}\tilde{f}_{n,k} - q^n \\
&= q^{n-k}\frac{\Big(x-a+\gamma(1-q^{n+k})\Big)}{\Big(\tau+\gamma(1-q^n)\Big)} - q^n \\
&= \frac{q^{n-k}\Big(x-a+\gamma(1-q^{n+k})\Big)}{\Big(\tau+\gamma(1-q^n)\Big)} - \frac{q^n\Big(\tau+\gamma(1-q^n)\Big)}{\Big(\tau+\gamma(1-q^n)\Big)} \\
&= \frac{q^{n-k}\Big(x-a+\gamma(1-q^{n+k})\Big)}{\Big(\tau+\gamma(1-q^n)\Big)} - \frac{q^n\Big(b-a+\gamma(1-q^n)\Big)}{\Big(\tau+\gamma(1-q^n)\Big)} \\
&= \frac{q^{n-k}x-q^{n-k}a+q^{n-k}\gamma(1-q^{n+k})-bq^n}{\Big(\tau+\gamma(1-q^n)\Big)} - \frac{q^n\Big(-a+\gamma(1-q^n)\Big)}{\Big(\tau+\gamma(1-q^n)\Big)} \\
&= \frac{q^{n-k}(x-bq^k)-aq^{n-k}+aq^n}{\Big(\tau+\gamma(1-q^n)\Big)} + \frac{q^{n-k}\gamma(1-q^{n+k})-\gamma q^n(1-q^n)}{\Big(\tau+\gamma(1-q^n)\Big)} \\
&= \frac{q^{n-k}(x-bq^k)-aq^{n-k}(1-q^k)}{\Big(\tau+\gamma(1-q^n)\Big)} + \frac{+q^{n-k}\gamma(1-q^{n+k})-\gamma q^n(1-q^n)}{\Big(\tau+\gamma(1-q^n)\Big)} \\
&= \frac{q^{n-k}(x-bq^k)-aq^{n-k}(1-q^k)}{\Big(\tau+\gamma(1-q^n)\Big)} + \frac{\gamma q^{n-k}\Big(1-q^{n+k}-q^k(1-q^n)\Big)}{\Big(\tau+\gamma(1-q^n)\Big)} \\
&= \frac{q^{n-k}(x-bq^k)-aq^{n-k}(1-q^k)+\gamma q^{n-k}\big(1-q^k\big)}{\Big(\tau+\gamma(1-q^n)\Big)} \\
&= \frac{q^{n-k}\Big(x-bq^k+(\gamma-a)(1-q^k)\Big)}{\Big(\tau+\gamma(1-q^n)\Big)}
\end{aligned} \quad (33)$$

From (32) and (33) we deduce:

$$f_{n,k} = \frac{[n+1-k]_q}{[n+1]_q}\Big(x-a+\gamma(1-q^{n+k})\Big) \quad (34)$$

$$g_{n,k} = \frac{[k+1]_q}{[n+1]_q}\, q^{n-k}\Big(x-bq^k+(\gamma-a)(1-q^k)\Big) \quad (35)$$

Using (24), the coefficient of B_k^{n+1} in the last line of (26) is given by:

$$\begin{aligned}
&P_{n,k}(x)f_{n,k} + P_{n,k-1}(x)g_{n,k-1} \\
&= \prod_{j=k}^{n-1}\Big(x-a+\gamma(1-q^i)\Big)\times \\
&\prod_{j=0}^{k-2}\Big(x-bq^i+(\gamma-a)(1-q^i)\Big)\times \\
&\Bigg[\Big(x-bq^{k-1}+(\gamma-a)(1-q^{k-1})\Big)f_{n,k}+ \\
&\Big(x-a+\gamma(1-q^{k-1})\Big)g_{n,k-1}\Bigg]
\end{aligned} \quad (36)$$

In this section, we use the following equalities: $[n+1-k]_q + [k]_q q^{n+1-k} = [n+1]_q$ and $q^n[n+1-k]_q + [k]_q = [n+1]_q$

Using (34) and (35) we obtain:

$$[n+1]_q\left[\left(x-bq^{k-1}+(\gamma-a)(1-q^{k-1})\right)f_{n,k}+\left(x-a+\gamma(1-q^{k-1})\right)g_{n,k-1}\right]$$
$$=\left(x-bq^{k-1}+(\gamma-a)(1-q^{k-1})\right)\times[n+1-k]_q\left(x-a+\gamma(1-q^{n+k})\right)+\left(x-a+\gamma(1-q^{k-1})\right)[k]_q q^{n+1-k}\times\left(x-bq^{k-1}+(\gamma-a)(1-q^{k-1})\right)$$
$$=\left(x-bq^{k-1}+(\gamma-a)(1-q^{k-1})\right)\times\left[[n+1-k]_q\left(x-a+\gamma(1-q^{n+k})\right)+[k]_q q^{n+1-k}\left(x-a+\gamma(1-q^{k-1})\right)\right]$$
(37)

We have:

$$\left[[n+1-k]_q\left(x-a+\gamma(1-q^{n+k})\right)+[k]_q q^{n+1-k}\left(x-a+\gamma(1-q^{k-1})\right)\right]$$
$$=\left([n+1-k]_q+[k]_q q^{n+1-k}\right)(x-a)+[n+1-k]_q\gamma\left(1-q^{n+k}\right)+[k]_q q^{n+1-k}\gamma\left(1-q^{k-1}\right)$$
$$=[n+1]_q\left(x-a\right)+\left([n+1-k]_q\gamma(1-q^{n+k})\right)+[k]_q q^{n+1-k}\left(\gamma(1-q^{k-1})\right)$$
$$=[n+1]_q(x-a)+\gamma\left([n+1-k]_q-q^{n+k}[n+1-k]_q+[k]_q q^{n+1-k}-[k]_q q^n\right)$$
$$=[n+1]_q(x-a)+\gamma\left([n+1-k]_q+[k]_q q^{n+1-k}-q^{n+k}[n+1-k]_q-[k]_q q^n\right)$$
$$=[n+1]_q(x-a)+\gamma\left([n+1-k]_q+[k]_q q^{n+1-k}-q^n\left[q^k[n+1-k]_q)+[k]_q\right]\right)$$
$$=[n+1]_q(x-a)+\gamma\left([n+1]_q-q^n[n+1]_q\right)$$
$$=[n+1]_q\left((x-a)+\gamma(1-q^n)\right)$$
(38)

From (37) and (38) we deduce that:

$$\left(x-bq^{k-1}+(\gamma-a)(1-q^{k-1})\right)f_{n,k}+\left(x-a+\gamma(1-q^{k-1})\right)g_{n,k-1}=\left(x-bq^{k-1}+(\gamma-a)(1-q^{k-1})\right)\times\left((x-a)+\gamma(1-q^n)\right)\quad(39)$$

From (24); (36) and (39), we have :

$$P_{n,k}(x)f_{n,k}+P_{n,k-1}(x)g_{n,k-1}=\prod_{j=k}^{n}\left(x-a+\gamma(1-q^i)\right)\times\prod_{j=0}^{k-1}\left(x-bq^i+(\gamma-a)(1-q^i)\right)=P_{n+1,k}(x)$$
(40)

Thus from the formula (26) and (40) we have shown that (25) is true at rank $n+1$. This completes the proof of the (q,γ)-Marsden identity.

□

10 Conclusion

In this paper, we have defined (q,γ)-Bernstein basis functions depending on the real parameter γ, and proved several important properties of these functions using mathematical induction. We also introduced the Marsden identity and implemented De Casteljau's algorithm. Finally, we provided some graphics of (q,γ)-Bernstein polynomials and (q,γ)-Bezier curve. Note that these proofs are based

on the work [11].
Our results show that (q, γ)-Bernstein basis functions are powerful tools for the representation and manipulation of polynomial curves, offering increased flexibility thanks to the second γ parameter. Marsden identity provides an essential link between Bernstein basis functions and Bezier curves, reinforcing their usefulness in a variety of areas, from computer-aided design to geometric modelling. In this case, we can use such a basis to approximate functions in the sense of least-squares problems, such as isogeometrics methods. This work is in progress. Looking to the future, there is still much to explore.
Further research will focus on extending this concept to multidimensional contexts, as well as developing more efficient algorithmic techniques for optimal data manipulation and representation. In addition, integrating this basis into practical applications would open up new avenues for technological innovation.

Declaration of Generative AI and AI assisted Technologies in the Writing Process
During the preparation of this work, the authors used chatgpt in order to improve the readability and language of manuscript. Having use this tool, the authors have reviewed and corrected the content where necessary and take full responsibility for the content of the publication.

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APPENDIX

Some graphical examples on the segment [2;5], highlighting the effect of parameters γ and q.

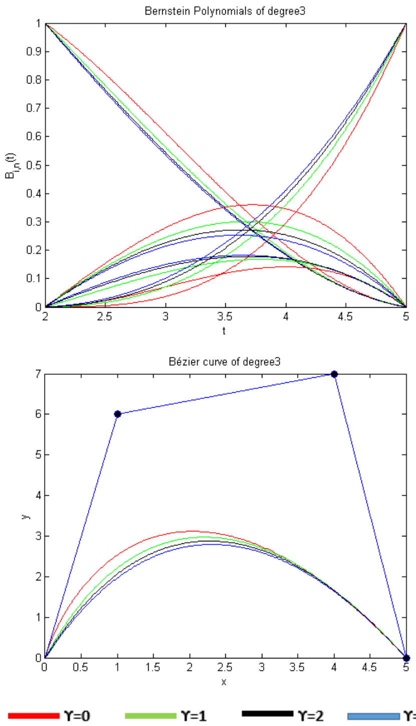

Figure 1: Effect of γ-parameter for $q = -0.95$

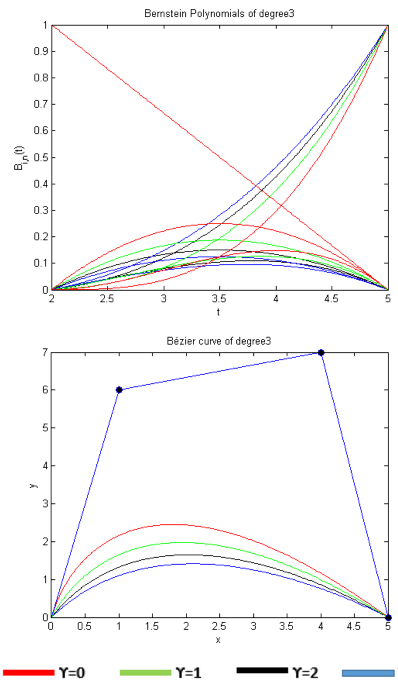

Figure 2: Effect of γ-parameter for $q = 0$

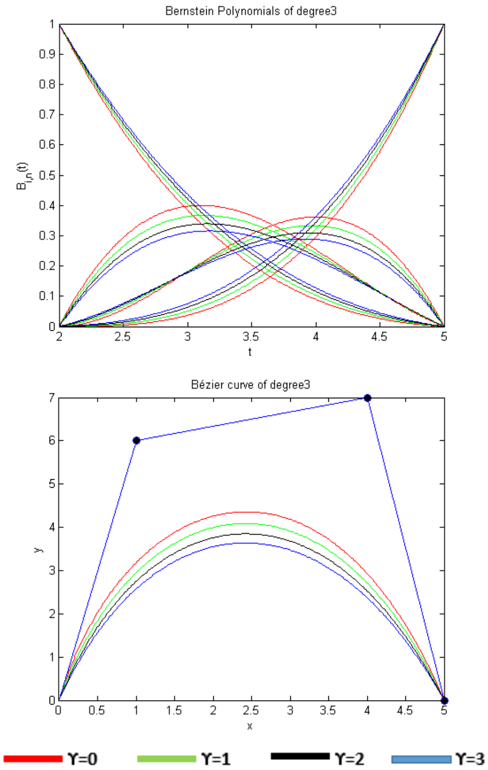

Figure 3: Effect of γ-parameter for $q = 0.8$

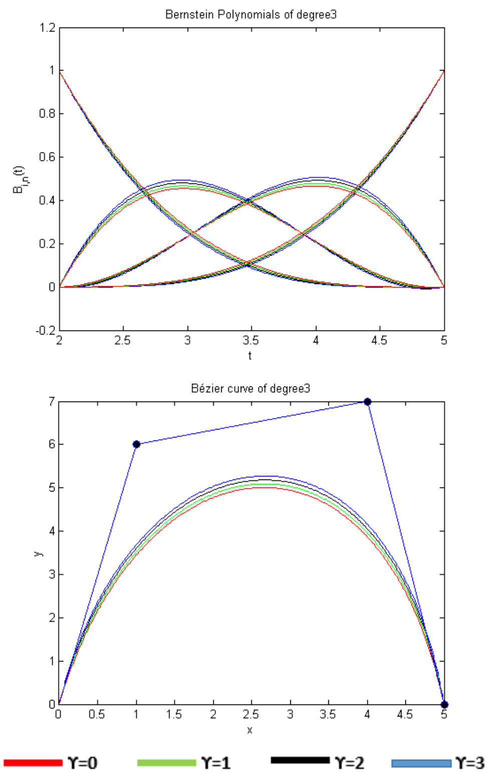

Figure 4: Effect of γ-parameter for $q = 1.05$

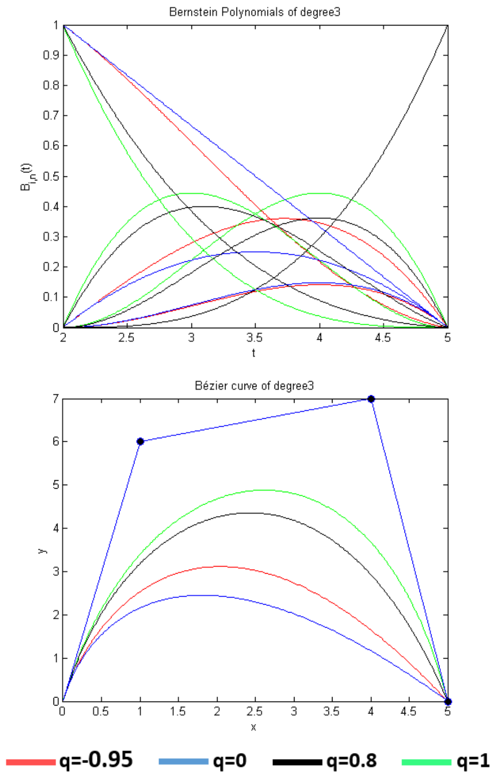

Figure 5: Effect of q-parameter for $\gamma = 0$

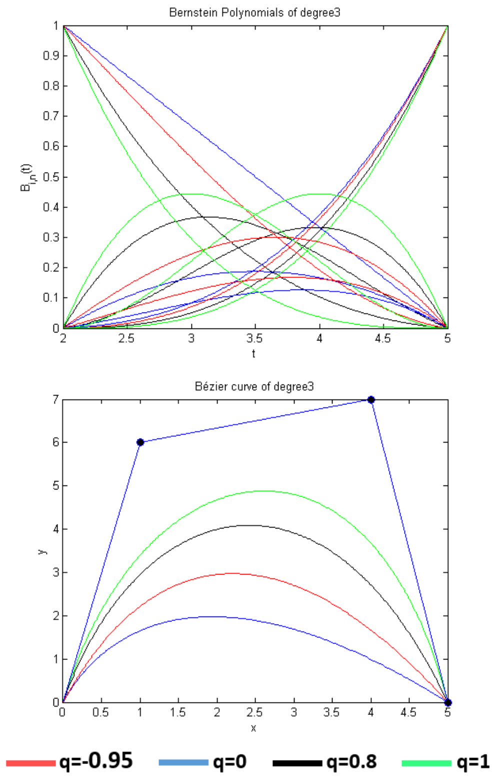

Figure 6: Effect of q-parameter for $\gamma = 1$

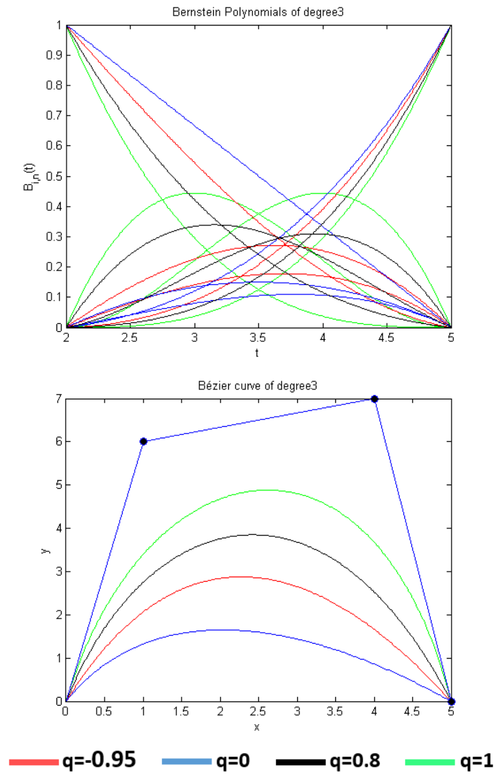

Figure 7: Effect of q-parameter for $\gamma = 2$

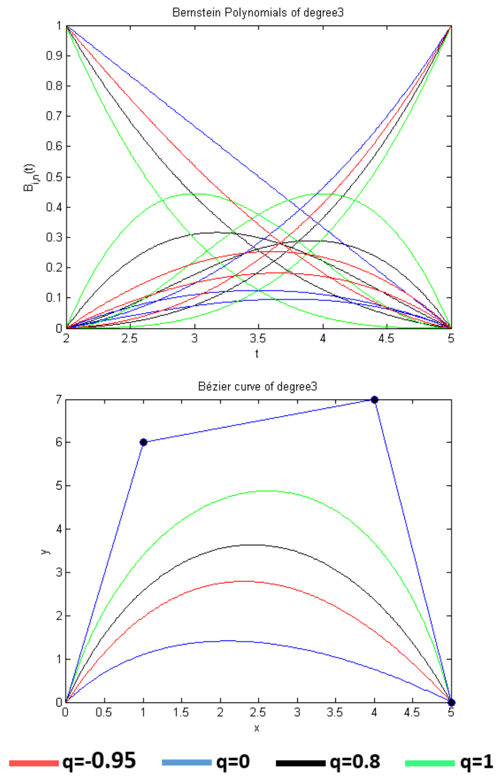

Figure 8: Effect of q-parameter for $\gamma = 3$

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Once we had the results of our article, I was asked (SORO SIONFON SIMON) to write it. I was helped in this task by Professor KOUA BROU JEAN CLAUDE. Doctor HAUDIÉ JEAN STEPHANE INKPÉ wrote the codes to produce the graphs.

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Conflict of Interest

The authors state that they have no financial interests or personal relationships that could have influenced the work presented in this paper.