Vitali Theorems in Non-Newtonian Sense and Non-Newtonian Measurable Functions

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Abstract: In this paper, we first state the $\nu-$Vitali theorems in the non-Newtonian sense. In the second part, we give the definition of the non-Newtonian measurable function and the relation between $\nu-$measurable and real measurable functions. We also study some basic properties of $\nu-$-measurable functions.

Key-Words: Non-Newtonian measurable set, non-Newtonian Vitali set, non-Newtonian measurable function

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1 Introduction

Non-Newtonian calculus, which has found applications in various fields such as engineering, mathematics, finance, economics, medicine, and biomedical sciences, was developed between 1967 and 1970 as an alternative to the classical calculus of Newton and Leibniz, [1], [2]. The foundational book titled Non-Newtonian Calculus, which laid the groundwork for this alternative calculus, was published in 1972 by [3]. The concepts of derivative and integral in the context of metacalculus were explored by [4], while geometric calculus and its applications were examined in [5]. The non-Newtonian Lebesgue measure for non-Newtonian open sets was defined and studied in[6]. Finally, the non-Newtonian measure for closed non-Newtonian sets, along with some related theorems, was defined and studied in [7]. For more details see, [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22].

Let ν be a generator, which means ν is a one-to-one function whose domain is real numbers and whose range is a subset $\mathbb{A}$ of $\mathbb{R}$. Let $\dot{p}, \dot{q} \in A$. Then, $\nu-$arithmetics are defined as follows;

$\nu-$ *addition*	$\dot{p}\dot{+}\dot{q} = \nu\{\nu^{-1}(\dot{p}) + \nu^{-1}(\dot{q})\}$
$\nu-$ *subtraction*	$\dot{p}\dot{-}\dot{q} = \nu\{\nu^{-1}(\dot{p}) - \nu^{-1}(\dot{q})\}$
$\nu-$ *multiplicative*	$\dot{p}\dot{\times}\dot{q} = \nu\{\nu^{-1}(\dot{p}) \times \nu^{-1}(\dot{q})\}$
$\nu-$ *division*	
$(\nu^{-1}(\dot{q}) \neq 0)$	$\dot{p}\dot{/}\dot{q} = \nu\{\nu^{-1}(\dot{p})/\nu^{-1}(\dot{q})\}$
$\nu-$ *order*	$\dot{p}\dot{\leq}\dot{q} \Leftrightarrow \nu^{-1}(\dot{p}) \leq \nu^{-1}(\dot{q})$

Numbers with $x\dot{>}\dot{0}$ are called $\nu-$positive numbers, and numbers with $x\dot{<}\dot{0}$ are called $\nu-$negative numbers. The set of $\nu-$integers is

$$\mathbb{Z}_\nu = \mathbb{Z}(N) = \ldots, \nu(-2), \nu(-1), \nu(0), \nu(1), \nu(2), \ldots.$$

The set $\mathbb{R}_\nu = \mathbb{R}(N) = \{\nu(a) : a \in \mathbb{R}\}$ is called the set of non-Newtonian real numbers.

The absolute non-Newtonian value of $\dot{a} \in A$ in the subset $A \subset \mathbb{R}_\nu$ is denoted by $|\dot{a}|_N$ and define as follows;

$$|\dot{a}|_\nu = \begin{cases} \dot{a} & , \dot{a}\dot{>}\nu(0) \\ \nu(0) & , \dot{a} = \nu(0) \\ \nu(0)\dot{-}\dot{a} & , \dot{a}\dot{<}\nu(0) \end{cases}$$

Accordingly,

$$\sqrt{\dot{a}^{2_N}}^N = |\dot{a}|_N = \nu\left\{|\nu^{-1}(\dot{a})|\right\}$$

is written for each $\dot{u}$ in the set $A \subset \mathbb{R}_\nu$ [3], [8].

Definition 1. *The non-Newtonian outer measure of a nonempty $\nu-$bounded set K is the largest lower bound of the measures of all $\nu-$bounded, $\nu-$open sets containing the set K. So it is defined by*

$$m_N^* K =^\nu \inf_{K \subset G} \{m_N G\}$$

[7].

Definition 2. *The non-Newtonian interior measure of a nonempty $\nu-$bounded set K is the smallest upper bound of the measures of all $\nu-$closed sets contained in the set K. So it is defined by*

$$m_{*N} K =^\nu \sup_{F \subset K} \{m_N F\}$$

[7].

Theorem 1. *Let be given a $\nu-$bounded set K. If Δ is a $\nu-$open set containing the set K, then we have the following equation;*

$$m_N^* K \dot{+} m_{*N}\left[C_\Delta^K\right] = m_N \Delta$$

[7].

Definition 3. *If the non-Newtonian interior and exterior measure of a $\nu-$bounded set K are equal, the set K is called a non-Newtonian Lebesgue measurable set, or simply the $\nu-$measurable set, [7].*

Theorem 2. *If the set K is the $\nu-$measurable set in $\mathbb{R}_\nu$, then $\nu^{-1}(K)$ is the measurable set in $\mathbb{R}$, [7].*

Theorem 3. *Let be given a $\nu-$bounded set E. If the set E can be written as a combination of finite or countably infinite sets of pairwise disjoint $\nu-$measurable E_k sets, then E is $\nu-$measurable and*

$$m_N E =_\nu \sum_k m_N E_k$$

equality is satisfied, [23].

2 Main Results

2.1 Vitali Theorems

Definition 4. *Let K be a set of $\nu-$points and B a family of $\nu-$closed intervals, none of which are single points.If for every $x \in K$ point and for every $\dot{\epsilon} \dot{>} \dot{0}$ there is a $\nu-$closed $b \in B$ interval such that*

$$x \in b, \quad m_N b \dot{<} \dot{\epsilon},$$

then, the set K is said to be contained by the family B in the $\nu-$Vitali sense.
In other words, if every point of the set K lies in arbitrarily small $\nu-$closed intervals belonging to the family B , then the set K is covered by the family B in the $\nu-$Vitali sense.

If the set K is contained by the B in the $\nu-$Vitali sense, then the set $\nu^{-1}(K)$ is also contained by a family in the Vitali sense. Let B is the family of $\nu-$closed sets b which do not consist of a single point and let B_1 be the family of $\nu^{-1}(b)$ closed sets that do not consist of a single point. Then, for $\forall x \in K$ and for each $\dot{\epsilon} \dot{>} \dot{0}$, there is an $\nu-$closed interval $b \in B$ such that

$$x \in b, \quad m_N b \dot{<} \dot{\epsilon}.$$

Then, we have

$$\nu^{-1}(x) \in \nu^{-1}(b)$$

and $\nu^{-1}(b) \in B_1$ since $b \in B$ so we get

$$\nu^{-1}(m_N(b)) < \nu^{-1}(\dot{\epsilon})$$
$$\nu^{-1}\left(\nu\{m(\nu^{-1}(b))\}\right) < \epsilon$$
$$m(\nu^{-1}(b)) < \epsilon \quad (\epsilon > 0)$$

$$\Rightarrow \nu^{-1}(K) \quad \text{Vitali}$$

Theorem 4. *If a $\nu-$bounded set K is covered by a family of closed intervals B in the $\nu-$Vitali sense, then it is possible to find a finite or countable family of $\nu-$closed intervals b_k in the set B such that*

$$b_k \cap b_i = \emptyset (k \neq i) \quad m_N^* \left[K \setminus \bigcup_k b_k\right] = \dot{0}.$$

Proof. Since the set K is $\nu-$bounded, the set $\nu^{-1}(K)$ is bounded and is covered by the family B_1 which consist of closed intervals. By the Vitali's theorem it is possible to find a finite or countable closed interval family $\nu^{-1}(b_k)$ in the set B_1, such that

$$\Rightarrow \nu^{-1}(b_k) \cap \nu^{-1}(b_i) = \emptyset (k \neq i)$$
$$m^* \left[\nu^{-1}(K) \setminus \bigcup_k \nu^{-1}(b_k)\right] = 0$$
$$\Rightarrow \nu\{\nu^{-1}(b_k \cap b_i)\} = \nu\{\emptyset\} (k \neq i)$$
$$m^* \left[\nu^{-1}(K) \setminus \nu^{-1}\left(\bigcup_k b_k\right)\right] = 0$$
$$\Rightarrow b_k \cap b_i = \emptyset (k \neq i)$$
$$\nu\left\{m^*\left[\nu^{-1}\left(K \setminus \bigcup_k b_k\right)\right]\right\} = \nu(0)$$
$$\Rightarrow b_k \cap b_i = \emptyset (k \neq i)$$
$$m_N^* \left[K \setminus \bigcup_k b_k\right] = \dot{0}$$

which gives the proof. □

Theorem 5. *Under the hypotheses of Theorem 4, for every $\dot{\epsilon} \dot{>} \dot{0}$ there is a finite system $b_1, b_2, \ldots, b_n$ consisting of pairwise disjoint $\nu-$closed intervals of the system B such that*

$$m_N^* \left[K \setminus \bigcup_k^n b_k\right] \dot{<} \dot{\epsilon}.$$

Proof. If the set K is covered by a family of closed intervals B in the sense of $\nu-$Vitali, then the set $\nu^{-1}(K)$ is also covered by a family of closed intervals B_1 in the sense of Vitali. The B_1 system has a finite

$$\nu^{-1}(b_1), \nu^{-1}(b_2), \ldots, \nu^{-1}(b_n)$$

system of pairwise disjoint closed intervals. Thus we get

$$\Rightarrow m^*\left[\nu^{-1}(K)\setminus\bigcup_k^n \nu^{-1}(b_k)\right] < \epsilon$$

$$\Rightarrow m^*\left[\nu^{-1}(K)\setminus\nu^{-1}\left(\bigcup_k^n b_k\right)\right] < \epsilon$$

$$\Rightarrow \nu\left\{m^*\left[\nu^{-1}\left(K\setminus\bigcup_k^n b_k\right)\right]\right\} < \nu(\epsilon)$$

$$\Rightarrow m_N^*\left[K\setminus\bigcup_k^n b_k\right] < \dot\epsilon$$

$$\Rightarrow \nu^{-1}\left(m_N^*\left[K\setminus\bigcup_k^n b_k\right]\right) < \nu^{-1}(\dot\epsilon)$$

$$\Rightarrow m_N^*\left[K\setminus\bigcup_k^n b_k\right] \dot< \dot\epsilon$$

. □

2.2 Measurable Functions

Definition 5. *Let*

$$f_\nu : X\subset\mathbb{R}_\nu \to \mathbb{R}_\nu$$
$$\dot a \to f_\nu(\dot a).$$

If for $\forall\dot\beta\in\mathbb{R}_\nu$*, the set*

$$A = \{\dot a\in X : f_\nu(\dot a)\dot>\dot\beta\}$$

is $\nu-$*measurable, that is,*

$$m_N^* A = m_{*N} A$$

then, the function f_ν *is called a non-Newtonian measurable function, or simply a* $\nu-$*measurable function.*

Theorem 6. *Let* $f_\nu : X\subset\mathbb{R}_\nu\to\mathbb{R}_\nu$ *be a function. The following expressions are equivalent; for* $\forall\dot\beta\in\mathbb{R}_\nu$

a) the set $A_{\dot\beta} = \{\dot a\in X : f_\nu(\dot a)\dot>\dot\beta\}$ *is the* $\nu-$*measurable set,*

b) the set $B_{\dot\beta} = \{\dot a\in X : f_\nu(\dot a)\dot\le\dot\beta\}$ *is the* $\nu-$*measurable set,*

c) the set $C_{\dot\beta} = \{\dot a\in X : f_\nu(\dot a)\dot\ge\dot\beta\}$ *is the* $\nu-$*measurable set,*

b) the set $D_{\dot\beta} = \{\dot a\in X : f_\nu(\dot a)\dot<\dot\beta\}$ *is the* $\nu-$*measurable set.*

Proof. It is obvious that $A_{\dot\beta} = X\setminus B_{\dot\beta},\ B_{\dot\beta} = X\setminus A_{\dot\beta}$.
$(a) \Rightarrow (b)$:Since $A_{\dot\beta}$ is $\nu-$measurable, its complement, $B_{\dot\beta}$, is also $\nu-$measurable.
$(b) \Rightarrow (a)$: Since $B_{\dot\beta}$ is $\nu-$measurable, its complement, $A_{\dot\beta}$, is also $\nu-$measurable.
Thus, we get $(a)\Leftrightarrow(b)$.
Since $C_{\dot\beta} = X\setminus D_{\dot\beta}$ is $D_{\dot\beta} = X\setminus C_{\dot\beta}$ $(c)\Leftrightarrow(d)$.
$(a)\Rightarrow(c)$: For $\forall\dot\beta\in\mathbb{R}_\nu$, let $A_{\dot\beta} = \{\dot a\in X : f_\nu(\dot a)\dot>\dot\beta\}$ be the $\nu-$measurable set.
For every $\dot m$ $\nu-$positive integer, we have $\dot\beta\dot-\frac{\dot 1}{m}\in\mathbb{R}_\nu$ since $\dot\beta\in\mathbb{R}_\nu$ and $\frac{\dot 1}{m}\in\mathbb{R}_\nu$ and so $A_{\dot\beta\dot-\frac{\dot 1}{m}}$ is a $\nu-$measurable set.
Thus

$$A_{\dot\beta\dot-\frac{\dot 1}{m}} = \{\dot a\in X : f_\nu(\dot a)\dot>\dot\beta\dot-\frac{\dot 1}{m}\}$$

and we get

$$\bigcap_{m=1}^{\infty} A_{\dot\beta\dot-\frac{\dot 1}{m}} = \bigcap_{m=1}^{\infty}\left\{\dot a\in X : f_\nu(\dot a)\dot>\dot\beta\dot-\frac{\dot 1}{m}\right\}$$
$$= \{\dot a\in X : f_\nu(\dot a)\dot\ge\dot\beta\} = C_{\dot\beta}$$

is the $\nu-$measurable set.
$(c)\Rightarrow(a)$: For $\forall\dot\beta\in\mathbb{R}_\nu$, $C_{\dot\beta} = \{\dot a\in X : f_\nu(\dot a)\dot\ge\dot\beta\}$ be a $\nu-$measurable set.
For every $\dot m$ $\nu-$positive integer, we have $\dot\beta\dot+\frac{\dot 1}{m}\in\mathbb{R}_\nu$ since $\dot\beta\in\mathbb{R}_\nu$ and $\frac{\dot 1}{m}\in\mathbb{R}_\nu$ and so $C_{\dot\beta\dot+\frac{\dot 1}{m}}$ is the $\nu-$measurable set.
Again

$$C_{\dot\beta\dot+\frac{\dot 1}{m}} = \{\dot a\in X : f_\nu(\dot a)\dot\ge\dot\beta\dot+\frac{\dot 1}{m}\}$$

and we get

$$\bigcup_{n=1}^{\infty} C_{\dot\beta\dot+\frac{\dot 1}{m}} = \bigcup_{m=1}^{\infty}\left\{\dot a\in X : f_\nu(\dot a)\dot\ge\dot\beta\dot+\frac{\dot 1}{m}\right\}$$
$$= \{\dot a\in X : f_\nu(\dot a)\dot>\dot\beta\} = A_{\dot\beta}$$

is the $\nu-$measurable set which gives $(a)\Leftrightarrow(c)$. □

Theorem 7. *If*

$$f_\nu : X\subset\mathbb{R}_\nu\to\mathbb{R}_\nu$$
$$\dot a\to f_\nu(\dot a)$$

is a non-Newtonian measurable function, then

$$\nu^{-1}\circ f_\nu\circ\nu : \nu^{-1}(X)\subset\mathbb{R}\to\mathbb{R}$$
$$a\to(\nu^{-1}\circ f_\nu\circ\nu)(a)$$

is a measurable function.

Proof. Since f_ν is a non-Newtonian measurable function, we have
$\forall\dot{\beta} \in \mathbb{R}_\nu$, $\{\dot{a} \in X : f_\nu(\dot{a})\dot{>}\dot{\beta}\}$ is the $\nu-$measurable set. For $\forall\beta \in \mathbb{R}$, the set

$$\begin{aligned}&\nu^{-1}(\{\dot{a} \in X : f_\nu(\dot{a})\dot{>}\dot{\beta}\})\\ &\Leftrightarrow \{\nu^{-1}(\dot{a}) \in \nu^{-1}(X) : f_\nu(\dot{a})\dot{>}\dot{\beta}\}\\ &\Leftrightarrow \{a \in \nu^{-1}(X) : \nu^{-1}(f_\nu(\nu(a))) > \nu^{-1}(\dot{\beta})\}\\ &\Leftrightarrow \{a \in \nu^{-1}(X) : (\nu^{-1} \circ f_\nu \circ \nu)(a) > \beta\}\end{aligned}$$

is measurable. This completes the proof. □

Example 1. *The non-Newtonian constant function*

$$\begin{aligned}f_\nu : X \subset \mathbb{R}_\nu &\to \mathbb{R}_\nu\\ \dot{a} &\to f_\nu(\dot{a}) = \dot{c}, \quad \dot{c} \in \mathbb{R}_\nu\end{aligned}$$

is a $\nu-$measurable.

Proof. For $\forall\dot{\beta} \in \mathbb{R}_\nu$, it can be shown that the set

$$\{\dot{a} \in X : f_\nu(\dot{a}) = \dot{c}\dot{>}\dot{\beta}\}$$

is $\nu-$measurable.

(i) Let $\dot{\beta}\dot{\geq}\dot{c}$. Then, the set

$$f_\nu(\dot{a})\dot{>}\dot{\beta}\dot{\geq}\dot{c}$$

$$\{\dot{a} \in X : f_\nu(\dot{a})\dot{>}\dot{\beta}\} = \emptyset$$

is $\nu-$measurable.

(ii) Let $\dot{\beta}\dot{<}\dot{c}$. Thus, the set

$$\{\dot{a} \in X : f_\nu(\dot{a})\dot{>}\dot{\beta}\} = X$$

is $\nu-$measurable.
Then the non-Newtonian constant function is $\nu-$measurable. □

Example 2. *For $\forall\dot{a} \in \mathbb{R}_\nu$, the set $\{\dot{a} \in X : f_\nu(\dot{a}) = \dot{\beta}\}$ is a-measurable if*

$$\begin{aligned}f_\nu : X \subset \mathbb{R}_\nu &\to \mathbb{R}_\nu\\ \dot{a} &\to f_\nu(\dot{a})\end{aligned}$$

is a non-Newtonian measurable function.

Proof. It is easy to see the following equality:

$$\begin{aligned}&\{\dot{a} \in X : f_\nu(\dot{a}) = \dot{\beta}\}\\ &= \{\dot{a} \in X : f_\nu(\dot{a})\dot{\geq}\dot{\beta}\} \cap \{\dot{a} \in X : f_\nu(\dot{a})\dot{\leq}\dot{\beta}\}.\end{aligned}$$

Since finite number of intersections of $\nu-$measurable sets are $\nu-$measurable the proof is completed. □

Definition 6. *Given a set E, the $\nu-$characteristic function of E is denoted by ${}^{\nu}\chi$ and defined by*

$${}^{\nu}\chi_E = \begin{cases} \dot{1}, & x \in E\\ \dot{0}, & x \notin E\end{cases}$$

Example 3. *If E is $\nu-$measurable set, the the function ${}^{\nu}\chi_E$ is a $\nu-$measurable function.*

Proof. For $\forall\beta \in \mathbb{R}_\nu$ we show that the set $\left\{x \in X : {}^{\nu}\chi_E(X)\dot{>}\dot{\beta}\right\}$ is a $\nu-$measurable.

i) If $\dot{\beta}\dot{<}\dot{0}$, then the set

$$\left\{x \in X : {}^{\nu}\chi_E(X)\dot{>}\dot{\beta}\right\} = X$$

is $\nu-$measurable.

ii) Let $\dot{0}\dot{\leq}\dot{\beta}\dot{<}\dot{1}$. Then the set

$$\left\{x \in X : {}^{\nu}\chi_E(X)\dot{>}\dot{\beta}\right\} = E$$

is $\nu-$measurable.

iii) Let $\dot{\beta}\dot{\geq}\dot{1}$.

$$\left\{x \in X : {}^{\nu}\chi_E(X)\dot{>}\dot{\beta}\right\} = \emptyset$$

set is $\nu-$measurable.
Hence, the function ${}^{\nu}\chi_E$ is $\nu-$measurable function when the set E is $\nu-$measurable. □

Theorem 8. *If the function f_ν is $\nu-$measurable non-Newtonian real-valued function and $\dot{c} \in \mathbb{R}_\nu$, then the function $\dot{c}\dot{\times}f_\nu$ is $\nu-$measurable.*

Proof. To show that the function $(\dot{c}\dot{\times}f_\nu)(\dot{a}) = \dot{c}\dot{\times}f_\nu(\dot{a})$ is $\nu-$measurable, the set

$$\{\dot{a} \in X : (\dot{c}\dot{\times}f_\nu)(\dot{a})\dot{>}\dot{\beta}\}$$

must be shown to be $\nu-$measurable.

i) If $\dot{c} = \dot{0}$, the set

$$\{\dot{a} \in X : (\dot{c}\dot{\times}f_\nu)(\dot{a})\dot{>}\dot{\beta}\} = X \qquad \text{if}\dot{\beta}\dot{<}\dot{0}$$

is $\nu-$measurable and the set

$$\{\dot{a} \in X : (\dot{c}\dot{\times}f_\nu)(\dot{a})\dot{>}\dot{\beta}\} = \emptyset \quad \text{if}\dot{\beta}\dot{\geq}\dot{0}$$

is $\nu-$measurable.
which shows that the $\dot{c}\dot{\times}f_\nu$ function is $\nu-$measurable.

ii) Let $\dot{c}\dot{>}\dot{0}$. We write

$$\{\dot{a} \in X : \dot{c}\dot{\times}f_\nu(\dot{a})\dot{>}\dot{\beta}\} = \{\dot{a} \in X : f_\nu(\dot{a})\dot{>}\dot{\beta}\dot{/}\dot{c}\}.$$

Since $\dot{\beta}\dot{/}\dot{c} \in \mathbb{R}_\nu$ and the f_ν function is $\nu-$measurable, the set $\{\dot{a} \in X : \dot{c}\dot{\times}f_\nu(\dot{a})\dot{>}\dot{\beta}\}$ is $\nu-$measurable.

iii) Let $\dot{c}\dot{<}\dot{0}$. The set

$$\{\dot{a} \in X : \dot{c}\dot{\times}f_\nu(\dot{a})\dot{<}\dot{\beta}\} = \{\dot{a} \in X : f_\nu(\dot{a})\dot{<}\dot{\beta}\dot{/}\dot{c}\}$$

is $\nu-$measurable since $\dot{\beta}\dot{/}\dot{c} \in \mathbb{R}_\nu$ and the function f_ν is $\nu-$measurable. □

Theorem 9. *If the function f_ν is $\nu-$measurable, then the non-Newtonian real-valued function $f_\nu^{2_N}$ is $\nu-$measurable.*

Proof.

i) If $\dot{\beta}\dot{<}\dot{0}$ then $\{\dot{a} \in X : [f_\nu(\dot{a})]^{2_N} \dot{>} \dot{\beta}\} = X$ set is $\nu-$measurable.

ii) If $\dot{\beta}\dot{\geq}\dot{0}$ then we have

$$\begin{aligned}&\{\dot{a} \in X : [f_\nu(\dot{a})]^{2_N} \dot{>} \dot{\beta}\}\\ &= \{\dot{a} \in X : |f_\nu(\dot{a})|_N \dot{>} \sqrt{\dot{\beta}}^N\}\\ &= \{\dot{a} \in X : f_\nu(\dot{a}) \dot{>} \sqrt{\dot{\beta}}^N\} \cup \{\dot{a} \in X : f_\nu(\dot{a}) \dot{<} \dot{-} \sqrt{\dot{\beta}}^N\}\end{aligned}$$

which shows the function $f_\nu^{2_N}$ is $\nu-$measurable. □

Theorem 10. *If non-Newtonian real-valued functions f_ν, g_ν are $\nu-$measurable, then the function $f_\nu \dot{+} g_\nu$ is $\nu-$measurable.*

Proof. Since the functions f_ν, g_ν are $\nu-$measurable, then we have $\nu^{-1} \circ f_\nu \circ \nu,\ \nu^{-1} \circ g_\nu \circ \nu$ are real-valued measurable functions. Therefore, we get $(\nu^{-1} \circ f_\nu \circ \nu) + (\nu^{-1} \circ g_\nu \circ \nu)$ is measurable.
Thus, since

$$\begin{aligned}&(\nu^{-1} \circ f_\nu \circ \nu)(a) + (\nu^{-1} \circ g_\nu \circ \nu)(a)\\ &= \nu^{-1}(\nu\{(\nu^{-1} \circ f_\nu \circ \nu)(a) + (\nu^{-1} \circ g_\nu \circ \nu)(a)\})\\ &= \nu^{-1}(\nu\{\nu^{-1}(f_\nu(\nu(a))) + \nu^{-1}(g_\nu(\nu(a)))\})\\ &= \nu^{-1}(f_\nu(\nu(a)) \dot{+} g_\nu(\nu(a)))\\ &= \nu^{-1}((f_\nu \dot{+} g_\nu)(\nu(a)))\\ &= (\nu^{-1} \circ (f_\nu \dot{+} g_\nu) \circ \nu)(a)\end{aligned}$$

is a measurable function for $\forall a \in \nu^{-1}(X)$, then the function $f_\nu \dot{+} g_\nu$ is $\nu-$measurable. □

Theorem 11. *If f_ν, g_ν are $\nu-$measurable, then the function $f_\nu \dot{\times} g_\nu$ is $\nu-$measurable.*

Proof. If f_ν and g_ν are $\nu-$measurable, then $\nu^{-1} \circ f_\nu \circ \nu,\ \nu^{-1} \circ g_\nu \circ \nu$ are real-valued measurable functions. Therefore, the $(\nu^{-1} \circ f_\nu \circ \nu) \times (\nu^{-1} \circ g_\nu \circ \nu)$ function is measurable. Thus, since

$$\begin{aligned}&(\nu^{-1} \circ f_\nu \circ \nu)(a) \times (\nu^{-1} \circ g_\nu \circ \nu)(a)\\ &= \nu^{-1}(\nu\{(\nu^{-1} \circ f_\nu \circ \nu(a)) \times (\nu^{-1} \circ g_\nu \circ \nu(a))\})\\ &= \nu^{-1}(\nu\{\nu^{-1}(f_\nu(\nu(a))) \times \nu^{-1}(g_\nu(\nu(a)))\})\\ &= \nu^{-1}(f_\nu(\nu(a)) \dot{\times} g_\nu(\nu(a)))\\ &= \nu^{-1}((f_\nu \dot{\times} g_\nu)(\nu(a)))\\ &= (\nu^{-1} \circ (f_\nu \dot{\times} g_\nu) \circ \nu)(a)\end{aligned}$$

is a measurable function for $\forall a \in \nu^{-1}(X)$, then the function $f_\nu \dot{\times} g_\nu$ is $\nu-$measurable. □

Theorem 12. *If the function f_ν is $\nu-$measurable, then the function $|f_\nu|_N$ is $\nu-$measurable.*

Proof.

i) If $\dot{\beta}\dot{<}\dot{0}$, the set $\{\dot{a} \in X : |f_\nu(\dot{a})|_N \dot{>} \dot{\beta}\} = X$ is $\nu-$measurable.

ii) Let $\dot{\beta}\dot{\geq}\dot{0}$. Then the set $\{\dot{a} \in X : |f_\nu(\dot{a})|_N \dot{>} \dot{\beta}\}$ is $\nu-$measurable since

$$\begin{aligned}&\{\dot{a} \in X : |f_\nu(\dot{a})|_N \dot{>} \dot{\beta}\}\\ &= \{\dot{a} \in X : f_\nu(\dot{a}) \dot{>} \dot{\beta}\} \cup \{\dot{a} \in X : f_\nu(\dot{a}) \dot{<} \dot{-}\dot{\beta}\}\end{aligned}$$

is $\nu-$measurable. This shows that the function $|f_\nu|_N$ is $\nu-$measurable. □

3 Conclusion

In this study, we first give the $\nu-$Vitali theorems in the non-Newtonian sense. In the second part, we give the definition of the non-Newtonian measurable function. Also, we show that a function $\nu-$measurable if and only if the function $\nu^{-1} \circ f_\nu \circ \nu$ is a measurable function.This can be seen as the crucial step in the definition of the Lebesgue integral in the non-Newtonian sense. We also investigate some basic properties of $\nu-$measurable functions.

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