Subgroups and Homomorphism Structures of Complex Pythagorean Fuzzy Sets

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Abstract: This research introduces the notion of complex Pythagorean fuzzy subgroup (CPFSG). Both complex fuzzy subgroup (CFSG) and complex intuitionistic fuzzy subgroup (CIFSG) have significance in assigning membership grades in the unit disk in the complex plane. CFSG has a limitation solved by CIFSG, while CIFSG deals with a limited range of values. The important novelty of the CPFSG lies in its ability to solve the above limitations simultaneously and gets a wider range of values to be engaged in CPFSG. This work has introduced and investigated CPFSG as a new algebraic structure via the conditions that the sum of the square membership and non-membership lies on the unit interval for both the amplitude term and phase term. The result as any CIFSG is CPFSG but the convers is not true has been proved. Complex Pythagorean fuzzy coset has been defined and complex Pythagorean fuzzy normal subgroup (CPFNSG) and their algebraic characteristic has been demonstrated. Homomorphism on the CPFSG is shown. Some results as the inverse image of CPFSG and CPFNSG under isomorphism function are also a CPFSG and CPFNSG, respectively.

Key-Words: - Complex Pythagorean fuzzy subgroup, complex Pythagorean fuzzy coset, complex Pythagorean fuzzy normal subgroup, Fuzzy subgroups, complex fuzzy subgroups, Intuitionistic fuzzy subgroups, Complex Intuitionistic fuzzy subgroups. Pythagorean fuzzy subgroups, Pythagorean fuzzy Normal subgroup, homomorphism of Pythagorean fuzzy subgroup.

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1 Introduction

Since, [1], owned the first introduction to the fuzzy set (FS) in 1965, researchers have developed it in various domains to better uncertainty and vagueness representation. FS can convey uncertainty and vagueness, by using membership degrees between 0 and 1. For example, a membership degree of 0.8 does not always imply a membership degree of 0.2. Contrarily, FS has been described in several works, where the concept was working in conjunction with many applications of fuzzy technology including artificial intelligence, computer science, control engineering, decision theory, expert systems, logic management science, operation research, robotics, and others. Numerous theoretical advancements have been made, see, [2], Fuzzy Set Theory and Its Applications. In 1986, [3], introduced the intuitionistic fuzzy set (IFS), where a non-membership degree has been added. It is independent of the membership degree, and the range of the membership degree plus non-membership degree is 0 to 1. IFSs were improved by various methods and applications, see, [4], [5]. Also, as an improvement is the Pythagorean fuzzy set (PFS) in 2013, [6]. Whereas the condition of PFS is the square of the membership degree plus the square of the non-membership degree between 0 and 1, then it can be proven that any IFS is PFS. The Pythagorean fuzzy set was used in a variety of applications, for example, it has been suggested as a creative solution to a decision-making (DM) dilemma using PFS in 2018, by [7]. In addition, Ejegwa provided a PFS application that included career placement based on academic achievement utilizing a Max-min-Max composition, [8].

Ambiguity and uncertainty in the data may be handled by the fuzzy set, but they are unable to demonstrate a prospective ignorance of the information and its fluctuations at a particular point in time during their execution. Changes in the phase (periodicity) of the data in real life coincide with ambiguity and uncertainty that are present in the data. The study, [9], presented as a result a novel concept is the complex fuzzy set (CFS), in which an element's membership grade is a complex integer from the unit circle. However, many researchers, [10], [11], [12], focus on CFS instead. Later, [13], [14], developed the idea of CFS as a complex intuitionistic fuzzy set (CIFS) by illustrating the complexity of complex-valued non-membership functions and introducing the concepts of complex intuitionistic fuzzy relation and distance measure in CIFS environments. Nevertheless, the range of CIFS may lose some information, where $p + q$ may exceed one but $p^2 + q^2$ may not (and that for both phase term and amplitude term). [15], established the idea of a CPFS to handle Pythagorean fuzzy values and extended several distance measures, where the partner recognition issue is addressed using these newly specified distance measures.

The main motivation for using the Pythagorean fuzzy set (PFS) is when the IFS fails to deal with uncertain and vague information in some fuzzy systems, [6]. A CPFS has been introduced to cover uncertain and periodicity information under a multiple attributes decision-making (MADM) problem, [16]. In 2023, [17], improved and incorporated the notion of Aczel-Alsin t-norm and t-conorm under the system of CPFS. CPFS has been applied aggregation operators to select a suitable candidate for a vacant post in a multinational company. Therefore, the development of algebraic structures under CPFSG can be an initial and applicable framework to handle the special type of information and application in the MADM field. One of the most crucial aspects of algebra is group theory, which provides a useful framework for analyzing objects that appear to be symmetrical. Classifying the symmetries of molecules, atoms, regular polyhedral, and crystal structures is essential. This idea has evolved into a common and effective tool for research on the behavior of codon sequences and the genetic code as a whole.

[18], in 1971, merged group theory with fuzzy sets, for the first time, and presented a fuzzy subgroup (FSG). Eighteen years later 1989, [19], introduced the intuitionistic fuzzy subgroup (IFSG) notion and presented the intuitionistic fuzzification algebraic structure after IFS presentation by [3]. In manuscripts, [20], [21], the intuitionistic fuzzy subgroup has further developed. In 2020, [22], introduced and studied the algebraic characteristics of anti-intuitionistic fuzzy subgroups on a selected averaging operator. [23], introduced the concept of soft expert symmetric group. They applied their concept to the multiple criteria decision-making problems. Moreover, in 2024, [24], studied the notion of intuitionistic fuzzy soft expert groups.

Pythagorean fuzzy subgroup (PFSG) was defined recently, by [25], in 2020, who also discussed the algebraic features of the subgroup in the fuzzy model and investigated related properties. [25], described some of the notion such Pythagorean fuzzy subgroup (PFSG), Pythagorean fuzzy coset and Pythagorean fuzzy normal subgroup. Also, they introduced Pythagorean fuzzy level subgroup and homomorphism on Pythagorean fuzzy subgroup. After that, some researchers generalized PFSG. In 2021, [26], [27], presented the concept of (α, β)-Pythagorean fuzzy sets and characterized (α, β)-Pythagorean fuzzy subgroups. [28], introduced Pythagorean fuzzy order of elements of groups and they discussed the algebraic properties of the Pythagorean fuzzy subgroup. Also, in [28], introduced the notion of a Pythagorean fuzzy quotient group and prove Lagrange's theorem

Also, [29], offered broad research on the normal subgroups and isomorphisms property under Pythagorean fuzzy sets. It should be noted that several developments have been widely presented in the field of Algebra because Pythagorean fuzzy subsets fail to work in some cases. This research emerged after defining the q-rung orthopair fuzzy set by [30]. This limitation leads development of the notion of q-rung orthopair fuzzy set to the complex plane and employing it in the algebra field as future research. [31], defined the complex fuzzy subgroup (CFSG) in 2021 and described the subgroup's relevant attributes. As a generalization of CFSG, [32], defined the theory of complex intuitionistic fuzzy subgroup (CIFSG), by employing a phase term and an amplitude term to the subgroup structure. In 2023, a new structure of complex Pythagorean fuzzy subfield (CPFSF) was presented by [33]. Also, they gave an application to demonstrate that the direct product of two CPFSFs is also a CPFSF. Furthermore, they studied the homomorphic images and inverse images of CPFSFs. [34], introduced the (ϵ, δ)-complex anti fuzzy subgroups and their applications.

The progressive development of FSG and IFSG to PFSG and the existing notions of CFSG and CIFSG introduced the present work of CPFSG. This work investigates the CPFSG which is a continuation of earlier research. This study has been designed according to the following structure, section 2 introduces the literature review. Section 3 generalizes the notion of complex Pythagorean fuzzy subgroups by adding two phase terms to the membership and non-membership functions of the PFSG structure. The results highlighted some inherited conditions from the characteristics of PFSG, (i.e. the sum of square phase terms of membership and non-membership values is less than or equal to 1). The definitions of complex Pythagorean coset and complex Pythagorean fuzzy normal subgroups are followed by an analysis of their algebraic properties in section 4. Then a discussion of homomorphism and a demonstration of the Pythagorean fuzzy subgroup's attributes in the complex form, in section 5. Finally, a general discussion and future work suggestions are presented in section 6.

2 Preliminaries

Zadeh defined FS in 1965, [1].

Definition 1. *Let $\mathbb{U}$ be a crisp set. Then $\mathbb{M} : \mathbb{U} \to [0,1]$ is called a fuzzy set, denoted by FS, of $\mathbb{U}$. Here $\mathbb{M}(\varpi)$ is called a degree of membership.*

Rosenfeld was the first who worked on fuzzy graphs and defined fuzzy subgroups in 1971, [18].

Definition 2. *Let $\mathbb{M} : \mathbb{U} \to [0,1]$ be a fuzzy subset of a group $(\mathbb{U}, \Box)$. Then $\mathbb{M}$ is said to be a fuzzy subgroup of $(\mathbb{U}, \Box)$, if the following conditions hold:*
i) $\mathbb{M}(\varpi \Box \kappa) \geq \mathbb{M}(\varpi) \wedge \mathbb{M}(\kappa)$.
ii) $\mathbb{M}(\varpi^{-1}) \geq \mathbb{M}(\varpi)$, *for all* $\varpi, \kappa \in \mathbb{U}$

Ramot et. al. defined CFS on a crisp set in 2002, [9].

Definition 3. *Let $\mathbb{U}$ be a crisp set and define $\mathbb{M}$ on $\mathbb{U}$ to be complex fuzzy set, where $\mathbb{M} = \{(\varpi, \mathbb{M}(\varpi)) : \varpi \in \mathbb{U}\}$ such that $\mathbb{M}(\varpi) : \mathbb{U} \to \{\zeta_1 : \zeta_1 \in \mathbb{C}, |\zeta_1| \leq 1\}$, provided that:*
$\mathbb{M}(\varpi) = p(\varpi)e^{2\pi i\alpha(\varpi)}$, *where* $p(\varpi)$ *and* $\alpha(\varpi) \in [0,1]$.

Recently in 2021, [31], defined CFSG.

Definition 4. *Let $\mathbb{M}(\varpi) : \mathbb{U} \to \{\zeta_1 : \zeta_1 \in \mathbb{C}, |\zeta_1| \leq 1\}$ be a complex fuzzy subset of a group $(\mathbb{U}, \Box)$. Then $\mathbb{M}$ is said to be a complex fuzzy subgroup, of $(\mathbb{U}, \Box)$, if the following conditions hold:*
i) $\mathbb{M}(\varpi \Box \kappa) \geq \mathbb{M}(\varpi) \wedge \mathbb{M}(\kappa)$.
ii) $\mathbb{M}(\varpi^{-1}) \geq \mathbb{M}(\varpi)$, *for all* $\varpi, \kappa \in \mathbb{U}$

Equivalently, for any $\varpi, \kappa \in \mathbb{U}$ and $\mathbb{M}(\varpi) = p(\varpi)e^{2\pi i\alpha(\varpi)}$, we have:
i) $p(\varpi \Box \kappa) \geq p(\varpi) \wedge p(\kappa)$ *and* $\alpha(\varpi \Box \kappa) \geq \alpha(\varpi) \wedge \alpha(\kappa)$.
ii) $p(\varpi^{-1}) \geq p(\varpi)$ *and* $\alpha(\varpi^{-1}) \geq \alpha(\varpi)$.

In 2013, [6], presented PFS of crisp set.

Definition 5. *Let $\mathbb{U}$ be a crisp set and define $\mathbb{P}$ on $\mathbb{U}$ to be pythagorean fuzzy set, where $\mathbb{P} = \{(\varpi, \mathbb{M}(\varpi), \mathbb{N}(\varpi)) : \varpi \in \mathbb{U}\}$. Such that $\mathbb{M}(\varpi) \in [0,1]$ and $\mathbb{N}(\varpi) \in [0,1]$ are the degree of membership and non-membership of $\varpi \in \mathbb{U}$, which satisfying the condition:*
$0 \leq \mathbb{M}^2(\varpi) + \mathbb{N}^2(\varpi) \leq 1$, *for all* $\varpi \in \mathbb{U}$.

Note that, in the previous definition, if the condition was $0 \leq \mathbb{M}(\varpi) + \mathbb{N}(\varpi) \leq 1$, for all $\varpi \in \mathbb{U}$, then $\mathbb{P}$ define an IFS on $\mathbb{U}$, see, [3].

The following is the definition of PFSG, by the 2020 study, [25].

Definition 6. *Let $(\mathbb{U}, \Box)$ be a group and $\mathbb{P} = (\mathbb{M}, \mathbb{N})$, be a PFS of $\mathbb{U}$. Then $\mathbb{P}$ is said to be a pythagorean fuzzy subgroup of $\mathbb{U}$ if the following conditions hold:*

1. $\mathbb{M}^2(\varpi \Box \kappa) \geq \mathbb{M}^2(\varpi) \wedge \mathbb{M}^2(\kappa)$ *and* $\mathbb{N}^2(\varpi \Box \kappa) \leq \mathbb{N}^2(\varpi) \vee \mathbb{N}^2(\kappa)$.
2. $\mathbb{M}^2(\varpi^{-1}) \geq \mathbb{M}^2(\varpi)$ *and* $\mathbb{N}^2(\varpi^{-1}) \leq \mathbb{N}^2(\varpi)$

, $\forall\, \varpi,\ \kappa \in \mathbb{U}$

Note that, if we apply the previous conditions without square, we get IFSG, [19].

The study developed the concept of CPFS in 2019, [15].

Definition 7. *Let $\mathbb{U}$ be a crisp set and define φ on $\mathbb{U}$ to be complex pythagorean fuzzy set, where $\varphi = \{(\varpi, \mathbb{M}(\varpi), \mathbb{N}(\varpi)) : \varpi \in \mathbb{U}\}$. Such that $\mathbb{M}(\varpi) : \mathbb{U} \to \{\zeta_1 : \zeta_1 \in \mathbb{C}, |\zeta_1| \leq 1\}$ and $\mathbb{N}(\varpi) : \mathbb{U} \to \{\zeta_2 : \zeta_2 \in \mathbb{C}, |\zeta_2| \leq 1\}$, are the degree of membership and non-membership of $\varpi \in \mathbb{U}$. Moreover, $\mathbb{M}(\varpi) = p(\varpi)e^{2\pi i\alpha(\varpi)}$, $\mathbb{N}(\varpi) = q(\varpi)e^{2\pi i\gamma(\varpi)}$ are satisfying the conditions; $0 \leq p^2(\varpi) + q^2(\varpi) \leq 1$ and $0 \leq \alpha^2(\varpi) + \gamma^2(\varpi) \leq 1$.*

For the preceding definition, if we let conditions be $0 \leq p(\varpi) + q(\varpi) \leq 1$ and $0 \leq \alpha(\varpi) + \gamma(\varpi) \leq 1$, for all $\varpi \in \mathbb{U}$. Then φ define a CIFS on $\mathbb{U}$, [13].

In [15], [16], operations on CPFS, such as union, intersection and complement, were defined.

Definition 8. *[16], let, $\varphi_1 = (\mathbb{M}_1, \mathbb{N}_1)$ and $\varphi_2 = (\mathbb{M}_2, \mathbb{N}_2)$ are CPFS that defined on $\mathbb{U}$, where: $\mathbb{M}_j(\varpi) : \mathbb{U} \to \{p_j(\varpi)e^{2\pi i\alpha_j(\varpi)} : 0 \leq p_j(\varpi),\ \alpha_j(\varpi) \leq 1\}$, and $\mathbb{N}_j(\varpi) : \mathbb{U} \to \{q_j(\varpi)e^{2\pi i\gamma_j(\varpi)} : 0 \leq q_j(\varpi),\ \gamma_j(\varpi) \leq 1\}$, for $j = 1, 2$, then:*

1. $\varphi_1 \cap \varphi_2 = (\mathbb{M}_1 \cap \mathbb{M}_2, \mathbb{N}_1 \cap \mathbb{N}_2)$, where:
a. $(\mathbb{M}_1 \cap \mathbb{M}_2)(\varpi) = (p_1(\varpi) \wedge p_2(\varpi))e^{2\pi i(\alpha_1(\varpi)\wedge\alpha_2(\varpi))}$.
b. $(\mathbb{N}_1 \cap \mathbb{N}_2)(\varpi) = (q_1(\varpi) \vee q_2(\varpi))e^{2\pi i(\gamma_1(\varpi)\vee\gamma_2(\varpi))}$.

2. $\varphi_1 \cup \varphi_2 = (\mathbb{M}_1 \cup \mathbb{M}_2, \mathbb{N}_1 \cup \mathbb{N}_2)$, where:
a. $(\mathbb{M}_1 \cup \mathbb{M}_2)(\varpi) = (p_1(\varpi) \vee p_2(\varpi))e^{2\pi i(\alpha_1(\varpi)\vee\alpha_2(\varpi))}$.
b. $(\mathbb{N}_1 \cup \mathbb{N}_2)(\varpi) = (q_1(\varpi) \wedge q_2(\varpi))e^{2\pi i(\gamma_1(\varpi)\wedge\gamma_2(\varpi))}$.

Definition 9. *[15], let, $\varphi = (\mathbb{M}, \mathbb{N})$ be CPFS defined on $\mathbb{U}$, where: $\mathbb{M}(\varpi) : \mathbb{U} \to \{p(\varpi)e^{2\pi i\alpha(\varpi)} : 0 \leq p(\varpi),\ \alpha(\varpi) \leq 1\}$, and $\mathbb{N}(\varpi) : \mathbb{U} \to \{q(\varpi)e^{2\pi i\gamma(\varpi)} : 0 \leq q(\varpi),\ \gamma(\varpi) \leq 1\}$. Then the complement of φ defined by:*

$\varphi^c = \bar{\varphi} = (\mathbb{N}, \mathbb{M})$, where:
a. $\mathbb{M}^c(\varpi) = \mathbb{N} = q(\varpi)e^{2\pi i\gamma(\varpi)}$.
b. $\mathbb{N}^c(\varpi) = \mathbb{M} = p(\varpi)e^{2\pi i\alpha(\varpi)}$.

3 Complex Pythagorean Fuzzy Subgroups

A generalization of PFSG and CPFS is introduced in the proceeding definition.

Definition 10. *Let $(\mathbb{U}, \Box)$ be a group and $\varphi = (\mathbb{M} = p\ e^{2\pi i\alpha}, \mathbb{N} = q\ e^{2\pi i\gamma})$, be a CPFS of $\mathbb{U}$. Then φ is said to be a complex Pythagorean fuzzy subgroups (CPFSG) of $\mathbb{U}$, where $p^2 + q^2 \leq 1$ and $\alpha^2 + \gamma^2 \leq 1$, if $\mathbb{M}$ and $\mathbb{N}$ have the following property:*
1a. $\mathbb{M}^2(\varpi\Box\kappa) = p^2(\varpi\Box\kappa)e^{2\pi i\alpha^2(\varpi\Box\kappa)} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} \wedge p^2(\kappa)e^{2\pi i\alpha^2(\kappa)} = \mathbb{M}^2(\varpi)\wedge\mathbb{M}^2(\kappa)$. *where,* $p^2(\varpi\Box\kappa) \geq p^2(\varpi) \wedge p^2(\kappa)$ *and* $\alpha^2(\varpi\Box\kappa) \geq \alpha^2(\varpi) \wedge \alpha^2(\kappa)$.
1b. $\mathbb{N}^2(\varpi\Box\kappa) = q^2(\varpi\Box\kappa)e^{2\pi i\gamma^2(\varpi\Box\kappa)} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} \vee q^2(\kappa)e^{2\pi i\gamma^2(\kappa)} = \mathbb{N}^2(\varpi)\vee\mathbb{N}^2(\kappa)$ *where,* $q^2(\varpi\Box\kappa) \leq q^2(\varpi) \vee q^2(\kappa)$ *and* $\gamma^2(\varpi\Box\kappa) \leq \gamma^2(\varpi) \vee \alpha^2(\kappa)$

2a. $\mathbb{M}^2(\varpi^{-1}) = p^2(\varpi^{-1})e^{2\pi i\alpha^2(\varpi^{-1})} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} = \mathbb{M}^2(\varpi)$
where, $p^2(\varpi^{-1}) \geq p^2(\varpi)$ *and* $\alpha^2(\varpi^{-1}) \geq \alpha^2(\varpi)$.
2b. $\mathbb{N}^2(\varpi^{-1}) = q^2(\varpi^{-1})e^{2\pi i\gamma^2(\varpi^{-1})} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = \mathbb{N}^2(\varpi)$
where, $q^2(\varpi^{-1}) \leq q^2(\varpi)$ *and* $\gamma^2(\varpi^{-1}) \leq \gamma^2(\varpi)$

Note that, if we apply the previous conditions without square, we get a CIFSG, [32].

Proposition 1. *Let $\varphi = (\mathbb{M} = p\,e^{2\pi i\alpha}, \mathbb{N} = q\,e^{2\pi i\gamma})$, be a CPFSG of a group $(\mathbb{U}, \Box)$, then the following holds:*

1. $\mathbb{M}^2(id) = p^2(id)e^{2\pi i\alpha^2(id)} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} = \mathbb{M}^2(\varpi)$, *where* $p^2(id) \geq p^2(\varpi)$ *and* $\alpha^2(id) \geq \alpha^2(\varpi)$.
2. $\mathbb{N}^2(id) = q^2(id)e^{2\pi i\gamma^2(id)} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = \mathbb{N}^2(\varpi)$, *where* $q^2(id) \leq q^2(\varpi)$ *and* $\gamma^2(id) \leq \gamma^2(\varpi)$.
3. $\mathbb{M}^2(\varpi^{-1}) = p^2(\varpi^{-1})e^{2\pi i\alpha^2(\varpi^{-1})} = p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} = \mathbb{M}^2(\varpi)$, *where* $p^2(\varpi^{-1}) = p^2(\varpi)$ *and* $\alpha^2(\varpi^{-1}) = \alpha^2(\varpi)$.
4. $\mathbb{N}^2(\varpi^{-1}) = q^2(\varpi^{-1})e^{2\pi i\gamma^2(\varpi^{-1})} = q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = \mathbb{N}^2(\varpi)$, *where* $q^2(\varpi^{-1}) = q^2(\varpi)$ *and* $\gamma^2(\varpi^{-1}) = \gamma^2(\varpi)$.

for all $\varpi \in \mathbb{U}$, where id is the identity of all elements.

Proof. Since φ is CPFSG then by Definition 10: "1" and "2" can be proved as follow, $\mathbb{M}^2(id) = p^2(id)e^{2\pi i\alpha^2(id)} = p^2(\varpi\Box\varpi^{-1})\ e^{2\pi i\alpha^2(\varpi\Box\varpi^{-1})} \geq \min\{p^2(\varpi)e^{2\pi i\alpha^2(\varpi)},\ p^2(\varpi^{-1})e^{2\pi i\alpha^2(\varpi^{-1})}\} = \min\{p^2(\varpi), p^2(\varpi^{-1})\}\ e^{2\pi i\min\{\alpha^2(\varpi),\alpha^2(\varpi^{-1})\}} = p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} = \mathbb{M}^2(\varpi)$. In addition, $\mathbb{N}^2(id) = q^2(id)\ e^{2\pi i\gamma^2(id)} = q^2(\varpi\Box\varpi^{-1})e^{2\pi i\gamma^2(\varpi\Box\varpi^{-1})} \leq \max\{q^2(\varpi)e^{2\pi i\gamma^2(\varpi)},\ q^2(\varpi^{-1})e^{2\pi i\gamma^2(\varpi^{-1})}\} = \max\{q^2(\varpi), q^2(\varpi^{-1})\}e^{2\pi i\max\{\gamma^2(\varpi),\gamma^2(\varpi^{-1})\}} = q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = \mathbb{N}^2(\varpi)$. Similarly, "3" and "4" can be proved. ■

In the following theorem, we proved that any CIFSG is CPFSG, whereas CIFS is a subclass of CPFS, [15].

Theorem 3.1. *If φ is a CIFSG of the group $(\mathbb{U}, \Box)$, then φ is a CPFSG of the group $(\mathbb{U}, \Box)$.*

Proof. At first, to show that $p^2(\varpi\Box\kappa)e^{2\pi i\alpha^2(\varpi\Box\kappa)} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} \wedge p^2(\kappa)e^{2\pi i\alpha^2(\kappa)}$ and

$q^2(\varpi\Box\kappa)e^{2\pi i\gamma^2(\varpi\Box\kappa)} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} \vee q^2(\kappa)e^{2\pi i\gamma^2(\kappa)}$. We know that φ is a CIFSG, then $p(\varpi\Box\kappa)e^{2\pi i\alpha(\varpi\Box\kappa)} \geq p(\varpi)e^{2\pi i\alpha(\varpi)} \wedge p(\kappa)e^{2\pi i\alpha(\kappa)}$ and $q(\varpi\Box\kappa)e^{2\pi i\gamma(\varpi\Box\kappa)} \leq q(\varpi)e^{2\pi i\gamma(\varpi)} \vee q(\kappa)e^{2\pi i\gamma(\kappa)}$, where $p+q \leq 1$ and $\alpha+\gamma \leq 1$. Then, we have four cases to consider:

a) Let $p(\varpi)e^{2\pi i\alpha(\varpi)} \geq p(\kappa)e^{2\pi i\alpha(\kappa)}$ and $q(\varpi)e^{2\pi i\gamma(\varpi)} \geq q(\kappa)e^{2\pi i\gamma(\kappa)}$, then $p(\varpi\Box\kappa)e^{2\pi i\alpha(\varpi\Box\kappa)} \geq p(\kappa)e^{2\pi i\alpha(\kappa)}$. Now consider $p^2(\varpi\Box\kappa)e^{2\pi i\alpha^2(\varpi\Box\kappa)} \geq p^2(\kappa)e^{2\pi i\alpha^2(\kappa)} = p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} \wedge p^2(\kappa)e^{2\pi i\alpha^2(\kappa)}$. Moreover, $q(\varpi\Box\kappa)e^{2\pi i\gamma(\varpi\Box\kappa)} \leq q(\varpi)e^{2\pi i\gamma(\varpi)}$. Now consider $q^2(\varpi\Box\kappa)e^{2\pi i\gamma^2(\varpi\Box\kappa)} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} \vee q^2(\kappa)e^{2\pi i\gamma^2(\kappa)}$.

b) Let $p(\varpi)e^{2\pi i\alpha(\varpi)} \leq p(\kappa)e^{2\pi i\alpha(\kappa)}$ and $q(\varpi)e^{2\pi i\gamma(\varpi)} \leq q(\kappa)e^{2\pi i\gamma(\kappa)}$, then with same argument of case a, we get the result.

c) Let $p(\varpi)e^{2\pi i\alpha(\varpi)} \leq p(\kappa)e^{2\pi i\alpha(\kappa)}$ and $q(\varpi)e^{2\pi i\gamma(\varpi)} \geq q(\kappa)e^{2\pi i\gamma(\kappa)}$, then $p(\varpi\Box\kappa)e^{2\pi i\alpha(\varpi\Box\kappa)} \geq p(\varpi)e^{2\pi i\alpha(\varpi)}$. Now consider $p^2(\varpi\Box\kappa)e^{2\pi i\alpha^2(\varpi\Box\kappa)} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} = p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} \wedge p^2(\kappa)e^{2\pi i\alpha^2(y)}$. Moreover, $q(\varpi\Box\kappa)e^{2\pi i\gamma(\varpi\Box\kappa)} \leq q(\varpi)e^{2\pi i\gamma(\varpi)}$. Now consider $q^2(\varpi\Box\kappa)e^{2\pi i\gamma^2(\varpi\Box\kappa)} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} \vee q^2(\kappa)e^{2\pi i\gamma^2(\kappa)}$.

d) Let $p(\varpi)e^{2\pi i\alpha(\varpi)} \geq p(\kappa)e^{2\pi i\alpha(y)}$ and $q(\varpi)e^{2\pi i\gamma(\varpi)} \leq q(\kappa)e^{2\pi i\gamma(\kappa)}$, then with same argument of case c, we get the result.

Secondly, since $p(\varpi^{-1})e^{2\pi i\alpha(\varpi^{-1})} \geq p(\varpi)e^{2\pi i\alpha(\varpi)}$ and $q(\varpi^{-1})e^{2\pi i\gamma(\varpi^{-1})} \leq q(\varpi)e^{2\pi i\gamma(\varpi)}$, then $p^2(\varpi^{-1})e^{2\pi i\alpha^2(\varpi^{-1})} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)}$ and $q^2(\varpi^{-1})e^{2\pi i\gamma^2(\varpi^{-1})} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)}$ too. ■

The converse of Theorem 3.1 is not always true, please see the following example.

Example 1. *For the set $\mathbb{U} = \{1, -1, i, -i\}$, define a group $(\mathbb{U}, .)$, where $'.'$ is the known multiplication. Also define $\varphi = (\mathbb{M}, \mathbb{N})$ be a CPFS on $\mathbb{U}$, where:*
$\mathbb{M}(1) = 0.7e^{2\pi i(0.8)}$, $\mathbb{N}(1) = 0.1e^{2\pi i(0.2)}$, $\mathbb{M}(-1) = 0.6e^{2\pi i(0.6)}$, $\mathbb{N}(-1) = 0.2e^{2\pi i(0.3)}$, $\mathbb{M}(i) = 0.4e^{2\pi i(0.3)}$, $\mathbb{N}(i) = 0.7e^{2\pi i(0.4)}$, $\mathbb{M}(-i) = 0.4e^{2\pi i(0.3)}$, $\mathbb{N}(-i) = 0.7e^{2\pi i(0.4)}$.
Now, to check that Definition 7 is satisfied, we get for $\varpi = 1$ that $0.49 + 0.01$ and $0.64 + 0.04$ both are in the closed interval $[0, 1]$. It is easy to check that for all $\varpi \in \mathbb{U}$, the property satisfied and φ is CPFS. But, it is not CIFS, where at $\varpi = -i$ we have $p+q = 1.1 \nleq 1$.

In addition, this set $\varphi = (\mathbb{M}, \mathbb{N})$ is CPFSG, see Definition 10:
i) For first property consider:
a) $\mathbb{M}^2(-1\ .\ -1) = \mathbb{M}^2(1) = 0.49e^{2\pi i(0.64)}$, and $\mathbb{M}^2(-1) \wedge \mathbb{M}^2(-1) = \mathbb{M}^2(-1) = 0.36e^{2\pi i(0.36)}$. Since, $0.49 \geq 0.36$ and $0.64 \geq 0.36$, then $\mathbb{M}^2(-1\ .\ -1) \geq \mathbb{M}^2(-1) \wedge \mathbb{M}^2(-1)$. Similarly check $\mathbb{M}^2(i\ .\ i) = \mathbb{M}^2(-1) = 0.36e^{2\pi i(0.36)} \geq \mathbb{M}^2(i) = 0.16e^{2\pi i(0.09)}$. Hence, it is straight forward that $\mathbb{M}^2(\varpi\Box\kappa) \geq \mathbb{M}^2(\varpi) \wedge \mathbb{M}^2(\kappa)$.
b) $\mathbb{N}^2(-1\ .\ i) = \mathbb{N}^2(-i) = 0.49e^{2\pi i(0.16)}$, and $\mathbb{N}^2(-1) \vee \mathbb{N}^2(i) = \max\{0.04, 0.49\}$ $e^{2\pi i \max\{0.09, 0.16\}} = 0.49e^{2\pi i(0.16)}$. Then $\mathbb{N}^2(-1\ .\ i) \leq \mathbb{N}^2(-1) \vee \mathbb{N}^2(i)$. Similarly to check that $\mathbb{N}^2(\varpi\Box\kappa) \leq \mathbb{N}^2(\varpi) \vee \mathbb{N}^2(\kappa)$ for all ϖ, $\kappa \in \mathbb{U}$.

ii) For second property, we have $1 = 1^{-1}$, $-1 = -1^{-1}$ and $i = -i^{-1}$, hence for $\varpi = 1$, -1 are trivially true and since $\varphi(i) = \varphi(-i)$, the property satisfied too.

Proposition 2. *For a CPFS $\varphi = (\mathbb{M} = p\, e^{2\pi i\alpha}, \mathbb{N} = q\, e^{2\pi i\gamma})$ of a group $(\mathbb{U}, \Box)$, it is a CPFSG if and only if:*
1. $\mathbb{M}^2(\varpi\Box\kappa^{-1}) = p^2(\varpi\Box\kappa^{-1})e^{2\pi i\alpha^2(\varpi\Box\kappa^{-1})} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} \wedge p^2(\kappa)e^{2\pi i\alpha^2(\kappa)} = \mathbb{M}^2(\varpi) \wedge \mathbb{M}^2(\kappa)$, where $p^2(\varpi\Box\kappa^{-1}) \geq p^2(\varpi) \wedge p^2(\kappa)$ and $\alpha^2(\varpi\Box\kappa^{-1}) \geq \alpha^2(\varpi) \wedge \alpha^2(\kappa)$

2. $\mathbb{N}^2(\varpi\Box\kappa^{-1}) = q^2(\varpi\Box\kappa^{-1})e^{2\pi i\gamma^2(\varpi\Box\kappa^{-1})} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} \vee q^2(\kappa)e^{2\pi i\gamma^2(\kappa)} = \mathbb{N}^2(\varpi) \vee \mathbb{N}^2(\kappa)$, where $q^2(\varpi\Box\kappa^{-1}) \leq q^2(\varpi) \vee q^2(\kappa)$ and $\gamma^2(\varpi\Box\kappa^{-1}) \leq \gamma^2(\varpi) \vee \gamma^2(\kappa)$

Proof. ($\Longrightarrow$) According to Proposition 1, we have $p^2(\varpi^{-1})\ e^{2\pi i\alpha^2(\varpi^{-1})} = p^2(\varpi)\ e^{2\pi i\alpha^2(\varpi)}$ and $q^2(\varpi^{-1})e^{2\pi i\gamma^2(\varpi^{-1})} = q^2(\varpi)e^{2\pi i\gamma^2(\varpi)}$ for all $\varpi \in \mathbb{U}$, then results follow by Definition 10.

($\Longleftarrow$) At first, φ is CPFS and is defined on group $(\mathbb{U}, \Box)$, then:
(a1) $\mathbb{M}^2(id) = p^2(id)e^{2\pi i\alpha^2(id)} = p^2(\varpi\Box\varpi^{-1})e^{2\pi i\alpha^2(\varpi\Box\varpi^{-1})} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} = \mathbb{M}^2(\varpi)$, where $p^2(\varpi\Box\varpi^{-1}) \geq p^2(\varpi)$ and $\alpha^2(\varpi\Box\varpi^{-1}) \geq \alpha^2(\varpi)$.
(b1) $\mathbb{M}^2(\varpi^{-1}) = p^2(\varpi^{-1})e^{2\pi i\alpha^2(\varpi^{-1})} = p^2(id\ \Box\varpi^{-1})e^{2\pi i\alpha^2(id\ \Box\varpi^{-1})} \geq$
$\min\{p^2(id)e^{2\pi i\alpha^2(di)}, p^2(\varpi)e^{2\pi i\alpha^2(\varpi)}\}$
$= \min\{p^2(id), p^2(\varpi)\}e^{2\pi i \min\{\alpha^2(di), \alpha^2(\varpi)\}}$
$= p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} = \mathbb{M}^2(\varpi)$, by (a1).

(c1) $\mathbb{M}^2(\varpi\square\kappa) = \mathbb{M}^2(\varpi\square\ (\kappa^{-1})^{-1}) \geq \mathbb{M}^2(\varpi) \wedge \mathbb{M}^2(\kappa^{-1}) \geq \mathbb{M}^2(\varpi) \wedge \mathbb{M}^2(\kappa)$, by (b1).

Similarly, we have:
(a2) $\mathbb{N}^2(id) = q^2(id)e^{2\pi i\gamma^2(id)} = q^2(\varpi\square\varpi^{-1})e^{2\pi i\gamma^2(\varpi\square\varpi^{-1})} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = \mathbb{N}^2(\varpi)$, where $q^2(\varpi\square\varpi^{-1}) \leq q^2(\varpi)$ and $\gamma^2(\varpi\square\varpi^{-1}) \leq \gamma^2(\varpi)$.
(b2) $\mathbb{N}^2(\varpi^{-1}) = q^2(\varpi^{-1})e^{2\pi i\gamma^2(\varpi^{-1})} = q^2(id\ \square\varpi^{-1})e^{2\pi i\gamma^2(id\ \square\varpi^{-1})} \leq$
$\max\{q^2(id)e^{2\pi i\gamma^2(di)}, q^2(\varpi)e^{2\pi i\gamma^2(\varpi)}\}$
$= \max\{q^2(id), q^2(\varpi)\}e^{2\pi i\max\{\gamma^2(id),\gamma^2(\varpi)\}}$
$= q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = \mathbb{N}^2(\varpi)$, by (a2).
(c2) $\mathbb{N}^2(\varpi\square\kappa) = \mathbb{N}^2(\varpi\square\ (\kappa^{-1})^{-1}) \leq \mathbb{N}^2(\varpi) \vee \mathbb{N}^2(\kappa^{-1}) \leq \mathbb{N}^2(\varpi) \vee \mathbb{N}^2(\kappa)$, by (b2).

Finally, by (c1) and (c2) the first condition is satisfied, and by (b1) and (b2) the second condition is satisfied, hence φ is CPFSG of a group $(\mathbb{U}, \square)$. ■

Proposition 3. *The intersection of two CPFSGs of a group* $(\mathbb{U}, \square)$ *is a CPFSG.*

Proof. Let $A,\ B$ be two CPFSGs of $\mathbb{U}$ and using previous proposition, then:
i) $\mathbb{M}^2_{A\cap B}(\varpi\square\kappa^{-1})$
$= p^2_{A\cap B}(\varpi\square\kappa^{-1})e^{2\pi i\alpha^2_{A\cap B}(\varpi\square\kappa^{-1})}$
$= (p^2_A(\varpi\square\kappa^{-1}) \wedge\ p^2_B(\varpi\square\kappa^{-1}))$
$e^{2\pi i(\alpha^2_A(\varpi\square\kappa^{-1})\wedge\ \alpha^2_B(\varpi\square\kappa^{-1}))}$
$\geq (\min\{p^2_A(\varpi), p^2_A(\kappa)\} \wedge\ \min\{p^2_B(\varpi), p^2_B(\kappa)\})$
$e^{2\pi i(\min\{\alpha^2_A(\varpi),\alpha^2_A(\kappa)\}\wedge\ \min\{\alpha^2_B(\varpi),\alpha^2_B(\kappa)\})}$
$= (\min\{p^2_A(\varpi), p^2_B(\varpi)\} \wedge\ \min\{p^2_A(\kappa), p^2_B(\kappa)\})$
$e^{2\pi i(\min\{\alpha^2_A(\varpi),\alpha^2_B(\varpi)\}\wedge\ \min\{\alpha^2_A(\kappa),\alpha^2_B(\kappa)\})}$
$= (p^2_{A\cap B}(\varpi) \wedge\ p^2_{A\cap B}(\kappa))$
$e^{2\pi i(\alpha^2_{A\cap B}(\varpi)\wedge\ \alpha^2_{A\cap B}(\kappa))}$
$= p^2_{A\cap B}(\varpi)e^{2\pi i\alpha^2_{A\cap B}(\varpi)} \wedge\ p^2_{A\cap B}(\kappa)e^{2\pi i\alpha^2_{A\cap B}(\kappa)}$
$= (\mathbb{M}^2_{A\cap B}(\varpi) \wedge\ \mathbb{M}^2_{A\cap B}(\kappa))$.

ii) $\mathbb{N}^2_{A\cap B}(\varpi\square\kappa^{-1})$
$= q^2_{A\cap B}(\varpi\square\kappa^{-1})e^{2\pi i\gamma^2_{A\cap B}(\varpi\square\kappa^{-1})}$
$= (q^2_A(\varpi\square\kappa^{-1}) \vee\ q^2_B(\varpi\square\kappa^{-1}))$
$e^{2\pi i(\gamma^2_A(\varpi\square\kappa^{-1})\vee\ \gamma^2_B(\varpi\square\kappa^{-1}))}$
$\leq (\max\{q^2_A(\varpi), q^2_A(\kappa)\} \vee\ \max\{q^2_B(\varpi), q^2_B(\kappa)\})$
$e^{2\pi i(\max\{\gamma^2_A(\varpi),\gamma^2_A(\kappa)\}\vee\ \max\{\gamma^2_B(\varpi),\gamma^2_B(\kappa)\})}$
$= (\max\{q^2_A(\varpi), q^2_B(\varpi)\} \vee\ \max\{q^2_A(\kappa), q^2_B(\kappa)\})$
$e^{2\pi i(\max\{\gamma^2_A(\varpi),\gamma^2_B(\varpi)\}\vee\ \max\{\gamma^2_A(\kappa),\gamma^2_B(\kappa)\})}$
$= (q^2_{A\cap B}(\varpi) \vee q^2_{A\cap B}(y))e^{2\pi i(\gamma^2_{A\cap B}(\varpi)\vee\gamma^2_{A\cap B}(\kappa))}$
$= q^2_{A\cap B}(\varpi)e^{2\pi i\gamma^2_{A\cap B}(\varpi)} \vee\ q^2_{A\cap B}(\kappa)e^{2\pi i\gamma^2_{A\cap B}(y)}$
$= (\mathbb{N}^2_{A\cap B}(\varpi) \vee\ \mathbb{N}^2_{A\cap B}(\kappa))$.
■

The union of two CPFSG is not necessary a CPFSG, see the following example.

Example 2. *Let* $(\mathbb{U}, \square) = (\mathbb{Z}, +)$ *be a group, also* $\varphi_1 = 3\mathbb{Z}$ *and* $\varphi_2 = 2\mathbb{Z}$ *be two CPFSG of* $\mathbb{Z}$. *Where,* $\varphi_j = (\mathbb{M}_{\varphi_j} = p_{\varphi_j}(\varpi)e^{2\pi i\alpha_{\varphi_j}(\varpi)},\ \mathbb{N}_{\varphi_j} = q_{\varphi_j}(\varpi)e^{2\pi i\gamma_{\varphi_j}(\varpi)})$*;* $j = 1,\ 2$.

They defined by:
$$\mathbb{M}_{\varphi_1}(\varpi) = \begin{cases} 0.4e^{2\pi i\ 0.5}: & \varpi\ \in\ 3\mathbb{Z} \\ 0.0e^{2\pi i\ 0.0}: & elsewhere \end{cases}$$
$$\mathbb{N}_{\varphi_1}(\varpi) = \begin{cases} 0.0e^{2\pi i\ 0.0}: & \varpi\ \in\ 3\mathbb{Z} \\ 0.5e^{2\pi i\ 0.5}: & elsewhere \end{cases}$$
$$\mathbb{M}_{\varphi_2}(\varpi) = \begin{cases} 0.04e^{2\pi i\ 0.1}: & \varpi\ \in\ 2\mathbb{Z} \\ 0.0e^{2\pi i\ 0.0}: & elsewhere \end{cases}$$
$$\mathbb{N}_{\varphi_2}(\varpi) = \begin{cases} 0.3e^{2\pi i\ 0.2}: & \varpi\ \in\ 2\mathbb{Z} \\ 0.4e^{2\pi i\ 0.3}: & elsewhere \end{cases}$$
For $\varphi = \varphi_1 \cup \varphi_2$, *we get:*
$$\mathbb{M}_{\varphi}(\varpi) = \begin{cases} 0.4e^{2\pi i\ 0.5}: & \varpi\ \in\ 3\mathbb{Z} \\ 0.04e^{2\pi i\ 0.1}: & \varpi\ \in\ 2\mathbb{Z} - 3\mathbb{Z} \\ 0.0e^{2\pi i\ 0.0}: & elsewhere \end{cases}$$
$$\mathbb{N}_{\varphi}(\varpi) = \begin{cases} 0.0e^{2\pi i\ 0.0}: & \varpi\ \in\ 3\mathbb{Z} \\ 0.03e^{2\pi i\ 0.2}: & \varpi\ \in\ 2\zeta - 3\mathbb{Z} \\ 0.4e^{2\pi i\ 0.3}: & elsewhere \end{cases}$$
Here, $\varpi_1 = 9$ *and* $\varpi_2 = -2$, *then:*
$\mathbb{M}^2_{\varphi}(9\ +\ -2) = \mathbb{M}^2_{\varphi}(7) = 0.0e^{2\pi i\ 0.0}$, *and*

$$\begin{aligned}&\mathbb{M}^2_{\varphi}(9) \wedge \mathbb{M}^2_{\varphi}(-2)\\ &= \min\{(0.4)^2e^{2\pi i\ (0.5)^2}, (0.04)^2e^{2\pi i\ (0.1)^2}\}\\ &= \min\{0.16, 0.0016\}e^{2\pi i\ \min\{0.25,0.01\}}\\ &= 0.0016e^{2\pi i\ 0.01}\end{aligned}$$

Hence, $0.0e^{2\pi i\ 0.0} \ngeq 0.0016e^{2\pi i\ 0.01}$
i.e. $\mathbb{M}^2_{\varphi}(9\ +\ -2) \ngeq \mathbb{M}^2_{\varphi}(9) \wedge \mathbb{M}^2_{\varphi}(-2)$
Similarly for non-membership, we get that (after calculation):
$\mathbb{N}^2_{\varphi}(9\ +\ -2) \nleq \mathbb{N}^2_{\varphi}(9) \vee \mathbb{N}^2_{\varphi}(-2)$, *where* $0.16e^{2\pi i\ 0.09} \nleq 0.09e^{2\pi i\ 0.04}$. *Therefore,* $\varphi = \varphi_1 \cup \varphi_2 = (p_{\varphi}(\varpi)e^{2\pi i\alpha_{\varphi}(\varpi)},\ q_{\varphi}(\varpi)e^{2\pi i\gamma_{\varphi}(\varpi)})$ *is not a CPFSG of* $(\mathbb{Z}, +)$.

Proposition 4. *For a CPFS* $\varphi = (\mathbb{M} = pe^{2\pi i\alpha}, \mathbb{N} = qe^{2\pi i\gamma})$ *of a group* $(\mathbb{U}, \square)$. *Then* $\mathbb{M}^2(\varpi\square\varpi\square\ldots\square\varpi) = p^2(\varpi\square\varpi\square\ldots\square\varpi)e^{2\pi i\alpha^2(\varpi\square\varpi\square\ldots\square\varpi)}$
$\geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} = \mathbb{M}^2(\varpi)$, *where* $p^2(\varpi\square\varpi\square\ldots\square\varpi) \geq p^2(\varpi)$ *and* $\alpha^2(\varpi\square\varpi\square\ldots\square\varpi) \geq \alpha^2(\varpi)$. *Also,*
$\mathbb{N}^2(\varpi\square\varpi\square\ldots\square\varpi)$
$= q^2(\varpi\square\varpi\square\ldots\square\varpi)e^{2\pi i\gamma^2(\varpi\square\varpi\square\ldots\square\varpi)}$
$\leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)} = \mathbb{N}^2(\varpi)$,
where $q^2(\varpi\square\varpi\square\ \ldots\square\varpi) \leq q^2(\varpi)$ *and*
$\gamma^2(\varpi\square\varpi\square\ \ldots\square\varpi) \leq \gamma^2(\varpi)$.

Proof. By induction the results will follow, such that $p^2(\varpi\square\varpi)e^{2\pi i\alpha^2(\varpi\square\varpi)} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)}$, where $p^2(\varpi\square\varpi) \geq p^2(\varpi)$ and $\alpha^2(\varpi\square\varpi) \geq \alpha^2(\varpi)$. Also, $q^2(\varpi\square\varpi)e^{2\pi i\gamma^2(\varpi\square\varpi)} \leq q^2(\varpi)e^{2\pi i\gamma^2(\varpi)}$, where $q^2(\varpi\square\varpi) \leq q^2(\varpi)$ and $\gamma^2(\varpi\square\varpi) \leq \gamma^2(\varpi)$. ■

Theorem 3.2. *For a CPFS $\varphi = (\mathbb{M} = pe^{2\pi i\alpha}, \mathbb{N} = qe^{2\pi i\gamma})$ of a group $(\mathbb{U}, \square)$. The set $\mathbb{L} = \{\varpi \in \mathbb{U} : p^2(id)e^{2\pi i\alpha^2(id)} = p^2(\varpi)e^{2\pi i\alpha^2(\varpi)}$ and $q^2(id)e^{2\pi i\gamma^2(id)} = q^2(\varpi)e^{2\pi i\gamma^2(\varpi)}\}$, is a subgroup of $\mathbb{U}$, where id is the identity of it.*

Proof. At first, we have $id \in \mathbb{L}$, hence $\mathbb{L}$ is not empty. Moreover, we need to show that $\varpi\square\kappa^{-1} \in \mathbb{L}$ for all $x,\ kappa \in \mathbb{U}$.
Assume that $\varpi,\ kappa \in \mathbb{L}$, where φ is CPFSG of $\mathbb{U}$, then, by Proposition 2, $p^2(\varpi\square\kappa^{-1})e^{2\pi i\alpha^2(\varpi\square\kappa^{-1})} \geq p^2(\varpi)e^{2\pi i\alpha^2(\varpi)} \wedge p^2(\kappa)e^{2\pi i\alpha^2(y)} = p^2(id)e^{2\pi i\alpha^2(id)}$, according to definition of $\mathbb{L}$. But, $\varpi\square\kappa^{-1} \in \varphi$, hence $p^2(id)e^{2\pi i\alpha^2(id)} \geq p^2(\varpi\square\kappa^{-1})e^{2\pi i\alpha^2(\varpi\square\kappa^{-1})}$, by Proposition 1. So that equality holds and $p^2(id)e^{2\pi i\alpha^2(id)} = p^2(\varpi\square\kappa^{-1})e^{2\pi i\alpha^2(\varpi\square\kappa^{-1})})$. Similarly, we can prove that $q^2(id)e^{2\pi i\gamma^2(id)} = q^2(\varpi\square\kappa^{-1})e^{2\pi i\gamma^2(\varpi\square\kappa^{-1})}$, by Proposition 1 and Proposition 2. So that $\varpi\square\kappa^{-1} \in \mathbb{L}$ and $\mathbb{L}$ is subgroup of $\mathbb{U}$. ■

4 Complex Pythagorean Fuzzy Normal Subgroup

In this section, we define complex Pythagorean fuzzy normal subgroup (CPFNSG) and gives equivalent conditions and some properties for it.

Definition 11. *Let $\varphi = (\mathbb{M} = pe^{2\pi i\alpha}, \mathbb{N} = qe^{2\pi i\gamma})$ be a CPFSG of a group $(\mathbb{U}, \square)$. Then for $\kappa \in \mathbb{U}$, the complex Pythagorean fuzzy left coset of φ is the CPFS $\kappa\varphi = (\kappa\mathbb{M} = \kappa pe^{2\pi i\ \kappa\alpha}, \kappa\mathbb{N} = \kappa q\ e^{2\pi i\ \kappa\gamma})$, which defined for membership by, $(\kappa\mathbb{M})^2(\varpi) = \kappa p^2(\varpi)e^{2\pi i\ \kappa\alpha^2(\varpi)} = p^2(\kappa^{-1}\ \square\varpi)e^{2\pi i\alpha^2(\kappa^{-1}\ \square\varpi)} = \mathbb{M}^2(\kappa^{-1}\ \square\ \varpi)$. Also, for nonmembership it defined by, $(\kappa\mathbb{N})^2(\varpi) = \kappa q^2(\varpi)e^{2\pi i\ \kappa\gamma^2(\varpi)} = q^2(\kappa^{-1}\ \square\varpi)e^{2\pi i\gamma^2(\kappa^{-1}\ \square\varpi)} = \mathbb{N}^2(\kappa^{-1}\ \square\ \varpi)$.*

In the same manner, the complex Pythagorean fuzzy right coset of φ is the CPFS $\varphi\kappa = (\mathbb{M}\kappa, \mathbb{N}\kappa)$ and is defined by $(\mathbb{M}\kappa)^2(\varpi) = \mathbb{M}^2(\varpi\square\kappa^{-1})$ and $(\mathbb{N}\kappa)^2(\varpi) = \mathbb{N}^2(\varpi\square\kappa^{-1})$, for membership and non-membership, respectively.

Definition 12. *Let $\varphi = (\mathbb{M}(\varpi) = p(\varpi)e^{2\pi i\alpha(\varpi)}, \mathbb{N}(\varpi) = q(\varpi)e^{2\pi i\gamma(\varpi)})$ be a CPFSG of a group $(\mathbb{U}, \square)$. Then φ is a CPFNSG, of the group $(\mathbb{U}, \square)$ if every complex Pythagorean fuzzy left coset is complex pythagorean fuzzy right coset of φ in $\mathbb{U}$, equivalently, $\kappa\varphi = \varphi\kappa$.*

Example 3. *Let $(\mathbb{U}, \square) = (\mathbb{Z}_5, +_5)$ be a group with addition integer modulo 5. Define a CPFS $\varphi = (\mathbb{M}(\varpi) = p(\varpi)e^{2\pi i\alpha(\varpi)}, \mathbb{N}(\varpi) = q(\varpi)e^{2\pi i\gamma(\varpi)})$, as follows:*

$$\mathbb{M}(\varpi) = \begin{cases} 0.8e^{2\pi i\ 0.5} : & \varpi = 0 \\ 0.7e^{2\pi i\ 0.6} : & \varpi = 1 \\ 0.7e^{2\pi i\ 0.8} : & \varpi = 2 \\ 0.2e^{2\pi i\ 0.5} : & \varpi = 3 \\ 0.9e^{2\pi i\ 0.8} : & \varpi = 4 \end{cases}$$

$$\mathbb{N}(\varpi) = \begin{cases} 0.1e^{2\pi i\ 0.3} : & \varpi = 0 \\ 0.6e^{2\pi i\ 0.5} : & \varpi = 1 \\ 0.5e^{2\pi i\ 0.4} : & \varpi = 2 \\ 0.5e^{2\pi i\ 0.7} : & \varpi = 3 \\ 0.3e^{2\pi i\ 0.3} : & \varpi = 4 \end{cases}$$

Now, we need to show that φ is a CPFSG of $\mathbb{U}$. For $\varpi = 1 \in \mathbb{U}$, the complex Pythagorean fuzzy left coset of φ is, the CPFS $1\varphi = (1\mathbb{M},\ 1\mathbb{N})$ and defined by:

$$\begin{aligned}(1\mathbb{M})^2(\varpi) &= \mathbb{M}^2(1^{-1}+_5\varpi) \\ &= p^2(1^{-1}+_5\varpi)e^{2\pi i\alpha^2(1^{-1}+_5\varpi)}\end{aligned}$$

and

$$\begin{aligned}(1\mathbb{N})^2(\varpi) &= \mathbb{N}^2(1^{-1}+_5\varpi) \\ &= q^2(1^{-1}+_5\varpi)e^{2\pi i\gamma^2(1^{-1}+_5\varpi)}\end{aligned}$$

Similarly, the CPF right coset of φ is, the CPFS $\varphi 1 = (\mathbb{M}1,\ \mathbb{N}1)$ and defined by:

$$\begin{aligned}(\mathbb{M}1)^2(\varpi) &= \mathbb{M}^2(\varpi+_5 1^{-1}) \\ &= p^2(\varpi+_5 1^{-1})e^{2\pi i\alpha^2(\varpi+_5 1^{-1})}\end{aligned}$$

and

$$\begin{aligned}(\mathbb{N}1)^2(\varpi) &= \mathbb{N}^2(\varpi+_5 1^{-1}) \\ &= q^2(\varpi+_5 1^{-1})e^{2\pi i\gamma^2(\varpi+_5 1^{-1})}\end{aligned}$$

Assume that $\varpi = 0$, then:

$$\begin{aligned}(1\mathbb{M})^2(0) &= \mathbb{M}^2(1^{-1}+_5 0) \\ &= p^2(1^{-1}+_5 0)e^{2\pi i\alpha^2(1^{-1}+_5 0)} \\ &= p^2(4+_5 0)e^{2\pi i\alpha^2(4+_5 0)} = \\ p^2(4)e^{2\pi i\alpha^2(4)} &= 0.9e^{2\pi i\ 0.8}\end{aligned}$$

and

$$\begin{aligned}(1\mathbb{N})^2(0) &= \mathbb{N}^2(1^{-1}+_5 0) \\ &= q^2(1^{-1}+_5 0)e^{2\pi i\gamma^2(1^{-1}+_5 0)} \\ &= q^2(4+_5 0)e^{2\pi i\gamma^2(4+_5 0)} = \\ q^2(4)e^{2\pi i\gamma^2(4)} &= 0.3e^{2\pi i\ 0.3}\end{aligned}$$

In addition, we can find $(\mathbb{M}1)^2(0)$ equal $0.9e^{2\pi i\ 0.8}$, and $(\mathbb{N}1)^2(0)$ equal $0.3e^{2\pi i\ 0.3}$. Hence, $(1\mathbb{M})^2(0) = (\mathbb{M}1)^2(0)$ and $(1\mathbb{N})^2(0) = (\mathbb{N}1)^2(0)$. For $\varpi = 1, 2, 3, 4$, it is easy to check that $(1\mathbb{M})^2(\varpi) = (\mathbb{M}1)^2(\varpi)$ and $(1\mathbb{N})^2(\varpi) = (\mathbb{N}1)^2(\varpi)$, that is $1\varphi = \varphi 1$.

In the same manner, we can show that, $\varpi\varphi = \varphi\varpi$ for all $\varpi \in \mathbb{U}$. Hence, $\varphi = (\mathbb{M},\ \mathbb{N})$ is a CPFNSG of the group $(\mathbb{Z}_5, +_5)$.

Note that, φ is not CIFS, where $\mathbb{M}(1)+\mathbb{N}(1)=0.7e^{2\pi i\ 0.6}+0.6e^{2\pi i\ 0.5}$ does not satisfied the conditions; $0.7+0.6=1.3\nleq 1$ and $0.6+0.5=1.1\nleq 1$.

Proposition 5. *Let* $\varphi=(\mathbb{M}=pe^{2\pi i\alpha},\mathbb{N}=qe^{2\pi i\gamma})$ *be a CPFSG of a group* $(\mathbb{U},\square)$. *Then* φ *is a CPFNSG of* $\mathbb{U}$ *if and only if* $\mathbb{M}^2(\varpi\square\kappa)=p^2(\varpi\square\kappa)e^{2\pi i\alpha^2(\varpi\square\kappa)}=p^2(\kappa\square\varpi)e^{2\pi i\alpha^2(\kappa\square\varpi)}=\mathbb{M}^2(\kappa\square\varpi)$ *and* $\mathbb{N}^2(\varpi\square\kappa)=q^2(\varpi\square\kappa)e^{2\pi i\gamma^2(\varpi\square\kappa)}=q^2(\kappa\square\varpi)e^{2\pi i\gamma^2(\kappa\square\varpi)}=\mathbb{N}^2(\kappa\square\varpi)$.

Proof. $\Rightarrow$ Assume that φ is a CPFNSG of $\mathbb{U}$, then $(\kappa\mathbb{M})^2(\varpi)=(\mathbb{M}\kappa)^2(\varpi)$, for all $\varpi,\kappa\in\mathbb{U}$. Equivalently, $\mathbb{M}^2(\kappa^{-1}\square\varpi)=p^2(\kappa^{-1}\square\varpi)e^{2\pi i\alpha^2(\kappa^{-1}\square\varpi)}=p^2(\varpi\square\kappa^{-1})e^{2\pi i\alpha^2(\varpi\square\kappa^{-1})}=\mathbb{M}^2(\varpi\square\kappa^{-1})$. Hence, $\mathbb{M}^2(\kappa\square\varpi)=p^2((\kappa^{-1})^{-1}\square\varpi)e^{2\pi i\alpha^2((\kappa^{-1})^{-1}\square\varpi)}=p^2(\varpi\square(\kappa^{-1})^{-1})e^{2\pi i\alpha^2(\varpi\square(\kappa^{-1})^{-1})}=p^2(\varpi\square\kappa)e^{2\pi i\alpha^2(\varpi\square\kappa)}=\mathbb{M}^2(\varpi\square\kappa)$. Moreover, by similar method we can verify that $\mathbb{N}^2(\varpi\square\kappa)=\mathbb{N}^2(\kappa\square\varpi)$.

$\Leftarrow$ Assume that $\zeta=\varpi^{-1}$, then for arbitrary $\varpi,\kappa\in\mathbb{U}$. We have $\mathbb{M}^2(\varpi\square\kappa)=\mathbb{M}^2(\kappa\square\varpi)$, hence $\mathbb{M}^2(\zeta^{-1}\square\kappa)=\mathbb{M}^2(\kappa\square\zeta^{-1})$ for any $\zeta,\ \kappa\in\mathbb{U}$. So that, $(\zeta\mathbb{M})^2(\kappa)=(\mathbb{M}\zeta)^2(\kappa)$. Similarly, we can prove that $(\zeta\mathbb{N})^2(\kappa)=(\mathbb{N}\zeta)^2(\kappa)$, then $\zeta\varphi=\varphi\zeta$ for all $\zeta\in\mathbb{U}$, which implies that φ is CPFNSG of a group $(\mathbb{U},\square)$. ■

Proposition 6. *For a group* $(\mathbb{U},\square)$ *that is defined on CPFSG,* $\varphi=(\mathbb{M}=pe^{2\pi i\alpha},\mathbb{N}=qe^{2\pi i\gamma})$. *Then* φ *is a CPFNSG of* $\mathbb{U}$ *if and only if* $\mathbb{M}^2(\varpi)=p^2(\varpi)e^{2\pi i\alpha^2(\varpi)}=p^2(a\square\varpi\square a^{-1})e^{2\pi i\alpha^2(a\square\varpi\square a^{-1})}=\mathbb{M}^2(a\square\varpi\square a^{-1})$*, and* $\mathbb{N}^2(\varpi)=q^2(\varpi)e^{2\pi i\gamma^2(\varpi)}=q^2(a\square\varpi\square a^{-1})e^{2\pi i\gamma^2(a\square\varpi\square a^{-1})}=\mathbb{N}^2(a\square\varpi\square a^{-1})$*, for all* $a,\varpi\in\mathbb{U}$

Proof. At first consider, $\mathbb{M}^2(x)=p^2(\varpi)e^{2\pi i\alpha^2(\varpi)}=p^2(\varpi\square id)e^{2\pi i\alpha^2(\varpi\square id)}=p^2(\varpi\square a\square a^{-1})e^{2\pi i\alpha^2(\varpi\square a\square a^{-1})}=\mathbb{M}^2(\varpi\square(a\square a^{-1}))=\mathbb{M}^2((\varpi\square a)\square a^{-1})=p^2((\varpi\square a)\square a^{-1})e^{2\pi i\alpha^2((\varpi\square a)\square a^{-1})}=p^2(a^{-1}\square(\varpi\square a))e^{2\pi i\alpha^2(a^{-1}\square(\varpi\square a))}=\mathbb{M}^2(a^{-1}\square(\varpi\square a))$, whereas φ is CPFNSG of $\mathbb{U}$. But $a=(a^{-1})^{-1}$ and by similarity $\mathbb{M}^2(\varpi)=\mathbb{M}^2(a\square\varpi\square a^{-1})$. Also, it is easy to show that $\mathbb{N}^2(\varpi)=\mathbb{N}^2(a\square\varpi\square a^{-1})$ too.

Conversely, $\mathbb{M}^2(a\square\varpi)=p^2(a\square\varpi\square id)e^{2\pi i\alpha^2(a\square\varpi\square id)}=p^2(a\square(\varpi\square a)\square a^{-1})e^{2\pi i\alpha^2(a\square(\varpi\square a)\square a^{-1})}=p^2(\varpi\square a)e^{2\pi i\alpha^2(\varpi\square a)}=\mathbb{M}^2(\varpi\square a)$. Also, it is easy to show that $\mathbb{N}^2(a\square\varpi)=\mathbb{N}^2(\varpi\square a)$. Then by previous proposition, φ is CPFNSG of $\mathbb{U}$. ■

Theorem 4.1. *Let* φ *be a CPFNSG of a group* $(\mathbb{U},\square)$. *Then the set* $\mathbb{L}=\{\varpi\in\mathbb{U}:\ p^2(id)e^{2\pi i\alpha^2(id)}=p^2(\varpi)e^{2\pi i\alpha^2(\varpi)}$ *and* $q^2(id)e^{2\pi i\gamma^2(id)}=q^2(\varpi)e^{2\pi i\gamma^2(\varpi)}\}$*, is a normal subgroup of* $\mathbb{U}$*, where id is the identity of it.*

Proof. At first $id\in\mathbb{L}$, i.e. $\mathbb{L}$ is not empty. Moreover, it is subgroup of $\mathbb{U}$, by Theorem 3.2. So that, $p^2(id)e^{2\pi i\alpha^2(id)}=p^2(\varpi\square\kappa^{-1})e^{2\pi i\alpha^2(\varpi\square\kappa^{-1})}$ and $q^2(id)e^{2\pi i\gamma^2(id)}=q^2(\varpi\square\kappa^{-1})e^{2\pi i\gamma^2(\varpi\square\kappa^{-1})}$. But, φ is a CPFNSG of $(\mathbb{U},\square)$. Then $p^2(\varpi\square\kappa^{-1})e^{2\pi i\alpha^2(\varpi\square\kappa^{-1})}=p^2(\kappa^{-1}\square\varpi)e^{2\pi i\alpha^2(\kappa^{-1}\square\varpi)}$ and $q^2(\varpi\square\kappa^{-1})e^{2\pi i\gamma^2(\varpi\square\kappa^{-1})}=q^2(\kappa^{-1}\square\varpi)e^{2\pi i\gamma^2(\kappa^{-1}\square\varpi)}$. Hence, $(\kappa^{-1}\square\varpi)\in\mathbb{L}$ and $\mathbb{L}$ is a normal subgroup of $\mathbb{U}$. ■

5 Homomorphism on Complex Pythagorean Fuzzy Subgroup

In this section, we discuss the effect of homomorphism on CPFSG.

Definition 13. *A homomorphism function* $f:\mathbb{U}\to\mathbb{V}$ *from group* $\mathbb{U}$ *to group* $\mathbb{V}$. *Let* **A** *be CPFSG of* $\mathbb{U}$ *and* **B** *be CPFSG of* $\mathbb{V}$. *Let* $\varpi\in\mathbb{U}$ *and* $\zeta\in\mathbb{V}$*, then we have:*
$f(\mathbf{A})(\zeta)=\{(\zeta,f(\mathbb{M}_A)(\zeta),f(\mathbb{N}_A)(\zeta))\}$*, is the image of* **A***, where:*

$$f(\mathbb{M}_A^2)=\begin{cases}\sup\limits_{\varpi\in f^{-1}(\zeta)}\mathbb{M}_A^2(\varpi) & ,\ f(\varpi)=\zeta\\ 0 & ,\ \textit{otherwise.}\end{cases}$$

$$=\begin{cases}(\sup\limits_{\varpi\in f^{-1}(\zeta)}p_A^2(\varpi))\cdot e^{2\pi i(\sup\limits_{\varpi\in f^{-1}(\zeta)}\alpha_A^2(\varpi))} & ,f(\varpi)=\zeta\\ 0\ e^{2\pi i\ 0} & ,\textit{otherwise.}\end{cases}$$

$$f(\mathbb{N}_A^2)=\begin{cases}\inf\limits_{\varpi\in f^{-1}(\zeta)}\mathbb{N}_A^2(\varpi) & ,\ f(\varpi)=\zeta\\ 1 & ,\ \textit{otherwise.}\end{cases}$$

$$=\begin{cases}(\inf\limits_{\varpi\in f^{-1}(\zeta)}q_A^2(\varpi))\cdot e^{2\pi i(\inf\limits_{\varpi\in f^{-1}(\zeta)}\gamma_A^2(\varpi))} & ,f(\varpi)=\zeta\\ 1 & ,\textit{otherwise.}\end{cases}$$

And the set of pre-image of $\mathbf{B}$ *is* $f^{-1}(\mathbf{B})(\varpi)$ $= \{(\varpi, f^{-1}(\mathbb{M}_B)(\varpi), f^{-1}(\mathbb{N}_B)(\varpi))\}$, *where:*

$$f^{-1}(\mathbb{M}_B^2)(\varpi) = (\mathbb{M}_B)^2(f(\varpi)) = p_B^2(f(\varpi))e^{2\pi\alpha_B^2(f(\varpi))},$$

$$f^{-1}(\mathbb{N}_B^2)(\varpi) = (\mathbb{N}_B)^2(f(\varpi)) = q_B^2(f(\varpi))e^{2\pi\gamma_B^2(f(\varpi))}, \forall \varpi \in \mathbb{U}.$$

Lemma 5.1. *Let* $f: \mathbb{U} \to \mathbb{V}$ *be a homomorphism from group* $\mathbb{U}$ *to group* $\mathbb{V}$, *and let* $\mathbf{A}$ *be CPFSG of* $\mathbb{U}$, $\mathbf{B}$ *be CPFSG of* $\mathbb{V}$. *Then:*

1) $f(\mathbb{M}_A^2)(\zeta) = f(p_A^2)(\zeta)e^{2\pi i f(\alpha_A^2)(\zeta)}$ $\forall \zeta \in \mathbb{V}$.

2) $f(\mathbb{N}_A^2)(\zeta) = f(q_A^2)(\zeta)e^{2\pi i f(\gamma_A^2)(\zeta)}$ $\forall \zeta \in \mathbb{V}$.

3) $f^{-1}(\mathbb{M}_B^2)(\varpi) = f^{-1}(p_B^2)(\varpi)e^{2\pi i f^{-1}(\alpha_B^2)(\varpi)}$ $\forall \varpi \in \mathbb{U}$.

4) $f^{-1}(\mathbb{N}_B^2)(\varpi) = f^{-1}(q_B^2)(\varpi)e^{2\pi i f^{-1}(\gamma_B^2)(\varpi)}$ $\forall \varpi \in \mathbb{U}$.

Proof.

1) $$f(\mathbb{M}_A^2)(\zeta) = \sup_{\varpi \in f^{-1}(\zeta)} \{\mathbb{M}_A^2(\varpi);\ f(\varpi) = \zeta\} = \sup_{\varpi \in f^{-1}(\zeta)} \{p_A^2(\varpi)e^{2\pi i\alpha_A^2(\varpi)};\ f(\varpi) = \zeta\} = \sup_{\varpi \in f^{-1}(\zeta)} \{p_A^2(\varpi)\}e^{2\pi i \sup_{\varpi \in f^{-1}(\zeta)}\{\alpha_A^2(\varpi)\}} = f(p_A^2)(\zeta)e^{2\pi i f(\alpha_A^2)(\zeta)}.$$

2) $$f(\mathbb{N}_A^2)(\zeta) = \inf_{\varpi \in f^{-1}(\zeta)} \{\mathbb{N}_A^2(\varpi);\ f(\varpi) = \zeta\} = \inf_{\varpi \in f^{-1}(\zeta)} \{q_A^2(\varpi)e^{2\pi i\gamma_A^2(\varpi)};\ f(\varpi) = \zeta\} = \inf_{\varpi \in f^{-1}(\zeta)} \{q_A^2(\varpi)\}e^{2\pi i \inf_{\varpi \in f^{-1}(\zeta)}\{\gamma_A^2(\varpi)\}} = f(q_A^2)(\zeta)e^{2\pi i f(\gamma_A^2)(\zeta)}.$$

3) $$f^{-1}(\mathbb{M}_B^2)(\varpi) = (\mathbb{M}_B)^2(f(\varpi)) = p_B^2(f(\varpi))e^{2\pi\alpha_B^2(f(\varpi))} = f^{-1}(p_B^2)(\varpi)e^{2\pi i f^{-1}(\alpha_B^2)(\varpi)}.$$

4) $$f^{-1}(\mathbb{N}_B^2)(x) = (\mathbb{N}_B)^2(f(x)) = q_B^2(f(\varpi))e^{2\pi\gamma_B^2(f(\varpi))} = f^{-1}(q_B^2)(\varpi)e^{2\pi i f^{-1}(\gamma_B^2)(\varpi)}.$$

■

Example 4. *Let* $(\mathbb{Z}_5, +_5)$ *and* $(\mathbb{Z}, +)$ *be complex Pythagorean fuzzy group (CPFG), where we define* $(\mathbb{Z}_5, +_5)$ *as in example 3.*
The map $f: (\mathbb{Z}, +) \to (\mathbb{Z}_5, +_5)$ *is complex Pythagorean fuzzy homomorphism. Consider* $\mathbf{A} = \{1, 2, 7, 9, 13, 14\} \subseteq \mathbb{Z}$, *then* $f(\mathbf{A}) = (\varpi, f(\mathbb{M}_A)(\varpi),\ f(\mathbb{N}_A)(\varpi))$. *Whereas,*

1) $$\begin{aligned} f(\mathbb{M}_A^2)(\zeta) &= \sup_{\varpi \in f^{-1}(\zeta)} \{\mathbb{M}_A^2(\varpi);\ f(\varpi) = \zeta\ (mod\ 5)\ \} \\ &= \sup\{\mathbb{M}_A^2(1), \mathbb{M}_A^2(2), \mathbb{M}_A^2(7), \mathbb{M}_A^2(9), \mathbb{M}_A^2(13), \mathbb{M}_A^2(14)\} \\ &= \sup\{p_A^2(1)e^{2\pi i\alpha_A^2(1)}, p_A^2(2)e^{2\pi i\alpha_A^2(2)}, p_A^2(7)e^{2\pi i\alpha_A^2(7)}, p_A^2(9)e^{2\pi i\alpha_A^2(9)}, p_A^2(13)e^{2\pi i\alpha_A^2(13)}, p_A^2(14)e^{2\pi i\alpha_A^2(14)}\} \\ &= \sup\{p_A^2(1), \ldots, p_A^2(14)\}e^{2\pi i \sup\{\alpha_A^2(1), \ldots, \alpha_A^2(14)\}} \\ &= \sup\{0.49, 0.81, 0.04\}e^{2\pi i\ \sup\{0.36, 0.64, 0.25\}} \\ &= 0.81e^{2\pi i\ 0.64}. \end{aligned}$$

2) $$\begin{aligned} f(\mathbb{N}_A^2)(\zeta) &= \inf_{\varpi \in f^{-1}(\zeta)} \{\mathbb{N}_A^2(\varpi);\ f(\varpi) = \zeta\ (mod\ 5)\ \} \\ &= \inf\{\mathbb{N}_A^2(1), \mathbb{N}_A^2(2), \mathbb{N}_A^2(7), \mathbb{N}_A^2(9), \mathbb{N}_A^2(13), \mathbb{N}_A^2(14)\} \\ &= \inf\{q_A^2(1)e^{2\pi i\gamma_A^2(1)}, q_A^2(2)e^{2\pi i\gamma_A^2(2)}, q_A^2(7)e^{2\pi i\gamma_A^2(7)}, q_A^2(9)e^{2\pi i\gamma_A^2(9)}, q_A^2(13)e^{2\pi i\gamma_A^2(13)}, q_A^2(14)e^{2\pi i\gamma_A^2(14)}\} \\ &= \inf\{q_A^2(1), \ldots, q_A^2(14)\}e^{2\pi i \inf\{\gamma_A^2(1), \ldots, \gamma_A^2(14)\}} \\ &= \inf\{0.36, 0.25, 0.09\}e^{2\pi i\ \inf\{0.25, 0.16, 0.49, 0.09\}} \\ &= 0.09e^{2\pi i\ 0.09} \end{aligned}$$

Theorem 5.2. *Let* $f: \mathbb{U} \xrightarrow{epimorphism} \mathbb{V}$, *from* $(\mathbb{U}, \square_1)$ *to* $(\mathbb{V}, \square_2)$, *and let* $\mathbf{A}$ *be CPFSG of* $\mathbb{U}$. *Then* $f(\mathbf{A})$ *is CPFSG of* $\mathbb{V}$.

Proof. For $\mathbf{A} = (\mathbb{M}_{\mathbb{A}} = p_A e^{2\pi i\alpha_A}, \mathbb{N}_{\mathbb{A}} = q_A e^{2\pi i\gamma_A})$ we want to show that $f(\mathbf{A}) = (f(\mathbb{M}_A), f(\mathbb{N}_A)) = (f(p_A)(\zeta)e^{2\pi i f(\alpha_A)(\zeta)}, f(q_A)(\zeta)e^{2\pi i f(\gamma_A)(\zeta)})$ is CPFSG.
Part 1: Since $\mathbf{A}$ is CPFSG, the set $S_1 = \{(\varpi, p_A(\varpi), q_A(\varpi)) : \varpi \in \mathbb{U},\ 0 \le p_A^2(\varpi) + q_A^2(\varpi) \le 1\}$ and $S_2 = \{(\varpi, \alpha_A(\varpi), \gamma_A(\varpi)) : \varpi \in \mathbb{U},\ 0 \le \alpha_A^2(\varpi) + \gamma_A^2(\varpi) \le 1\}$ are the amplitude and phase terms of CPFSG. Then by Theorem[6.1], [25]; f is epimorphism, we have:
i)
a) $$(f(p_A))^2(\zeta_1 \square_2\ \zeta_2) = f(p_A^2)(\zeta_1 \square_2\ \zeta_2) \ge f(p_A^2)(\zeta_1) \wedge f(p_A^2)(\zeta_2) = (f(p_A))^2(\zeta_1) \wedge (f(p_A))^2(\zeta_2).$$

b) $$(f(q_A))^2(\zeta_1 \square_2\ \zeta_2) = f(q_A^2)(\zeta_1 \square_2\ \zeta_2) \le f(q_A^2)(\zeta_1) \vee f(q_A^2)(\zeta_2) = (f(q_A))^2(\zeta_1) \vee (f(q_A))^2(\zeta_2).$$

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c) $(f(\alpha_A))^2(\zeta_1\square_2\,\zeta_2)$
$= f(\alpha_A^2)(\zeta_1\square_2\,\zeta_2)$
$\geq f(\alpha_A^2)(\zeta_1) \wedge f(\alpha_A^2)(\zeta_2)$
$= (f(\alpha_A))^2(\zeta_1) \wedge (f(\alpha_A))^2(\zeta_2)$.

d) $(f(\gamma_A))^2(\zeta_1\square_2\,\zeta_2)$
$= f(\gamma_A^2)(\zeta_1\square_2\,\zeta_2)$
$\leq f(\gamma_A^2)(\zeta_1) \vee f(\gamma_A^2)(\zeta_2)$
$= (f(\gamma_A))^2(\zeta_1) \vee (f(\gamma_A))^2(\zeta_2)$.

ii)
a) $(f(p_A))^2(\zeta^{-1})$
$= f(p_A^2)(z^{-1}) = f(p_A^2)(\zeta)$
$= (f(p_A))^2(\zeta)$.

b) $(f(q_A))^2(\zeta^{-1})$
$= f(q_A^2)(\zeta^{-1})$
$= f(q_A^2)(\zeta)$
$= (f(q_A))^2(\zeta)$.

c) $(f(\alpha_A))^2(\zeta^{-1})$
$= f(\alpha_A^2)(\zeta^{-1})$
$= f(\alpha_A^2)(\zeta)$
$= (f(\alpha_A))^2(\zeta)$.

d) $(f(\gamma_A))^2(\zeta^{-1})$
$= f(\gamma_A^2)(\zeta^{-1})$
$= f(\gamma_A^2)(\zeta)$
$= (f(\gamma_A))^2(\zeta)$.

Part 2: Using part 1 and Lemma 5.1, we have:

1) $(f(\mathbb{M}_A))^2(\zeta_1\square_2\,\zeta_2)$
$= f(\mathbb{M}_A^2)(\zeta_1\square_2\,\zeta_2)$
$= f(p_A^2)(\zeta_1\square_2\,\zeta_2)e^{2\pi i\ f(\alpha_A^2)(\zeta_1\square_2\,\zeta_2)}$
$\geq (f(p_A^2)(\zeta_1) \wedge f(p_A^2)(\zeta_2))e^{2\pi i(f(\alpha_A^2)(\zeta_1)\wedge f(\alpha_A^2)(\zeta_2))}$
$= f(p_A^2)(\zeta_1)e^{2\pi i\ f(\alpha_A^2)(\zeta_1)} \wedge f(p_A^2)(\zeta_2)e^{2\pi i\ f(\alpha_A^2)(\zeta_2)}$
$=f(\mathbb{M}_A^2)(\zeta_1) \wedge f(\mathbb{M}_A^2)(\zeta_2)$
$=\ (f(\mathbb{M}_A))^2(\zeta_1) \wedge (f(\mathbb{M}_A))^2(\zeta_2)$.

2) $(f(\mathbb{N}_A))^2(\zeta_1\square_2\,\zeta_2)$
$= f(\mathbb{N}_A^2)(\zeta_1\square_2\,\zeta_2)$
$=f(q_A^2)(\zeta_1\square_2\,\zeta_2)e^{2\pi i\ f(\gamma_A^2)(\zeta_1\square_2\,\zeta_2)}$
$\leq (f(q_A^2)(\zeta_1) \vee f(q_A^2)(\zeta_2))e^{2\pi i(f(\gamma_A^2)(\zeta_1)\vee f(\gamma_A^2)(\zeta_2))}$
$= f(q_A^2)(\zeta_1)e^{2\pi i\ f(\gamma_A^2)(\zeta_1)} \vee f(q_A^2)(\zeta_2)e^{2\pi i\ f(\gamma_A^2)(\zeta_2)}$
$= f(\mathbb{N}_A^2)(\zeta_1) \vee f(\mathbb{N}_A^2)(\zeta_2)$
$=\ (f(\mathbb{N}_A))^2(\zeta_1) \vee (f(\mathbb{N}_A))^2(\zeta_2)$.

3) $(f(\mathbb{M}_A))^2(\zeta^{-1})$
$= f(\mathbb{M}_A^2)(\zeta^{-1})$
$=f(p_A^2)(\zeta^{-1})e^{2\pi i\ f(\alpha_A^2)(\zeta^{-1})}$
$=f(p_A^2)(\zeta)e^{2\pi i\ f(\alpha_A^2)(\zeta)}$
$=f(\mathbb{M}_A^2)(\zeta) = (f(\mathbb{M}_A))^2(\zeta)$.

4) $(f(\mathbb{N}_A))^2(\zeta^{-1})$
$= f(\mathbb{N}_A^2)(\zeta^{-1})$
$= f(q_A^2)(\zeta^{-1})e^{2\pi i\ f(\gamma_A^2)(\zeta^{-1})}$
$=f(q_A^2)(\zeta)e^{2\pi i\ f(\gamma_A^2)(\zeta)}$
$=f(\mathbb{N}_A^2)(\zeta) = (f(\mathbb{N}_A))^2(\zeta)$.
Hence result is followed. ■

Theorem 5.3. *Let* $f : \mathbb{U} \xrightarrow{isomorphism} \mathbb{V}$*, from* $(\mathbb{U},\square_1)$ *to* $(\mathbb{V},\square_2)$*, and let* $\mathbf{B}$ *be CPFSG of* $\mathbb{V}$*. Then* $f^{-1}(\mathbf{B})$ *is CPFSG of* $\mathbb{U}$*.*

Proof. Let $\mathbf{B} = (\mathbb{M}_{\mathbb{B}} = p_B e^{2\pi i\alpha_B}, \mathbb{N}_{\mathbb{B}} = q_B e^{2\pi i\gamma_B})$ be CPFSG of $\mathbb{V}$, we want to show that $f^{-1}(\mathbf{B}) = (f^{-1}(\mathbb{M}_B), f^{-1}(\mathbb{N}_B))$ $= (f^{-1}(p_B)(\varpi)e^{2\pi i f^{-1}(\alpha_B)(\varpi)}$, $f^{-1}(q_B)(\varpi)e^{2\pi i f^{-1}(\gamma_B)(\varpi)})$ is CPFSG of $\mathbb{U}$, by following Definition 10 of CPFSG.
Part 1: Since $\mathbf{B}$ is CPFSG, the set $S_1 = \{(\zeta, p_B(\zeta), q_B(\zeta)) : \zeta \in \mathbb{V},\ 0 \leq p_B^2(\zeta) + q_B^2(\zeta) \leq 1\}$ and $S_2 = \{(\zeta, \alpha_B(\zeta), \gamma_B(\zeta)) : \zeta \in \mathbb{V},\ 0 \leq \alpha_B^2(\zeta) + \gamma_B^2(\zeta) \leq 1\}$ are the amplitude and phase terms of CPFSG. Then by Theorem[6.2], [25]; f is isomorphism, we have:
i) a) $(f^{-1}(p_B))^2(\varpi_1\square_1\,\varpi_2) = f^{-1}(p_B^2)(\varpi_1\square_1\,\varpi_2) \geq f^{-1}(p_B^2)(\varpi_1) \wedge f^{-1}(p_B^2)(\varpi_2) = (f^{-1}(p_B))^2(\varpi_1) \wedge (f^{-1}(p_B))^2(\varpi_2)$.
b) $(f^{-1}(q_B))^2(\varpi_1\square_1\,\varpi_2) = f^{-1}(q_B^2)(\varpi_1\square_1\,\varpi_2) \leq f^{-1}(q_B^2)(\varpi_1) \vee f^{-1}(q_B^2)(\varpi_2) = (f^{-1}(q_B))^2(\varpi_1) \vee (f^{-1}(q_B))^2(\varpi_2)$.
Hence, similarly for set S_2:
c) $(f^{-1}(\alpha_B))^2(\varpi_1\square_1\,\varpi_2) \geq (f^{-1}(\alpha_B))^2(\varpi_1) \wedge (f^{-1}(\alpha_B))^2(\varpi_2)$.
d) $(f^{-1}(\gamma_B))^2(\varpi_1\square_1\,\varpi_2) \leq (f^{-1}(\gamma_B))^2(\varpi_1) \vee (f^{-1}(\gamma_B))^2(\varpi_2)$.
ii) By same strategy in (i) we get:
a) $(f^{-1}(p_B))^2(\varpi^{-1}) = (f^{-1}(p_B))^2(\varpi)$.
b) $(f^{-1}(q_B))^2(\varpi^{-1}) = (f^{-1}(q_B))^2(\varpi)$.
c) $(f^{-1}(\alpha_B))^2(\varpi^{-1}) = (f^{-1}(\alpha_B))^2(\varpi)$.
d) $(f^{-1}(\gamma_B))^2(\varpi^{-1}) = (f^{-1}(\gamma_B))^2(\varpi)$.

Part 2: Using part 1 and Lemma 5.1, we have:

1) $(f^{-1}(\mathbb{M}_B))^2(\varpi_1\square_1\,\varpi_2)$
$= f^{-1}(\mathbb{M}_B^2)(\varpi_1\square_1\,\varpi_2)$
$= f^{-1}(p_B^2)(\varpi_1\square_1\,\varpi_2)e^{2\pi i\ f^{-1}(\alpha_B^2)(\varpi_1\square_1\,\varpi_2)}$
$\geq (f^{-1}(p_B^2)(\varpi_1) \wedge f^{-1}(p_B^2)(\varpi_2))$
$e^{2\pi i(f^{-1}(\alpha_B^2)(\varpi_1)\wedge f^{-1}(\alpha_B^2)(\varpi_2))}$
$= f^{-1}(p_B^2)(\varpi_1)e^{2\pi i\ f^{-1}(\alpha_B^2)(\varpi_1)} \wedge$
$f^{-1}(p_B^2)(\varpi_2)e^{2\pi i\ f^{-1}(\alpha_B^2)(\varpi_2)}$
$= f^{-1}(\mathbb{M}_B^2)(\varpi_1) \wedge f^{-1}(\mathbb{M}_B^2)(\varpi_2)$
$=\ (f^{-1}(\mathbb{M}_B))^2(\varpi_1) \wedge (f^{-1}(\mathbb{M}_B))^2(\varpi_2)$.

2) Similarly we can show that $(f^{-1}(\mathbb{N}_B))^2(\varpi_1\square_1\,\varpi_2) \leq$

$(f^{-1}(\mathbb{N}_B))^2(\varpi_1) \vee (f^{-1}(\mathbb{N}_B))^2(\varpi_2)$.

3) $(f^{-1}(\mathbb{M}_B))^2(\varpi^{-1})$
$= f^{-1}(\mathbb{M}_B^2)(\varpi^{-1})$
$= f^{-1}(p_B^2)(\varpi^{-1})e^{2\pi i\, f^{-1}(\alpha_B^2)(\varpi^{-1})}$
$= f^{-1}(p_B^2)(\varpi)e^{2\pi i\, f^{-1}(\alpha_B^2)(\varpi)}$
$= f^{-1}(\mathbb{M}_B^2)(\varpi) = (f^{-1}(\mathbb{M}_B))^2(\varpi)$.

4) Similarly we can show that
$(f^{-1}(\mathbb{N}_B))^2(\varpi^{-1}) = (f^{-1}(\mathbb{N}_B))^2(\varpi)$.
Hence the result is followed. ■

Theorem 5.4. *Let* $f : \mathbb{U} \xrightarrow{epimorphism} \mathbb{V}$, *from* $(\mathbb{U}, \Box_1)$ *to* $(\mathbb{V}, \Box_2)$, *and let* **A** *be CPFNSG of* $\mathbb{U}$. *Then* $f(A)$ *is CPFNSG of* $\mathbb{V}$.

Proof. Let **A** be a CPFNSG of $\mathbb{U}$, so that sets S_1 and S_2 are the amplitude and phase terms in the CPFNSG, as in part 1 of the proof of Theorem 5.2. Then, according to Definition 13 and Theorem[6.3], [25], we have:

$(f(\mathbb{M}_A))^2(\zeta_1\Box_2\zeta_2)$
$= f(\mathbb{M}_A^2)(\zeta_1\Box_2\zeta_2)$
$= \sup_{\varpi\in f^{-1}(\zeta_1\Box_2\zeta_2)} \{\mathbb{M}_A^2(\varpi);\ f(\varpi) = \zeta_1\Box_2\zeta_2\}$
$= \sup_{\varpi\in f^{-1}(\zeta_1\Box_2\zeta_2)} \{p_A^2(\varpi)e^{2\pi i\alpha_A^2(\varpi)};$
$f(\varpi) = \zeta_1\Box_2\zeta_2\}$
$= \sup_{\varpi\in f^{-1}(\zeta_1\Box_2\zeta_2)} \{p_A^2(\varpi)\}e^{2\pi i \sup_{\varpi\in f^{-1}(\zeta_1\Box_2\zeta_2)}\{\alpha_A^2(\varpi)\}}$
$= \sup_{\varpi\in f^{-1}(\zeta_2\Box_2\zeta_1)} \{p_A^2(\varpi)\}e^{2\pi i \sup_{\varpi\in f^{-1}(\zeta_2\Box_2\zeta_1)}\{\alpha_A^2(\varpi)\}}$
$= \sup_{\varpi\in f^{-1}(\zeta_2\Box_2\zeta_1)} \{p_A^2(\varpi)e^{2\pi i\alpha_A^2(\varpi)}; f(\varpi) =$
$\zeta_2\Box_2\zeta_1\}$
$= \sup_{\varpi\in f^{-1}(\zeta_2\Box_2\zeta_1)} \{\mathbb{M}_A^2(\varpi); f(\varpi) = \zeta_2\Box_2\zeta_1\}$
$= f(\mathbb{M}_A^2)(\zeta_2\Box_2\zeta_1) = (f(\mathbb{M}_A))^2(\zeta_2\Box_2\zeta_1)$
Also, by same strategy we can show that $(f(\mathbb{N}_A))^2(\zeta_1\Box_2\zeta_2) = (f(\mathbb{N}_A))^2(\zeta_2\Box_2\zeta_1)$. Hence, by Proposition 5 result is followed. ■

Theorem 5.5. *Let* $f : \mathbb{U} \xrightarrow{isomorphism} \mathbb{V}$, *from* $(\mathbb{U}, \Box_1)$ *to* $(\mathbb{V}, \Box_2)$, *and let* **B** *be CPFNSG of* $\mathbb{V}$. *Then* $f^{-1}(\mathbf{B})$ *is CPFNSG of* $\mathbb{U}$.

Proof. Let **B** be a CPFNSG of $\mathbb{V}$, so that sets S_1 and S_2 are the amplitude and phase terms in the CPFNSG, as in part 1 of the proof of Theorem 5.3. Then, according to Definition 13 and Theorem[6.4], [25], we have:

$f^{-1}(\mathbb{M}_B^2)(\varpi_1\Box_1\varpi_2) = (\mathbb{M}_B)^2(f(\varpi_1\Box_1\varpi_2))$
$= p_B^2(f(\varpi_1\Box_1\varpi_2))e^{2\pi\alpha_B^2(f(\varpi_1\Box_1\varpi_2))}$
$= p_B^2(f(\varpi_2\Box_1\varpi_1))e^{2\pi\alpha_B^2(f(\varpi_2\Box_1\varpi_1))}$
$= f^{-1}(p_B^2)(\varpi_2\Box_1\varpi_1)e^{2\pi i f^{-1}(\alpha_B^2)(\varpi_2\Box_1\varpi_1)}$
$= (\mathbb{M}_B)^2(f(\varpi_2\Box_1\varpi_1)) = f^{-1}(\mathbb{M}_B^2)(\varpi_2\Box_1\varpi_1)$
Also, by same strategy we can show that $f^{-1}(\mathbb{N}_B^2)(\varpi_1\Box_1\ \varpi_2) = f^{-1}(\mathbb{N}_B^2)(\varpi_2\Box_1\ \varpi_1)$
Hence, by Proposition 5 result is followed. ■

6 Conclusion

This research generalized the notion of CPFSG and discussed various algebraic attributes of CPFSG. This generalization happened by applying phase terms of complex numbers to the PFSG structure and its conditions. Therefore, CPFSG is considered a generalization of CIFSG and CFSG. Some results between the current concept and CIFSG were improved and discussed. Also, coset, normality, and homeomorphism under complex Pythagorean fuzzy subgroups were introduced and their properties investigated. The limitation of this research is that there are some values indicated out of the range of complex Pythagorean fuzzy sets and subgroups, which can be covered (as future research) by introducing the notion of complex Fearmatean fuzzy subgroups and Q-rung orthonormal fuzzy subgroups. Also, as future research and possible appropriate applications indicate, the need for secure communication between two sides that are using CPF information encourages us to use the algebraic structure of CPFSG to construct a suitable cryptographic primitive and system. Also, we may extend the presented works to some algebraic notions, such as factor groups, rings, fields, and integral domains.

References:

[1] Zadeh L.A., Fuzzy Sets, *Inform. and Control.*, Vol.8, 1965, pp. 338-353.

[2] Zimmermann, H.J., *Fuzzy Set Theory and Its Applications*, Springer Netherlands, 2013. DOI: 10.1007/978-94-010-0646-0

[3] Atanassov, K.T., Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, Vol.20, No.1, 1986, pp. 87-96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[4] Garg, H., A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems, *Appl. Soft Comput.*, Vol.38, 2016, pp. 988-999.

[5] Garg, H., and Arora, R., Generalized and group-based generalized intuitionistic fuzzy soft sets with applications in decision-making, *Applied Intelligence*, Vol. 48, 2018, pp. 343-356. DOI: 10.1007/s10489-017-0981-5

[6] Yager, Ronald R., Pythagorean fuzzy subsets, *2013 Joint IFSA World Congress and NAFIPS*

Annual Meeting (IFSA/NAFIPS), 2013, pp. 57-61.

[7] Naz, S., Ashraf, S., and Akram, M., A Novel Approach to Decision-Making with Pythagorean Fuzzy Information, *Mathematics*, Vol. 6, No. 6, 2018, 95. https://doi.org/10.3390/math6060095.

[8] Ejegwa, P.A., Pythagorean fuzzy set and its application in career placements based on academic performance using max–min–max composition, *Complex & Intelligent Systems*, Vol. 5, 2019, pp. 165–175. https://doi.org/10.1007/s40747-019-0091-6

[9] Ramot, D., Milo, R., Friedman, M., and Kandel, A., Complex fuzzy sets, *IEEE Transactions on Fuzzy Systems*, Vol. 10, No. 2, 2002, pp. 171-186. DOI: 10.1109/91.995119

[10] Greenfield, S., Chiclana, F., and Dick, S., Interval-valued complex fuzzy logic, *IEEE International Conference on Fuzzy Systems, Vancouver, BC, Canada*, 2016, pp. 1–6.

[11] Ramot, D., Friedman, M., Langholz, G., and Kandel, A., Complex fuzzy logic, *IEEE Transactions on Fuzzy Systems*, Vol. 11, No. 4, 2003, pp. 450–461. DOI: 10.1109/TFUZZ.2003.814832

[12] Omolbanin, Y., and Dick, S., A Systematic Review of Complex Fuzzy Sets and Logic, *Fuzzy Sets and Systems*, Vol. 338, 2018, pp. 1-22. https://doi.org/10.1016/j.fss.2017.01.010.

[13] Alkouri, A. U. M., and Salleh, A. R., Complex intuitionistic fuzzy sets, *AIP Conf. Proc.*, Vol. 1482, No. 1, 2012, pp. 464–470.

[14] Alkouri, A. U. M., and Salleh, A. R., Complex Atanassov's intuitionistic fuzzy relation, *Abstract and Applied Analysis*, Vol. 2013, 2013, pp. 1–18. https://doi.org/10.1155/2013/287382

[15] Ullah, K.; Mahmood, T.; Ali, Z.; Jan, N., On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition, *Complex & Intelligent Systems*, Vol. 6, 2019, pp. 15 - 27.

[16] Akram, M., and Naz, S., A novel decision-making approach under complex Pythagorean fuzzy environment, *Mathematical and Computational Applications*, Vol. 24, No. 73, 2019, pp. 1 - 33.

[17] Jin, H., Hussain, A., Ullah, K., and Javed, A., Novel Complex Pythagorean Fuzzy Sets under Aczel–Alsina Operators and Their Application in Multi-Attribute Decision Making, *Symmetry* , Vol.15, No. 1, 2023, 68. https://doi.org/10.3390/sym15010068

[18] Rosenfeld, A., Fuzzy Groups, *Journal of Mathematical Analysis and Applications*, Vol. 35, No. 3, 1971, pp. 512-517. https://doi.org/10.1016/0022-247X(71)90199-5

[19] Biswas, R., Intuitionistic fuzzy subgroup, *Mathematical Forum*, Vol. 10, 1989, pp. 39–44.

[20] Sharma, P. K., Homomorphism of intuitionistic fuzzy groups, *International Mathematical Forum*, Vol. 6, No. 64, 2011, pp. 3169-3178.

[21] Sharma, P. K., Intuitionistic fuzzy Groups, *International Journal of Data Warehousing & Mining*, Vol. 1, No. 1, 2011, pp. 86-94.

[22] Alghazzawi, D., Shuaib, U., Fatima, T., Razaq, A., and Binyamin, M. A., Algebraic Characteristics of Anti-Intuitionistic Fuzzy Subgroups Over a Certain Averaging Operator, *IEEE Access*, Vol. 8, 2020, pp. 205014-205021. doi: 10.1109/ACCESS.2020.3035590

[23] S. Kalaiselvan and S. Vijayabalaji, Soft expert symmetric group and its application in MCDM problem, Symmetry, 14(12) (2022), 2685.

[24] Vijayabalaji, S., Kalaiselvan, S., Atagün, A. O., & Kamaci, H. (2024). A study on intuitionistic fuzzy soft expert groups. Int. J. Open Problems Compt. Math, 17(2).

[25] S. Bhunia, G. Ghorai and Q. Xin, On the characterization of Pythagorean fuzzy subgroups, AIMS Math. 6(1) (2021), 962–978. https://doi.org/10.3934/math.2021058

[26] S. Bhunia and G. Ghorai, A new approach to fuzzy group theory using (α, β)-Pythagorean fuzzy sets, Songklanakarin Journal of Science and Technology 43(1) (2021), 295–306. https://doi. org/10.14456/sjst-psu.2021.38

[27] S. Bhunia, G. Ghorai and Q. Xin, On the fuzzification of Lagrange's theorem in (α, β)-Pythagorean fuzzy environment, AIMS Math. 6(9) (2021), 9290–9308. https://doi.org/10.3934/math. 2021540.

[28] Bhunia, S. U. P. R. I. Y. A., & Ghorai, G. A. N. E. S. H. (2024). An approach to Lagrange's theorem in Pythagorean fuzzy subgroups. Kragujevac J. Math., 48(6), 893-906.

[29] Razaq, A., Alhamzi, G., Razzaque, A., and Garg, H., A Comprehensive Study on Pythagorean Fuzzy Normal Subgroups and Pythagorean Fuzzy Isomorphisms, *Symmetry*, Vol. 14, No. 10, 2022, 2084. https://doi.org/10.3390/sym14102084.

[30] Yager, R. R., Generalized orthopair fuzzy sets, *IEEE Transactions on Fuzzy Systems*, Vol. 25, No. 5, 2017, pp. 1222–1230. DOI: 10.1109/TFUZZ.2016.2604005

[31] Abuhijleh, E.A., Massa'deh, M., Sheimat, A., and Alkouri, A., Complex fuzzy groups based on Rosenfeld's approach, *WSEAS TRANSACTIONS on MATHEMATICS*, Vol. 20, 2021, pp. 368–377. DOI: 10.37394/23206.2021.20.38

[32] Gulzar, M., Mateen, M. H., Algazzawi, D., and kausar, N., A Novel Applications of Complex Intuitionistic Fuzzy Sets in Group Theory, *IEEE Access*, Vol. 8, 2020, pp. 196075 - 196085. DOI:10.1109/ACCESS.2020.3034626

[33] Mateen, M. H., Alsuraiheed, T., & Hmissi, M. (2023). A New Algebraic Structure of Complex Pythagorean Fuzzy Subfield. IEEE Access.

[34] Ali, A., Mateen, M. H., Xin, Q., Alsuraiheed, T., & Alhamzi, G. (2024). (ϵ, δ)-complex anti fuzzy subgroups and their applications. AIMS Mathematics, 9(5), 11580-11595.

Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

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Abdulazeez Alkouri gave the idea. He reviewed the previous works and wrote the draft of the paper till section three. In addition, he wrote section three in the form of a first draft, then he revised the final version of the paper.

Eman A. Abuhijleh has co-operated with A. Alkouri in the draft of the paper till section three, then she checked the reference details.

Eman Almuhur investigated the new ideas and wrote the draft of sections four and five. Also, she finished the first draft of the paper in Latex format.

Ghada Alafifi Played the role of project administration and investigation in this research.

Sana Abu-Ghurra, Prepared the published work by those from the original research group and helped to create the final form of this research after revision stages

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