Linear State Optimal Control Problem with a Stochastic Switching Time

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Abstract: In this paper, we analyse an optimal control problem over a finite horizon with a stochastic switching time, assuming that the two optimal control problems present in its two stages have a particularly simple form called linear state. It is well known that linear state optimal control problems can be solved easily using the HJB equation approach and assuming that the value function is linear in the state. Unfortunately, this simplicity of solution does not extend to the problem with stochastic switching time. We prove that a necessary and sufficient condition for the problem to maintain a linear state structure is to assume that the hazard rate of the switching time depends only on the temporal variable. Finally, assuming that the hazard rate is constant, we completely characterise the solution of the obtained linear state optimal control problem.

Key-Words: Optimal control, Regime shifts, Stochastic switching time, Linear state structure.

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1 Introduction

An optimal control problem with stochastic switching time consists of a dynamic optimisation problem divided into two stages by an event that occurs at a random time. The switch may have one or more simultaneous effects on the system, such as a change in the running payoff, the salvage value function, the state dynamics, and eventually the control set. These types of problem, which essentially constitute a specific case of piecewise deterministic models [1], can be applied in many contexts, such as rational risk, [2], renewable resources, [3], and open source software, [4].

In the literature on dynamic optimisation, this class of problems has been extensively studied and two are the main methodologies applied to solve them: the *backward approach* and the *heterogeneous approach*. For more details on these techniques [5], [6], [7].

In this paper, we are interested in studying the structure of the two subproblems that define the two stages of an optimal control problem with a stochastic switching time. More precisely, we seek to verify whether the general optimal control problem, which takes into account the two stages generated from the switch, turns out to be linear state whenever the two subproblems are linear state in turn [1].

The search for formulations of optimal control problems that are particularly simple to solve is very important in applications, as it allows bypassing analytical complexity and directly obtaining explicit solutions that can be tested and evaluated. The most important example is related to the formulation of LQ optimal control problems, both deterministic and stochastic [1]. However, this research is still ongoing; for example, with respect to Markov chains, a recent work by Lefener can be found in [8].

As mentioned previously, in an optimal control problem with a stochastic switching time, it is assumed that there exists an event that occurs at random time τ, which abruptly changes the nature of the system and splits the time horizon into two stages: a Stage 1 before the occurrence of τ, and a Stage 2 afterwards.

To pursue our goal, in Section 2 we introduce the linear state structure, drawing from the literature on differential game theory. We then formulate an optimal control problem with a stochastic switching time where both stages have a linear state structure. Furthermore, in Section 3 we reformulate the problem as a deterministic optimal control problem equivalent to the original stochastic one and solve it by adopting the backward approach. We observe that, even though the structure of the original problem is quite simple - specifically, linear in the state and quadratic in the control - its deterministic reformulation is not, in turn, linear state. Finally, in Section 4 we determine the necessary and sufficient conditions to guarantee the linear state structure for the transformed problem.

2 Linear State Structure

The class of linear state optimal control problems has been introduced in the context of differential game theory. Originally defined as *state-separable games* by [9], furthermore, they have been denoted as *linear state games* in [1].

This class consists of games for which the state equations and the objective functions are linear in the state variables and no (multiplicative) interaction between control variables of one player and state variables of the opponent is present. State-separability is important in game theory because these games have the property that their open-loop Nash equilibria are Markov perfect and therefore are subgame perfect. In addition, these problems are "tractable" and easy to solve, even in the case of hierarchical moves.

In our paper, we adapt the linear state definition introduced in the game theory environment to the optimal control problems of the two stages of an optimal control problem with a stochastic switching time. To comprehend where the non-linearity comes from, for simplicity, we consider a specific and simple linear state structure for the two subproblems. We assume that all the functions involved are autonomous with respect to time and switching time; furthermore, we assume that they are linear in the states, while the payoffs are quadratic in the controls. Finally, we assume that there are no jumps in the state at the switching time and that switching costs are not considered. For a more general model the reader can refer to [7].

In the Table 1, we report the functions that characterise the problem and their notation. All parameters take real values, and we further assume $\kappa_1, \kappa_2 > 0$ to guarantee the second-order optimal sufficient conditions. With such a notation, and taking into account the stochasticity of τ, the switching time optimal control problem is equivalent to maximising the expectation of the following total payoff:

Table 1. Functions and notations

	Stage 1	**Stage 2**
Dynamics	$\alpha_1 x_1 + \gamma_1 u_1$	$\alpha_2 x_2 + \gamma_2 u_2$
Payoff	$\pi_1 x_1 - \kappa_1 u_1^2/2$	$\pi_2 x_2 - \kappa_2 u_2^2/2$
Salvage	$\sigma_1 x_1$	$\sigma_2 x_2$
Control sets	$U_1 = \mathbb{R}$	$U_2 = \mathbb{R}$

$$\max_{\substack{u_1(t)\in U_1\\ u_2(s,t)\in U_2}} \mathbb{E}\Bigg[\mathbf{1}_{\{\tau<T\}}\Big\{\int_0^\tau \pi_1 x_1(t) - \kappa_1 u_1^2(t)/2dt + \int_\tau^T \pi_2 x_2(\tau,t) - \kappa_2 u_2^2(\tau,t)/2dt + \sigma_2 x_2(\tau,T)\Big\} + \mathbf{1}_{\{\tau\geq T\}}\Big\{\int_0^T \pi_1 x_1(t) - \kappa_1 u_1^2(t)/2dt + \sigma_1 x_1(T)\Big\}\Bigg] \tag{1}$$

subject to:

$$\begin{cases} \dot{x}_1(t) = \alpha_1 x_1(t) + \gamma_1 u_1(t), & t\in[0,T] \\ x_1(0) = x_0 \\ \dot{x}_2(\tau,t) = \alpha_2 x_2(\tau,t) + \gamma_2 u_2(\tau,t), & t\in[\tau,T] \\ x_2(\tau,\tau) = x_1(\tau) \end{cases} \tag{2}$$

Observe that the last equation depends on the continuity assumption for the state trajectory at τ.

The stochastic switching time τ can be modelled as an absolutely continuous random variable taking values in $[0,+\infty)$. In line with most of the related literature [10], we introduce the so called *hazard rate* of τ as follows:

$$\lim_{h\to 0^+} \frac{\mathbb{P}(\tau \leq t+h \,|\, \tau > t)}{h} = \eta(t, x_1(t)) \tag{3}$$

and assume it depending on the time and on the state of the system.

3 Problem Reformulation

Analogously to what has been done in [11], and in [7], we introduce an auxiliary Stage 1 state variable $z(t) := \mathbb{P}(\tau > t)$, that is, the probability of still being in Stage 1 at time t, to reformulate the problem (1) in the following deterministic form.

$$\max_{\substack{u_1(t)\in U_1\\ u_2(s,t)\in U_2}} \Bigg[\int_0^T z(t)\Big\{\pi_1 x_1(t) - \kappa_1 u_1^2(t)/2 + \eta(t,x_1(t))\Big[\sigma_2 x_2(t,T) + \int_t^T \pi_2 x_2(t,\theta) - \kappa_2 u_2^2(t,\theta)/2\, d\theta\Big]\Big\}dt + z(T)\sigma_1 x_1(T)\Bigg] \tag{4}$$

subject to:

$$\begin{cases} \dot{x}_1(t) = \alpha_1 x_1(t) + \gamma_1 u_1(t), & t \in [0,T] \\ x_1(0) = x_0 \\ \dot{x}_2(s,t) = \alpha_2 x_2(s,t) + \gamma_2 u_2(s,t), & t \in [s,T] \\ x_2(s,s) = x_1(s) \\ \dot{z}(t) = -\eta(t, x_1(t)) z(t), & t \in [0,T] \\ z(0) = 1 \end{cases} \tag{5}$$

We can observe that this resulting problem is, in general, not trivial, even if we assumed a very simple form in the two stages.

The aim of this paper is to verify under which assumptions such a transformed deterministic optimal control problem can be more tractable, from the solving point of view. For example, a linear state structure would make the problem easier to solve and more consistent if included in a differential game context [1].

Observe that in the dynamics (5) the last ODEs (referred to z) contains a multiplicative term between the hazard rate (depending, in general, on x_1) and the state variable z itself. Moreover, $z(t)$ multiplies the entire objective function within the integral.

To verify that the loss of the linear state structure of the problem does not depend on the form of the two subproblems, let us apply the well-known *backward approach* to solve the optimal control problem with a switching time characterised in Table 1.

First, let us solve the Stage 2 problem with dynamic programming. Let us define the value function V_2 of the second Stage

$$V_2(t,x) := \sup_{u(\theta)\in U_2} \left[\int_t^T \pi_2 x_2(\theta) - \kappa_2 u_2^2(\theta)/2 \, d\theta + \sigma_2 x_2(T) \right] \tag{6}$$

subject to:

$$\begin{cases} \dot{x}(\theta) = \alpha_2 x_2(\theta) + \gamma_2 u_2(\theta) & \text{for } \theta \in [t,T] \\ x_2(t) = x \end{cases} \tag{7}$$

If $V_2(t,x)$ is differentiable, then it is the solution of the corresponding system of HJB:

$$\begin{cases} -\partial_t V_2(t,x) = & \max_{\mathtt{w}\in U_2} \{\pi_2 x - \kappa_2 \mathtt{w}^2/2 \\ & \quad + \partial_x V_2(t,x) \cdot (\alpha_2 x + \gamma_2 \mathtt{w})\} \\ V_2(T,x) = & \sigma_2 x \end{cases} \tag{8}$$

The optimal feedback strategy $\Phi_2(t,x)$ that maximises the RHS of the HJB equation (20) is degenerate, and equal to

$$\Phi_2(t,x) = \frac{\gamma_2}{\kappa_2} \partial_x V_2(t,x). \tag{9}$$

Due to the linear state structure of Stage 2 problem, the value function $V_2(t,x)$ is linear in the state. Therefore, we assume that it has the following linear form $V_2(t,x) = A_2(t)x + B_2(t)$. The two unknown functions $A_2(t)$ and $B_2(t)$ must satisfy the following system of decoupled ODEs.

$$\begin{cases} \dot{A}_2(t) = -\alpha_2 A_2(t) - \pi_2 \\ A_2(T) = \sigma_2 \\ \dot{B}_2(t) = -\gamma_2^2 (A_2(t))^2 / 2\kappa_2 \\ B_2(T) = 0 \end{cases} \tag{10}$$

that admits a unique solution. In particular, for all $t \in (0,T]$

$$A_2(t) = \frac{\pi_2(e^{\alpha_2(T-t)} - 1)}{\alpha_2} + \sigma_2 e^{\alpha_2(T-t)}, \tag{11}$$

$$B_2(t) = \int_t^T \frac{\gamma_2^2}{2\kappa_2} (A_2(s))^2 ds. \tag{12}$$

So that the degenerate feedback optimal control for Stage 2 is

$$\Phi_2(t,x) = \frac{\gamma_2}{\kappa_2} A_2(t). \tag{13}$$

Assuming optimal behaviour in Stage 2, and the continuity of the state function, we obtain the following objective for Stage 1:

$$\max_{u_1(t)\in U_1} \Bigg[\int_0^T z(t) \Big\{ \pi_1 x_1(t) - \kappa_1 u_1^2(t)/2 + \eta\big(t, x_1(t)\big)\big(A_2(t) x_1(t) + B_2(t)\big) \Big\} dt + z(T)\sigma_1 x_1(T) \Bigg] \tag{14}$$

subject to the following differential equations in the variables $x_1(t)$ and $z(t)$, for all $t \in [0,T]$:

$$\begin{cases} \dot{x}_1(t) = \alpha_1 x_1(t) + \gamma_1 u_1(t) \\ x_1(0) = x_0 \\ \dot{z}(t) = -\eta\big(t, x_1(t)\big) z(t) \\ z(0) = 1 \end{cases} \tag{15}$$

We can observe that the problem is not linear state due to the presence in the objective functional of the auxiliary variable $z(t)$ as a multiplicative term. Moreover, the two state equations are coupled because of the presence of the hazard rate η, which depends on $x_1(t)$, in the ODE for $z(t)$.

From this very simple example, we can guess that the loss of the linear state structure comes from the switching-time characteristic of the original problem. In the following section, we will determine a necessary and sufficient condition on the hazard rate to guarantee the linear state structure for the deterministic formulation of the optimal control problem with a random switching time.

4 Analysis of the Structure

We started with an optimal control problem with a stochastic switching time in which both optimal control problems comprising Stage 1 and Stage 2 have a particularly simple linear state structure. By applying the backward approach and adopting the Hamilton Jacobi Bellmann equation to solve the Stage 2 problem, we obtained the optimal control problem characterised by the objective functional (14) subject to the state equations (15) that clearly appear not to be linear state. Now, we ask ourselves under which assumptions the obtained problem retains the same structure as the two original sub-problems.

In the next theorem, we provide necessary and sufficient conditions under which (14) and (15) constitute a linear state optimal control problem.

Theorem 1. *The optimal control problem characterised by the objective functional* (14) *and the dynamics* (15) *is linear state if and only if the hazard rate of the stochastic switching time does not depend on the state function of Stage 1, i.e.*

$$\partial_{x_1}\eta(t,x_1)=0.$$

Proof. Let us first observe that the system of state equations in (15) is linear in the state variables if and only if the function η does not depend on the state variable x_1. Moreover, if we assume that $\eta(t,x)=\eta(t)$, then the differential equation for the state variable z is decoupled from the differential equation for the state variable x_1. The solution of the ODE for the state variable z allows us to rewrite the optimal control problem (14) and (15) as follows:

$$\max_{u_1(t)\in U_1}\Bigg[\int_0^T e^{-\int_0^t\eta(r)dr}\Big\{\pi_1x_1(t)-\kappa_1u_1^2(t)/2 + \eta(t)\big(A_2(t)x_1(t)+B_2(t)\big)\Big\}dt + e^{-\int_0^T\eta(r)dr}\sigma_1x_1(T)\Bigg] \tag{16}$$

subject to:

$$\begin{cases}\dot{x}_1(t)=\alpha_1x_1(t)+\gamma_1u_1(t)\\ x_1(0)=x_0\end{cases} \tag{17}$$

This immediately shows that the problem has the required linear state structure. □

It is worth observing that in (16) the function $z(t)=e^{-\int_0^t\eta(r)dr}$ plays the role of a discount factor, where the hazard rate represents a variable discount rate. Recalling, [12], in the case of non-constant discounting, the use of standard optimal control techniques gives rise to time-inconsistent solutions. This interpretation suggests to apply the Hamilton-Jacobian-Bellman approach to solve problem (16) and (17).

Under the further assumption of a constant hazard rate, i.e. $\eta(t)\equiv\eta>0$, the analytical solution of the problem can be easily obtained by defining the value function for the discounted problem

$$v_1(t,x):= \sup_{u_1(t)\in U_1}\Bigg[\int_t^T e^{-\eta(r-t)}\Big\{\pi_1x_1(r)-\kappa_1u_1^2(r)/2 + \eta\big(A_2(r)x_1(r)+B_2(r)\big)\Big\}dr + e^{-\eta(T-t)}\sigma_1x_1(T)\Bigg] \tag{18}$$

subject to:

$$\begin{cases}\dot{x}_1(r)=\alpha_1x_1(r)+\gamma_1u_1(r)\\ x_1(t)=x\end{cases} \tag{19}$$

If $v_1(t,x)$ is differentiable, then as shown in [1], it is the solution of the HJB equations:

$$\begin{cases}\eta\cdot v_1(t,x)-\partial_tv_1(t,x)=\\ \max_{\mathtt{w}\in U_1}\big\{(\pi_1+\eta A_2(t))x+\eta B_2(t)-\kappa_1\mathtt{w}^2/2 \\ \qquad +\partial_xv_1(t,x)\cdot(\alpha_1x+\gamma_1\mathtt{w})\big\}\\ v_1(T,x)=\sigma_1x\end{cases} \tag{20}$$

If we assume that $v_1(t,x)=a_1(t)x+b_1(t)$, then the two unknown functions $a_1(t)$ and $b_1(t)$ must satisfy the following system of decoupled ODEs.

$$\begin{cases}\dot{a}_1(t)=(\eta-\alpha_1)a_1(t)-\pi_1-\eta A_2(t)\\ a_1(T)=\sigma_1\\ \dot{b}_1(t)=\eta b_1(t)-\eta B_2(t)-\gamma_1^2a_1^2(t)/2\kappa_1\\ b_1(T)=0\end{cases} \tag{21}$$

This is a system of linear differential equations decoupled from each other, depending on the solution $A_2(t), B_2(t)$ of (10) already solved for Stage 2 in the previous section. The linearity of the system and the regularity of the coefficients (which are all continuous functions) ensure the existence and uniqueness of the solution to this system. The optimal control for the original problem is as follows:

$$u_1^*(t) = \gamma_1 a_1(t)/\kappa_1,$$

while

$$u_2^*(s,t) = \gamma_2 A_2(t)/\kappa_2.$$

We notice that the hazard rate affects only the optimal control of Stage 1. Moreover, the particularly simple form of the original problem makes the optimal control of Stage 2 independent from the instant at which the switch occurs.

5 Conclusion

In this paper, we analyse an optimal control problem with a stochastic switching time. To simplify the solution process and focus on identifying the real reason for the loss of the linear state structure, we assumed that both problems in Stage 1 and Stage 2 are linear state. We noticed that despite these stringent assumptions, the initial problem fails to maintain the linear state property. The structure is preserved only if the hazard rate function does not depend on the state of the system. This study highlights the difficulty of deriving an analytical solution for optimal control problems with stochastic switching time, even with very basic problem data assumptions.

A future line of research suggested by this work is the connection, which can be seen explicitly in formula (16), between optimal control problems with stochastic switching time and optimal control problems with heterogeneous discounting factors, [12]. Further studies are needed to better clarify the connection between these two classes of problems.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

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Conflict of Interest

The authors have no conflicts of interest to declare.

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