The Hermite-Birkhoff Problem and Local Spline Approximation

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Abstract: - This paper discusses the use of local spline approximations to solve the Hermite-Birkhoff problem. The solution to a specific problem using polynomial and non-polynomial local splines of the third order of approximation is considered. Here we discuss the case when the values of the function $u(x)$ and its derivative $u'(x)$ are given at the nodes of the grid in an alternative way: $\dots, u(x_j), u'(x_{j+1}), u(x_{j+2}), \dots$. Note that when using polynomial and non-polynomial spline approximations, it is possible to obtain acceptable solutions in several interesting cases that are impossible when we use the classical approach. In the case of using local basis splines, many previously unsolvable problems turn out to be solvable. The results of the numerical experiments are presented.

Key-Words: - Hermite-Birkhoff interpolation, approximation, polynomial local splines, non-polynomial local splines, exponential local splines, Hermite interpolation, Lagrange interpolation.

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1 Introduction

First of all, let us recall how the Hermite-Birkhoff algebraic interpolation problem is formulated. Paper [1] provides us with a formulation of the Hermite-Birkhoff interpolation problem. Let's take a rectangular matrix $I = \left[\varepsilon_{i0,}\varepsilon_{i1,}, \dots, \varepsilon_{in}\right]_{i=1}^{m}$ with m rows and $n+1$ columns. Here $\varepsilon_{i,j}$ equals 0 or 1, and, $\sum_{i,j} \varepsilon_{i,j} = n+1$. Let e be the set of ordered pairs $e = \{(i,j) | \varepsilon_{i,j} = 1\}$. Thus, the number of ones among the elements of matrix I is equal to the number of its columns. Matrix I is called the incidence matrix. Let $x_1 < x_2 < \dots < x_m$ be real numbers.

The Hermite-Birkhoff problem is formulated as follows. We have to find a polynomial $P(x)$ of degree not greater than n that satisfies the conditions

$$P^{(j)}(x_i) = f_i^{(j)}, (i,j) \in e.$$

Here $f_i^{(j)}$ are given numbers.

This problem was first considered by Birkhoff in 1906. Polya focused a lot of his attention on solving this problem. Of particular note, in 1931 he formulated the conditions for the existence of a solution to this problem in several important cases. Let us note that an important special case of the Hermite-Birkhoff problem is the Hermite interpolation problem.

Currently, many authors are engaged in constructing a solution to the Hermite–Birkhoff problem.

Paper [2], discusses various aspects of the Hermite–Birkhoff interpolation that involve prescribed values of a function and/or its first derivative. An algorithm is given that finds the unique polynomial satisfying the given conditions if it exists. A mean value type error term is developed which illustrates the ill-conditioning present when trying to find a solution to a problem that is close to a problem that does not have a unique solution. The author of paper [2], writes, that such problems may arise when using a collocation to solve two-point boundary value problems, [3], [4]. Another example arises in the numerical solution of ordinary differential equations with defect control when using Runge–Kutta methods, [5], [6].

As noted in [7], a Birkhoff interpolation problem is not always solvable even in the appropriate polynomial or rational space. In paper [7] the authors propose to split up a univariate unsolvable Hermite-Birkhoff interpolation problem into two or more solvable subproblems and to blend the local solutions by using multinode basis functions as blending functions.

The classical Hermit-Birkhoff problem in [8] is considered for trigonometrical polynomials.

Paper [9] studies the problem of the Hermite-Birkhoff interpolation with splines. In paper [9], interpolation knots and spline knots are considered as "dual" elements. This leads to a dual problem which is poised if and only if the original problem is poised. Estimations of the number of zeros in the appropriate interpolation kernel yield a Cauchy-type representation of the interpolation error for certain cases of Hermite problems.

In this paper, we discuss the solution of the Hermite-Birkhoff interpolation using local polynomial and non-polynomial splines. Note that when using polynomial and non-polynomial spline approximations, it is possible to obtain acceptable solutions in several interesting cases that are impossible when we use the classical approach.

2 Hermite-Birkhoff splines

Suppose n is an even integer, and a, b are real. Suppose we have the nodes of the ordered grid $\{x_j\}$:

$$a = x_0 <\ldots< x_{j-1} < x_j < x_{j+1} <\ldots< x_n = b.$$

We denote $u_j = u(x_j), u'_j = u'(x_j)$. Let the values of the function $u(x)$ and its derivative $u'(x)$ be given at the nodes of the grid, $\{x_j\}$, in an alternative way, such as $\ldots, u_j, u'_{j+1}, u_{j+2}, \ldots$.

We assume that the function $u \in C^3([a,b])$. First, we suppose that the values of the function $u(x)$ and its derivative $u'(x)$ are given in the following way $'_0, u_1, u'_2, u_3, \ldots$. On each interval $[x_j, x_{j+1})$, $j = 1, 3, 5, \ldots, n-1$, we can approximate the function $u(x)$ with the expression $\tilde{u}$:

$$\tilde{u}(x) = u'(x_{j-1})\omega_{j-1,1}(x) + u(x_j)\omega_{j,0}(x) + u'(x_{j+1})\omega_{j+1,1}(x)\,, [x_j, x_{j+1}). \quad (1)$$

Let $1, \varphi(x)$ and $\psi(x)$ be sufficiently smooth and linearly independent functions. Suppose the system $1, \varphi(x), \psi(x)$ forms a Chebyshev system on $[a, b] \subset R$. From the conditions

$$\tilde{u}(x) = u(x) \text{ with } u = 1, \varphi(x), \psi(x)$$

We obtain a system of linear algebraic equations

$$\omega_{j,0}(x) = 1,$$
$$\varphi'(x_{j-1})\omega_{j-1,1}(x) + \varphi(x_j)\omega_{j,0}(x) + \varphi'(x_{j+1})\omega_{j+1,1}(x) = \varphi(x),$$
$$\psi'(x_{j-1})\omega_{j-1,1}(x) + \psi(x_j)\omega_{j,0}(x) + \psi'(x_{j+1})\omega_{j+1,1}(x) = \psi(x).$$

We assume that the determinant of this system is not equal to 0. In the special case when $\psi(x)=\varphi^2(x)$, the determinant of the system Δ_j has the form:

$$\Delta_j = 2\varphi'_{j-1}\varphi'_{j+1}(\varphi_{j-1} - \varphi_{j+1}).$$

Here $\varphi_i = \varphi(x_i)$.
Under the assumption that the determinant of the system is not equal to 0, it is not difficult to obtain formulas for the basis splines on $[x_j, x_{j+1})$:

$$\omega_{j,0}(x) = 1,$$
$$\omega_{j-1,1}(x) = \varphi'_{j+1}\left(\varphi_j - \varphi(x)\right)\left(2\varphi_{j+1} - \varphi_j - \varphi(x)\right)/\Delta_j,$$
$$\omega_{j+1,1}(x) = \varphi'_{j-1}(\varphi(x) - \varphi_j)\left(2\varphi_{j-1} - \varphi(x) - \varphi_j\right)/\Delta_j.$$

Splines $\tilde{u}(x)$ we call Hermite-Birkhoff splines, and basis splines $\omega_{k,i}$ we call Hermite-Birkhoff basis splines.

Let us present expressions of the basis splines in several specific special cases:

1) If $\varphi(x) = x, \psi(x) = x^2$, then the formulas take the form:
$$\omega_{j,0}(x) = 1,$$
$$\omega_{j-1,1}(x) = (x_j - x)(2x_{j+1} - x_j - x)/\Delta_j,$$
$$\omega_{j+1,1}(x) = (x - x_j)(2x_{j-1} - x_j - x)/\Delta_j,$$
where $\Delta_j = 2(x_{j-1} - x_{j+1})$.

2) If $\varphi(x) = e^x, \psi(x) = e^{2x}$, then the formulas take the form:
$$\omega_{j,0}(x) = 1,$$
$$\omega_{j+1,1}(x) = (e^{2x+x_{j-1}} - e^{x_{j-1}+2x_j} + 2e^{2x_{j-1}+x_j} - 2e^{x+2x_{j-1}})/\Delta_j,$$
$$\omega_{j-1,1}(x) = -(e^{2x+x_{j+1}} - e^{x_{j+1}+2x_j} + 2e^{2x_{j+1}+x_j} - 2e^{x+2x_{j+1}})/\Delta_j,$$
where
$$\Delta_j = 2\left(e^{x_{j-1}+2x_{j+1}} - e^{2x_{j-1}+x_{j+1}}\right).$$

Let $x_j = 0$, $x_{j+1} = x_j + h = 0.12$. The plots of the basis polynomial splines are given in Figure 1, Figure 2 and Figure 3. The plots of the basis exponential splines are given in Figure 4, Figure 5 and Figure 6.

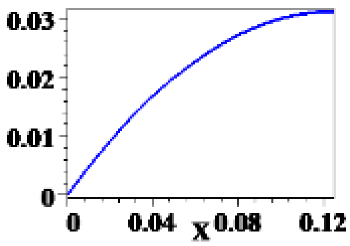

Fig. 1: The plot of the basis polynomial spline $\omega_{j+1,1}(x)$

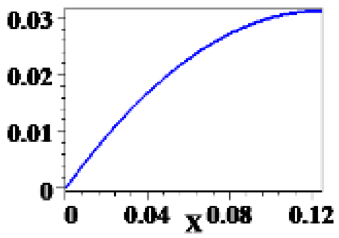

Fig. 2: The plot of the basis polynomial spline $\omega_{j-1,1}(x)$

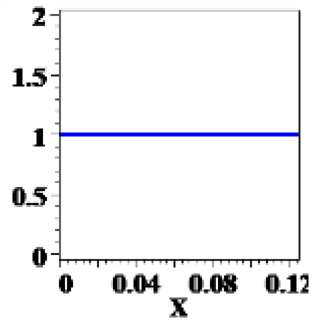

Fig. 3: The plot of the basis polynomial splines $\omega_{j,0}(x)$

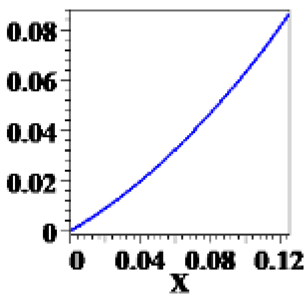

Fig. 4: The plot of the basis exponential splines $\omega_{j+1,1}(x)$

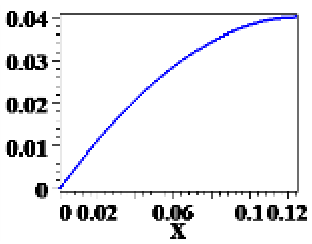

Fig. 5: The plot of the basis exponential splines $\omega_{j-1,1}(x)$

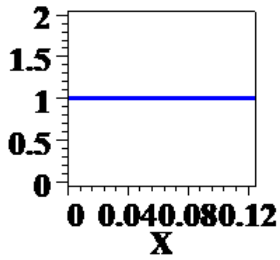

Fig. 6: The plot of the basis exponential splines $\omega_{j,0}(x)$

On each interval $[x_j, x_{j+1})$, $j = 0, 2, 4, \ldots, n-2$, we can approximate the function $u(x)$ with the expression $\tilde{u}$:

$$\tilde{u}(x) = u'(x_j)\omega_{j,1}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_{j+2})\omega_{j+2,1}(x). \qquad (2)$$

In the polynomial case, when $\varphi(x) = x$, $\psi(x) = x^2$, we get:

$$\omega_{j+1,0}(x) = 1,$$
$$\omega_{j,1}(x) = \frac{(x - x_{j+1})(x - 2x_{j+2} + x_{j+1})}{2(x_j - x_{j+2})},$$
$$\omega_{j+2,1}(x) = \frac{(x - x_{j+1})(x - 2x_j + x_{j+1})}{2(x_{j+2} - x_j)}.$$

Note, we can obtain the approximation of the first derivative with the formulae, obtained with (1) and (2):

$$\tilde{u}'(x) = u'(x_{j-1})\omega'_{j-1,1}(x) + u(x_j)\omega'_{j,0}(x) + u'(x_{j+1})\omega'_{j+1,1}(x),$$
$$\tilde{u}'(x) = u'(x_j)\omega'_{j,1}(x) + u(x_{j+1})\omega'_{j+1,0}(x) + u'(x_{j+2})\omega'_{j+2,1}(x).$$

There is another form of the approximation. Now, we suppose that the values of the function $u(x)$ and its derivative $u'(x)$ are given in the following way $u_0, u'_1, u_2, \ldots$.

On each interval $[x_j, x_{j+1})$, $j = 0, 2, 4, \ldots, n-2$, we can also approximate the function $u(x)$ with the expression $\tilde{u}$:

$$\tilde{u}(x) = u(x_j)\omega_{j,0}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + u'(x_{j+3})\omega_{j+3,1}(x).$$

From the conditions:
$\tilde{u}(x) = u(x)$ with $u = 1, \varphi(x), \psi(x)$

We obtain a system of linear algebraic equations:

$$\omega_{j,0}(x) = 1,$$
$$\varphi(x_j)\omega_{j,0}(x) + \varphi'(x_{j+1})\omega_{j+1,1}(x) + \varphi'(x_{j+3})\omega_{j+3,1}(x) = \varphi(x),$$

$$\psi(x_j)\omega_{j,0}(x) + \psi'(x_{j+1})\omega_{j+1,1}(x) + \psi'(x_{j+3})\omega_{j+3,1}(x) = \psi(x).$$

In the polynomial case, when $\varphi(x) = x$, $\psi(x) = x^2$, we get:

$$\omega_{j,0}(x) = 1,$$
$$\omega_{j+1,1}(x) = \frac{-(x - x_j)(x - 2x_{j+3} + x_j)}{2(x_{j+3} - x_{j+1})},$$
$$\omega_{j+3,1}(x) = \frac{(x - x_j)(x - 2x_{j+1} + x_j)}{2(x_{j+3} - x_{j+1})}.$$

On each interval $[x_j, x_{j+1})$, $j = 1, 3, 5, \dots, n-1$, we can approximate the function $u(x)$ with the expression $\tilde{u}$:

$$\tilde{u}(x) = u'(x_j)\omega_{j,1}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_{j+2})\omega_{j+2,1}(x),$$

where in the polynomial case basis splines have the form:

$$\omega_{j+1,0}(x) = 1,$$
$$\omega_{j,1}(x) = \frac{(x - x_{j+1})(x - 2x_{j+2} + x_{j+1})}{2(x_j - x_{j+2})},$$
$$\omega_{j+2,1}(x) = \frac{(x - x_{j+1})(x - 2x_j + x_{j+1})}{2(x_{j+2} - x_j)}.$$

In the exponential case the basis splines have the form:

$$\omega_{j+1,0}(x) = 1,$$
$$\omega_{j,1}(x) = \frac{q(x)}{2(\exp(x_j + 2x_{j+2}) - \exp(2x_j + x_{j+2}))},$$
$$\omega_{j+2,1}(x) = \frac{g(x)}{2(\exp(x_j + 2x_{j+2}) - \exp(2x_j + x_{j+2}))}.$$
$$q(x) = \exp(2x_{j+1} + x_{j+2}) - \exp(2x + x_{j+2}) - 2\exp(x_{j+1} + 2x_{j+2}) + 2\exp(x + 2x_{j+2}),$$
$$g(x) = 2\exp(2x_j + x_{j+1}) - 2\exp(x + 2x_j) - \exp(2x_{j+1} + x_j) + \exp(2x + x_j).$$

Note, that it is impossible to construct the approximation on the interval $[x_j, x_{j+1})$ in the form:

$$\tilde{u}(x) = u(x_j)\omega_{j,0}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + u(x_{j+2})\omega_{j+2,0}(x).$$

In next section we will consider estimates of approximation errors.

3 The Errors of the Approximation

In [10], error estimates using non-polynomial splines were obtained in the general case. Here we obtain estimates of the errors of approximation of the function $u(x)$ on the interval $[x_j, x_{j+1})$ with the spline expressions $\tilde{u}(x)$ with $\psi(x) = \varphi^2(x)$, when $\varphi(x) = e^x$ or $\varphi(x) = x$.

First, let $\varphi(x) = e^x, \psi(x) = e^{2x}$. Following the method proposed in [10], we construct a homogeneous linear differential equation having a fundamental system of solutions: 1, $\varphi(x)$, $\psi(x)$:

$$Lu = u''' - 3u'' + 2u' = 0.$$

Let us use approximation (1). Using the basis spline estimates

$$|\omega_{j-1,1}(x)| \le h/4, |\omega_{j+1,1}(x)| \le 3h/4,$$

We obtain the required estimate of the approximation error on the interval $[x_j, x_{j+1})$:

$$|\tilde{u}(x) - u(x)| \le Kh^3 \parallel u''' - 3u'' + 2u' \parallel_{[x_{j-1}, x_{j+1}]},$$

where $K \approx 0.58$.

Similarly, when $\varphi(x) = x, \psi(x) = x^2$, we obtain an estimate for the approximation error on the interval $[x_j, x_{j+1})$ in the form:

$$|\tilde{u}(x) - u(x)| \le \frac{h^3}{2} \parallel u''' \parallel_{[x_{j-1}, x_{j+1}]}.$$

Similar estimations can be found when we use expression (2).

4 Numerical Experiments

In this section we present the results of numerical experiments on the approximation of some of the functions on the interval $[-1,1]$. Suppose the values of the function $u(x)$ and its derivative $u'(x)$ be given at the nodes of the grid $\{x_j\}$, in an alternative way: $\dots, u_j, u'_{j+1}, u_{j+2}, \dots$. Using an equidistant grid with step h and expressions (1), (2), we construct the approximation of function u on $[-1,1]$. All calculations were done in the Maple environment. Figure 7 shows the plot of the error of approximation of $u(x) = \sin(x)$ with the polynomial Hermite-Birkhoff splines. Figure 8 shows the plot of the error of approximation of $u'(x) = \cos(x)$ with the Hermite-Birkhoff polynomial splines.

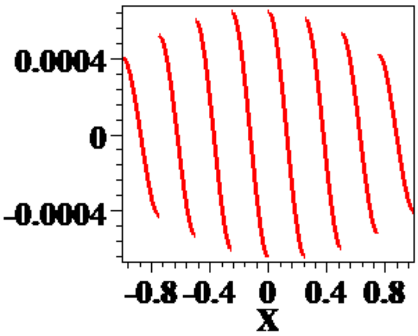

Fig. 7: The plot of the error of approximation of $u(x) = \sin(x)$ with the polynomial splines ($n = 16$)

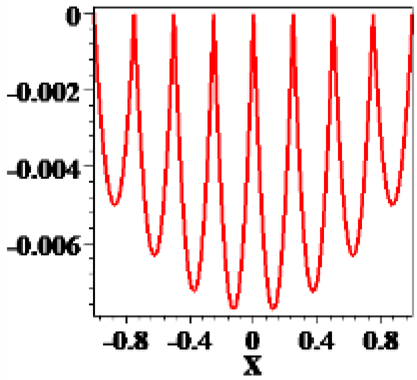

Fig. 8: The plot of the error of approximation of $u'(x) = \cos(x)$ with the polynomial splines

4.1 The Lagrange Interpolation and the Hermite Interpolation

In this subsection we discuss the Lagrange and the Hermite interpolation.

Let us construct the equidistant set of nodes $\{x_k\}$, $k = 0,1, \ldots, n$, with step $h = 2/n$ on $[-1,1]$. We construct the Lagrange interpolation of the function $u(x)$ with the interpolation polynomial $P_n(x) = a_n x^n + \cdots + a_0$. We construct $P_n(x)$ using $n + 1$ function values at the grid nodes x_k. The polynomial $P_n(x)$ has the form:

$$P_n(x) = \sum_{k=0}^{n} u(x_k) \frac{w(x)}{w'(x_k)(x - x_k)},$$
$$w(x) = (x - x_0)(x - x_1) \ldots (x - x_n).$$

As is known, Runge noted in 1901 that in this case the approximation of the function $u(x) = 1/(1 + 25x^2)$ gives us an unsatisfactory result. For example, let $n = 10$. The plot of the function $u(x) = 1/(1 + 25x^2)$ and its approximation with the interpolation polynomial is shown in Figure 9.

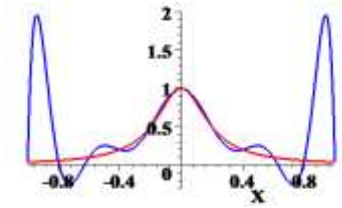

Fig. 9: The plot of the function $u(x) = 1/(1 + 25x^2)$ (red) and its approximation (blue) with the interpolation polynomial ($n = 10$).

Now we construct the approximation of the Runge function using the local cubic Hermite splines of the first level. Note that local spline approximations give good results.

Note that Hermitian cubic splines are well known. They are constructed separately on each grid interval in the form of a third-degree polynomial. These splines form a continuously differentiable piecewise function on the interval $[-1,1]$. Following Professor S.G. Mikhlin, we write Hermitian cubic splines of first height on the interval $[x_j, x_{j+1}]$ in the form:

$$\tilde{u}(x) = u(x_j)\omega_{j,0}(x) + u'(x_j)\omega_{j,1}(x)$$
$$+u(x_{j+1})\omega_{j+1,0}(x) + u'(x_{j+1})\omega_{j+1,1}(x).$$

The basis splines $\omega_{k,i}$, $k = j, j+1, i = 0,1$, are as follows:

$$\omega_{j,0}(x) = \frac{(x_{j+1} - x)^2}{(x_{j+1} - x_j)^2} + 2\frac{(x - x_j)(x_{j+1} - x)^2}{(x_{j+1} - x_j)^3},$$

$$\omega_{j,1}(x) = \frac{(x_j - x)^2}{(x_{j+1} - x_j)^2} + 2\frac{(x_{j+1} - x)(x_j - x)^2}{(x_{j+1} - x_j)^3},$$

$$\omega_{j,1}(x) = \frac{(x - x_j)(x_{j+1} - x)^2}{(x_{j+1} - x_j)^2}$$

$$\omega_{j+1,1}(x) = \frac{(x - x_{j+1})(x_j - x)^2}{(x_{j+1} - x_j)^2}$$

These splines are well known and widely used.

On each interval $[x_j, x_{j+1}]$ these splines provide the fourth order of approximation:

$$|\tilde{u}(x) - u(x)| \leq Kh^4,$$

Here $x_{j+1} - x_j = h$.

The plot of the of the error of approximation of function $u(x) = 1/(1 + 25x^2)$ with the interpolation cubic splines on $[-1,1]$ is shown in Figure 10.

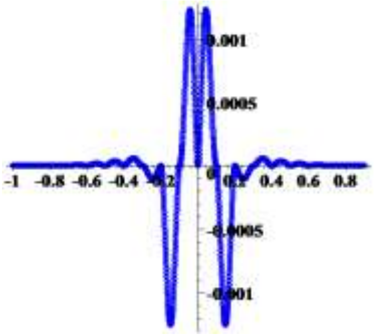

Fig. 10: The plot of the error of approximation of $u(x) = 1/(1 + 25x^2)$ with the polynomial splines $(n = 10)$

4.2 The Spline Approximation

Next, we present the results of approximation of function u with the Hermite-Birkhoff splines. We suppose that the values of the function $u(x)$ and its derivative $u'(x)$ are given in the following way $u'_0, u_1, u'_2, u_3, \ldots$. On each interval $[x_j, x_{j+1})$, $j = 1, 3, 5, \ldots, n-1$, we can approximate the function $u(x)$ with the expression (1) or (2). Figure 11, Figure 12, Figure 13, Figure 14 and Figure 15 show the plot of the Runge function, its approximation, and the error of approximation of the Runge function $u(x) = 1/(1 + 25x^2)$ with the Hermite-Birkhoff polynomial splines. Figure 11 shows the function and its approximation with Hermite-Birkhoff splines when $n = 16$. In this case, the approximation error turns out to be large and it is visible in Figure 11. Figure 12, Figure 13, Figure 14 and Figure 15 present only plots of the errors of approximation of the function and its first derivative. It can be seen that with an increase in the number of grid nodes, the approximation error decreases. This is consistent with the theoretical estimates obtained in the previous section.

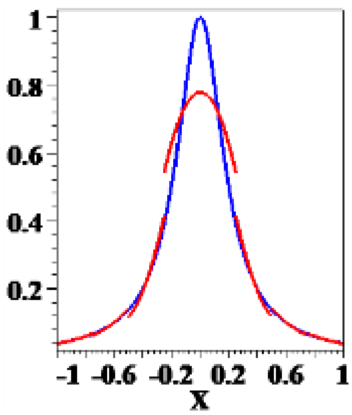

Fig. 11: The plot of the function $u(x) = 1/(1 + 25x^2)$ (blue) and its approximation (red) with the polynomial spline interpolation $(n = 16)$

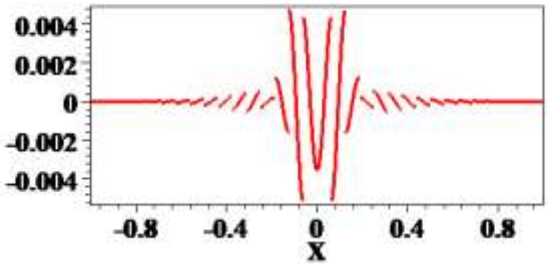

Fig. 12: The plot of the error of approximation of $u(x) = 1/(1 + 25x^2)$ with the polynomial splines $(n = 64)$

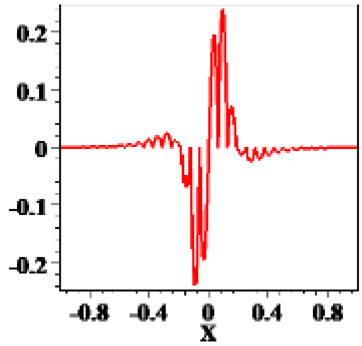

Fig. 13: The plot of the error of approximation of the first derivative of the function $u(x) = 1/(1 + 25x^2)$ with the polynomial splines $(n = 64)$

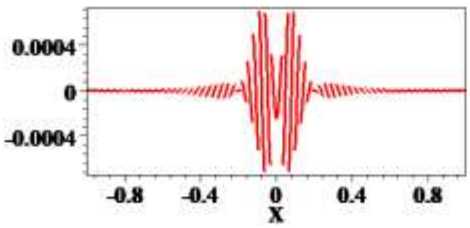

Fig. 14: The plot of the error of approximation of $u(x) = 1/(1 + 25x^2)$ with the polynomial splines $(n = 128)$

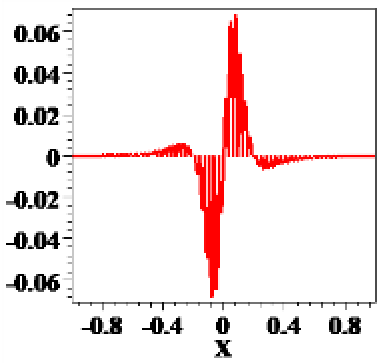

Fig. 15: The plot of the error of approximation of the first derivative of the function $u(x) = 1/(1 + 25x^2)$ with the polynomial splines $(n = 128)$

Figure 16 and Figure 17 show the plot of the error of approximation of the Runge function $u(x) = 1/(1 + 25x^2)$ with the Hermite-Birkhoff exponential splines.

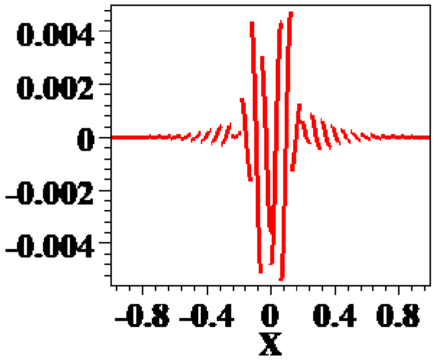

Fig. 16: The plot of the error of approximation of $u(x) = 1/(1 + 25x^2)$ with the exponential splines $(n = 64)$

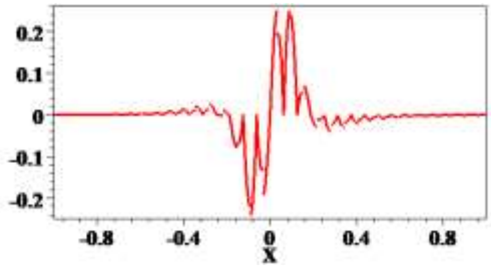

Fig. 17: The plot of the error of approximation of the first derivative of the function $u(x) = 1/(1 + 25x^2)$ with the exponential splines $(n = 64)$

Figure 18 and Figure 19 show the plot of the error of approximation of the Runge function $u(x) = \exp(x)$ with the Hermite-Birkhoff exponential splines (Digits=15).

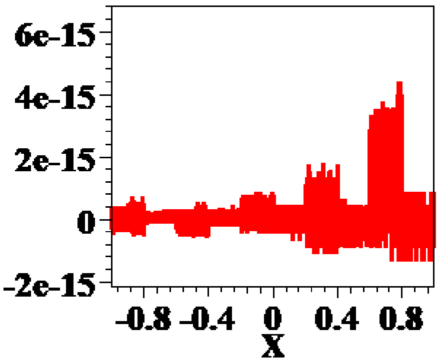

Fig. 18: The plot of the error of approximation of $u(x) = \exp(x)$ with the exponential splines $(n = 10, Digits = 15)$

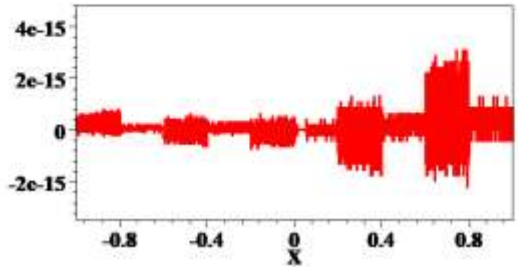

Fig. 19: The plot of the error of approximation of the first derivative of the function $u(x) = \exp(x)$ with the exponential splines $(n = 10, Digits = 15)$

Suppose the values of function u or its derivative are given on interval $[0,1]$ and $n = 10$. The values of function u and/or its derivative are specified at the nodes of a uniform grid with step $h = 0.1$.

Table 1 shows the actual approximation errors ($\max_{[0,1]}|\tilde{u}(x) - u(x)|$) when $\psi(x) = \varphi^2(x)$, obtained by solving the Hermite–Birkhoff problem in the Maple environment. At each grid interval, an approximation was constructed using Hermite-Birkhoff splines using formulas (1) or (2). Next, the maximum approximation error in absolute value was found at each grid interval, and then the maximum error among the values at each grid interval was selected over the entire interpolation interval. These values are given in the following Tables.

Table 1. The actual approximation errors

$\varphi\backslash u$	$\sin(x)$	e^x	$\cos(x)$	x^5
x	$3 \cdot 10^{-4}$	$7 \cdot 10^{-4}$	$2 \cdot 10^{-4}$	$1 \cdot 10^{-2}$
e^x	$9 \cdot 10^{-4}$	0	$9 \cdot 10^{-4}$	$4 \cdot 10^{-3}$

For comparison, in Table 2 we present the values of the theoretical error estimates. When constructing the data for Table 2, the theoretical estimates obtained in Section 3 were used at each grid interval. Next, the maximum value was selected, which was placed in Table 2.

Table 2. Theoretical estimates of the approximation error

$\varphi\backslash u$	$\sin(x)$	e^x	$\cos(x)$	x^5
x	$5 \cdot 10^{-4}$	$1 \cdot 10^{-3}$	$5 \cdot 10^{-4}$	$3 \cdot 10^{-2}$
e^x	$2 \cdot 10^{-3}$	0	$2 \cdot 10^{-3}$	$7 \cdot 10^{-3}$

The results presented in Table 1 and Table 2 confirm the accuracy of the found constant in the inequalities of approximation.

5 Conclusion

This paper discussed using local splines to solve Hermite-Birkhoff interpolation problems. Note that these splines can provide a solution to this problem even if the classical solution to the Hermite-Birkhoff problem has no solution.

This paper discussed spline interpolations using values of the functions and their first derivatives. In the future, we will consider other types of Hermite-Birkhoff splines.

Many papers are devoted to the construction of adaptive grids when solving various problems. The use of adaptive grids is advisable when applying grid methods to solve integral and differential equations. On one hand, adaptive grids allow us to reduce the number of grid nodes while on the other hand, it allows us to increase the accuracy of the solution. Using local Hermite-Birkhoff basis splines makes it easy to apply adaptive grids. It will be done in the next paper.

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References:

[1] I. J. Schoenberg, On Hermite-Birkhoff interpolation, *Journal of Mathematical Analysis and Applications*. Vol. 16, No. 3,1966, pp. *538–543*. doi*:*10.1016/0022-247X(66)90160-0

[2] W.F. Finden, An error term and uniqueness for Hermite–Birkhoff interpolation involving only function values and/or first derivative values, *Journal of Computational and Applied Mathematics,* Vol. 212, 2008, pp. 1–15.

[3] W. Finden, Higher order approximations using interpolation applied to collocation solutions of two point boundary value problems, *J. Comput. Appl. Math.*, Volume 206, Iss.1, 2007, pp 99-115, doi:10.1016/j.cam.2006.06.003.

[4] H. Jin, S. Pruess, Uniformly superconvergent approximations for linear two-point boundary value problems, *SIAM J. Numer. Anal.*, Vol. 35, No 1, 1998, pp. 363–375.

[5] D.J. Higham, Runge–Kutta defect control using Hermite–Birkhoff interpolation, *SIAM J. Sci. Comput.* , Vol. 12, 1991, pp. 991–999.

[6] W.H. Enright, The relative efficiency of alternative defect control schemes for high-order continuous Runge–Kutta formulas, *SIAM J. Numer. Anal.*, Vol. 30, No 5, 1993, pp. 1419–1445.

[7] Francesco Dell'Accio, Filomena Di Tommaso, Kai Horman, Reconstruction of a function from Hermite–Birkhoff data, *Journal Applied Mathematics and Computation,* Vol. 318, February 2018, pp. 51–69.

[8] Darell J. Johnson, The Trigonometric Hermite-Birkhoff Interpolation Problem, *Transactions of the American Mathematical Society,* Vol. 212, Oct. 1975, pp. 365-374, https://doi.org/10.2307/1998632

[9] Kurt Jetter, Duale Hermite-Birkhoff problem, *Journal of Approximation Theory*, Vol. 17, No 2, June 1976, pp. 119-134.

[10] I.G. Burova, On left integro-differential splines and Cauchy problem, *International Journal of Mathematical Models and Methods in Applied Sciences*, Vol.9, 2015, pp. 683-690.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Conflict of Interest
No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.