Gauss–Legendre Numerical Integrations for Average Run Length Running on EWMA Control Chart with Fractionally Integrated MAX Process

WILASINEE PEERAJIT
Department of Applied Statistics, Faculty of Applied Science,
King Mongkut's University of Technology,
North Bangkok, Bangkok 10800,
THAILAND

Abstract: - The performance of a process running on an exponentially weighted moving average (EWMA) control chart is contingent upon the ability to detect changes in the process mean rapidly. This entails determining the shortest average run length (ARL) for when a process becomes out-of-control (ARL_1). Herein, we propose a numerical integral equation (NIE) method to approximate the ARL for a long-memory fractionally integrated moving-average process with an exogenous variable (FI-MAX) with underlying exponential white noise running on an EWMA control chart using the Gauss-Legendre quadrature. In a numerical evaluation to compare its performance with that derived by using explicit formulas for this scenario, both performed equally well in terms of accuracy percentage (> 95%) and showed very consistent ARL_1 values. Therefore, the NIE approach is acceptable for approximating the ARL for this specific situation. In addition, comparing their standard deviations of the run length (SDRLs) illustrates that the NIE method performed better in rapidly detecting a shift in the process mean. Real data consistent with an FI-MAX process were also analyzed to demonstrate the applicability of using the proposed method for FI-MAX processes on EWMA charts.

Key-Words: - exponentially weighted moving average (EWMA) control chart, long-memory, average run length (ARL), fractionally integrated moving-average process with an exogenous variable, Gauss-Legendre quadrature, explicit formulas, exponential white noise.

Received: March 23, 2024. Revised: July 13, 2024. Accepted: August 29, 2024. Published: September 27, 2024.

1 Introduction

Statistical process control (SPC) refers to various analytical and statistical techniques to increase production efficiency. The control chart is an essential tool in SPC that is considered a significant instrument for monitoring manufacturing processes and detecting specific statistical variations. Walter A. Shewhart invented the control chart to identify manufacturing process variability, [1]. Despite its simplicity, the memoryless Shewhart control chart fails to detect small-to-moderate changes in process parameters. Researchers then developed memory-type control charts like cumulative sum (CUSUM) and exponentially weighted moving average (EWMA).

The study [2], proposed a CUSUM control chart as an alternative to the Shewhart control chart for detecting small-to-moderate changes in a process parameter. The study [3], introduced the EWMA statistic to improve the efficiency of control charts for small changes. In addition, EWMA statistics consider current and past information, resulting in superior performance compared to statistics that rely on current information alone. Other study, [4], used time-varying control limits instead of asymptotic ones to quickly detect mean changes on an EWMA control chart.

Average run length (ARL) is the average number of samples taken before the process changes until it reaches an out-of-control state. We refer to the process determination of an in-control ARL ARL_0 and the process determination of an out-of-control ARL ARL_1. Several researchers have evaluated the performance of EWMA control charts using ARL-based Monte Carlo simulations, Markov chain methods, or integral equations, [5], [6], [7].

The literature provides ARL computations to evaluate process performance on an EWMA control chart, [8], [9], [10]. Many have solved integral equations using analytical ARL with explicit formulas and approximated ARL with numerical integral equations. This study uses an NIE technique to approximate the ARL and measures its efficacy using the standard deviation of the run length (SDRL). We compared the effectiveness of this approach strategy to the explicit formulas documented in [11], [12], [13].

Time series analysis examines the autocorrelation functions, which decrease progressively in a hyperbolic manner. Other types of data, such as wind speed, air temperature, air quality, econometrics, and hydrologic phenomena, also exhibit these patterns. These processes exhibit a long-memory component and are often analyzed using the autoregressive fractionally integrated moving-average (ARFIMA) model, which is characterized by the fractional differencing parameter *d*. For a thorough comprehension of long-memory processes, consult the sources cited as [14], [15], [16], [17], [18]. In addition to the primary time series data, exogenous factors may have a strong link with the initial time series. These variables may be readily available or easily acquired. According to a research study, [19], incorporating these external factors into time series models enhances their performance and increases the accuracy of their predictions. The focus of the current work is on the fractionally integrated MA model with an exogenous variable (FI-MAX).

Prior research has utilized control charts to detect shifts in the mean of long-memory processes characterized by white noise following a normal distribution, [20], [21], [22]. However, white noise following an exponential distribution, [23], [24] is another process worthy of investigation.

Herein, we propose a solution to approximate the ARL of a long-memory FI-MAX process with underlying exponential white noise running on an EWMA control chart based on an NIE method using the Gauss-Legendre quadrature. To the best of our knowledge, this approach has not previously been investigated. Furthermore, the applicability of the suggested approach is compared with that of the ARL derived using explicit formulas.

The rest of the paper is organized as follows. In Section 2 , we provide the basic structure of the EWMA control chart running a long-memory FI-MAX process with exponential white noise. In Section 3, the approximation of the ARL for the specific situation mentioned above using the NIE method is derived and compared with the explicit formula approach. In Section 4 , the results of the computed ARLs and SDRLs for the proposed NIE and established empirical formula methods are discussed. Section 5 consists of an example of the implementation of the proposed method for approximating the ARL. Finally, conclusions are provided in Section 6.

2 The EWMA Statistic and the FI-MAX Process with Exponential White Noise

In [3] first provided a derivation of EWMA control charts by considering prior observations in the decision process, which is in contrast to the Shewhart control chart and better for detecting small-to-moderate changes in the process mean, [8]. The EWMA statistic is defined as follows:

$$D_t = (1-\lambda)D_{t-1} + \lambda Y_t,\ t = 1, 2, ... \quad (1)$$

where $t=1,$ the initial values of D_0 are set equal to φ, and λ is a smoothing parameter with a range of values from zero to one, [25]. In practice, its value is set between 0.05 and 0.3. For the in-control process, mean μ_0 and variance σ^2 of the EWMA statistic are

$$E(D_t) = \mu_0,\ V(D_t) = \sigma^2 \frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right], \quad (2)$$

respectively.

The center line (CL), upper control limit (UCL), and lower control limit (LCL) for the EWMA control chart are:

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]},$$
$$CL = \mu_0,$$
$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}, \quad (3)$$

respectively, where L is the width of the control limits. In practice, the value of the in-control ARL ARL_0 is predetermined. In this study, monitoring statistics D_t are plotted in relation to their corresponding control limits. If D_t is in the range of the LCL to the UCL, the process is in-control, meaning that no change has been noticed in the process mean and the process is proceeding within acceptable parameters. Conversely, if the value of D_t is greater than or equal to the UCL or LCL, the process is out-of-control.

The stopping time (τ) of the upper-sided EWMA control chart is given by:

$$\tau = \inf\{t \ge 0;\ D_t > \text{UCL}\},\ \text{for } \varphi \le \text{UCL}, \qquad (4)$$

In this study, we are concerned with a long-memory $\text{FI-MAX}(d,q,k)$ process, where d is the fractionally integrated/differentiated order, q is the MA order, and k is the order of the exogenous variables. The process is stationary and invertible when $0 < d < 0.5$. Now, we consider the long-memory FI-MAX process in the context of identifying changes in the process mean on an EWMA control chart, which can be written as

$$(1-B)^d Y_t = (1 - \sum_{i=1}^{q} \theta_i B^i)\varepsilon_t + \sum_{j=1}^{k} \beta_j X_{jt}, \qquad (5)$$

where θ_i is the i^{th} MA coefficient, β_j the *j-th* coefficient corresponding to k, and ε_t is a white noise process assumed to be exponentially distributed as $\varepsilon_t \sim Exp(v)$, when shift parameter $v > 0$. The fractional difference operator $(1-B)^d$ is:

$$\sum_{p=0}^{\infty} \binom{d}{p} (-B)^p = 1 - dB - \frac{d(1-d)}{2}B^2 - \frac{d(1-d)(2-d)}{6}B^3 - ...,$$

where d is the degree of the differencing parameter, B is a backward-shift operator, and $B^p Y_t = Y_{t-p}$ for order p. Therefore, the $\text{FI-MAX}(d,q,k)$ process can be expressed in the following general form:

$$\begin{aligned} Y_t &= \varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-j} + dY_{t-1} + \frac{d(1-d)}{2} Y_{t-2} \\ &\quad + \frac{d(1-d)(2-d)}{6} Y_{t-3} + ... + \sum_{j=1}^{k} \beta_j X_{jt} \end{aligned} \qquad (6)$$

where $|\theta_i| < 1;\ i = 1,2,...,q$ are MA coefficients and $X_{jt};\ j = 1,2,...,k.$ are exogenous variables, $\beta_j;\ j = 1,2,...,k.$ are coefficients depending on exogenous variable. The initial value of a long-memory $\text{FI-MAX}(d,q,k)$ process must satisfy $Y_{t-1}, Y_{t-2}, Y_{t-3},...,$ and $X_{1t}, X_{2t},..., X_{kt}$ = 1. For exponential white noise, the initial value of $\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3},...,$ is 1.

The process in Eq. (6) in conjunction with the EWMA statistic can be mathematically expressed as:

$$\begin{aligned} D_t &= (1-\lambda)D_{t-1} + \lambda(\varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-j} + dY_{t-1} + \frac{d(1-d)}{2} Y_{t-2} \\ &\quad + \frac{d(1-d)(2-d)}{6} Y_{t-3} + ... + \sum_{j=1}^{k} \beta_j X_{jt}) \\ &= (1-\lambda)D_{t-1} + \lambda\varepsilon_t - \lambda\sum_{i=1}^{q} \theta_i \varepsilon_{t-j} + \lambda dY_{t-1} + \frac{d(1-d)}{2} \lambda Y_{t-2} \\ &\quad + \frac{d(1-d)(2-d)}{6} \lambda Y_{t-3} + ... + \sum_{j=1}^{k} \beta_j X_{jt}) \end{aligned} \qquad (7)$$

The EWMA statistic for a long-memory $\text{FI-MAX}(d,q,k)$ process being in-control when LCL = 0 is as follows. If Y_1 provides the control state for D_1, then:

$$\begin{aligned} 0 &< (1-\lambda)D_0 + \lambda\varepsilon_1 - \lambda\theta_1\varepsilon_0 - \lambda\sum_{i=2}^{q} \theta_i \varepsilon_{t-j} + \lambda dY_0 + \frac{d(1-d)}{2} \lambda Y_{t-2} \\ &\quad + \frac{d(1-d)(2-d)}{6} \lambda Y_{t-3} + ... + \sum_{j=1}^{k} \beta_j X_{jt}) < \text{UCL} \end{aligned} \qquad (8)$$

Eq. (8) provides the initial value of $D_0 = \varphi$, which can be reformulated according to ε_1 as follows:

$$\begin{aligned} &\left((1-\lambda)\varphi + \lambda\theta_1\varepsilon_0 + \lambda\sum_{i=2}^{q} \theta_i \varepsilon_{t-j} - \lambda dY_0 - \frac{d(1-d)}{2} \lambda Y_{t-2} \right. \\ &\quad \left. - \frac{d(1-d)(2-d)}{6} \lambda Y_{t-3} - ... - \lambda\sum_{j=1}^{k} \beta_j X_{jt} \right) \Big/ \lambda \\ &\qquad < \varepsilon_1 < \\ &\left(\text{UCL} - (1-\lambda)\varphi + \lambda\theta_1\varepsilon_0 + \lambda\sum_{i=2}^{q} \theta_i \varepsilon_{t-j} - \lambda dY_0 - \frac{d(1-d)}{2} \lambda Y_{t-2} \right. \\ &\quad \left. - \frac{d(1-d)(2-d)}{6} \lambda Y_{t-3} - ... - \lambda\sum_{j=1}^{k} \beta_j X_{jt} \right) \Big/ \lambda \end{aligned}$$

or $L < \varepsilon_t < H$. According to the bounds in the equation above, the probability distribution function ε_1 can be rewritten as

$$P(L < \varepsilon_1 < H) = \int_L^H f(z)\,dz,$$

where $f(z)$ represents the probability density function of an exponential distribution. Thus, the approximated ARL can now be computed using this structure for the EWMA statistic.

3 Approximating the ARL

Fredholm's approximation of the ARL using an integral equation of the second kind can theoretically represent the ARL. [26], was the first to use this approach for a process on an EWMA chart. In this section, we approximate the ARL by solving an integral equation using the Gauss-Legendre quadrature, [27].

3.1 The NIE Method

Let $L(\varphi)$ represent the ARL, which detects small changes in the mean of the long-memory FI-MAX(d,q,k) process, starting from the initial value $(D_0 = \varphi)$. Subsequently, $L(\varphi) = E_\infty(\tau) \geq \gamma$ representing the solution to the integral equation can be rewritten as:

$$L(\varphi) = 1 + \frac{1}{\lambda}\int_0^{UCL} L(z) f\Big(\frac{z-(1-\lambda)\varphi}{\lambda} + \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} - dY_{t-1} - \frac{d(1-d)}{2}Y_{t-2} - \frac{d(1-d)(2-d)}{6}Y_{t-3} - \dots - \sum_{j=1}^{k}\beta_j X_{jt}\Big)dz \quad (9)$$

Since the integration interval can become infinite under Gaussian rules, the weight function $W(z)$ must not equal 1 and the set of points $z_r, r = 1,2,..,n$ are spaced equally. The Gaussian function is in the form:

$$\int_0^{UCL} W(z) f(z) dz \approx \sum_{r=1}^{m} w_r f(a_r)$$

where interval [0, UCL] is partitioned into a sequence of points $0 \leq a_1 \leq a_2 \leq .... \leq a_m \leq \text{UCL}$, where: $a_r = \frac{UCL}{m}\left(r - \frac{1}{2}\right),\ w_r = \frac{UCL}{m};\ r = 1,2,...,m.$ represents a set of constant weights.

A numerical approximation for an integral equation can be obtained by applying the Gauss-Legendre quadrature, which involves solving the following system of algebraic linear equations:

$$\tilde{L}(a_l) \approx 1 + \frac{1}{\lambda}\sum_{r=1}^{m} w_r \tilde{L}(a_r) f\left(\frac{a_r - (1-\lambda)a_l}{\lambda} - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + dY_{t-1} + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + \dots + \sum_{j=1}^{k}\beta_j X_{jt}\right)$$

for $l = 1,2,...,m.$ Thus,

$$\tilde{L}(a_1) \approx 1 + \frac{1}{\lambda}\sum_{r=1}^{m} w_r \tilde{L}(a_r) f\left(\frac{a_r - (1-\lambda)a_1}{\lambda} - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + dY_{t-1} + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + \dots + \sum_{j=1}^{k}\beta_j X_{jt}\right)$$

$$\tilde{L}(a_2) \approx 1 + \frac{1}{\lambda}\sum_{r=1}^{m} w_r \tilde{L}(a_r) f\left(\frac{a_r - (1-\lambda)a_2}{\lambda} - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + dY_{t-1} + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + \dots + \sum_{j=1}^{k}\beta_j X_{jt}\right)$$

$$\vdots$$

$$\tilde{L}(a_m) \approx 1 + \frac{1}{\lambda}\sum_{r=1}^{m} w_r \tilde{L}(a_r) f\left(\frac{a_r - (1-\lambda)a_m}{\lambda} - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + dY_{t-1} + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + \dots + \sum_{j=1}^{k}\beta_j X_{jt}\right).$$

This can be reformulated in matrix format as:

$$\mathbf{L}_{m\times1} = \mathbf{1}_{m\times1} + \mathbf{R}_{m\times m}\mathbf{L}_{m\times1} \quad (10)$$

where $\mathbf{L}_{m\times1} = \left[\hat{L}(a_1), \hat{L}(a_2), ..., \hat{L}(a_m), \right]'$ is a column vector of $\tilde{L}(a_l);\ l = 1, 2, ..., m,$ $\mathbf{I}_{m\times1} = diag(1,1,\dots,1)$ is the unit matrix order m, $\mathbf{1}_{m\times1} = [1,1,...,1]'$ is a column vector of ones, and $\mathbf{R}_{m\times m}$ is a matrix having the element $(m,m)^{th}$, which can be expressed as:

$$\mathbf{R}_{m\times m} = \begin{bmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ R_{21} & R_{22} & \dots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{m1} & R_{m2} & \dots & Rr_{mm} \end{bmatrix},$$

with $R_{lr} \approx \frac{1}{\lambda} w_r f\left(\frac{a_r-(1-\lambda)a_l}{\lambda}-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+dY_{t-1}\right.$

$\left.+\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...+\sum_{j=1}^{k}\beta_j X_{jt}\right)$

where $R_{lr};\ l,r=1,2,....,m.$

As the inverse of $(\mathbf{I}_m-\mathbf{L}_{m\times m})^{-1}$ both exists and is invertible, approximating the ARL using the NIE method can be mathematically represented as the following system of linear equations in matrix form:

$$\mathbf{L}_{m\times 1}=(\mathbf{I}_m-\mathbf{L}_{m\times m})^{-1}\mathbf{1}_{m\times 1}, \quad (11)$$

Finally, a_l is replaced with φ in $\tilde{L}(a_l)$. Therefore, approximating the ARL for a long-memory FI-MAX(d,q,k) process on a EWMA control chart using the NIE method becomes:

$$\tilde{L}(\varphi)\approx 1+\frac{1}{\lambda}\sum_{r=1}^{m}w_r\tilde{L}(a_r)\,f\left(\frac{a_r-(1-\lambda)\varphi}{\lambda}-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+dY_{t-1}+\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...+\sum_{j=1}^{k}\beta_j X_{jt}\right) \quad (12)$$

where $a_r=\frac{UCL}{m}\left(r-\frac{1}{2}\right),\ w_r=\frac{UCL}{m};\ r=1,2,...,m.$

3 .2 The Explicit Formulas for the Analytical ARL

The ARL computed via explicit formulas used to verify the proposed method covered in the previous subsection is obtained by solving the integral equation as follows:

$$L(\varphi)=1-\frac{\lambda e^{\frac{(1-\lambda)\varphi}{\lambda\nu}}\left(e^{-\frac{\mathrm{UCL}}{\lambda\nu}}-1\right)}{\lambda e^{-\frac{1}{\nu}\left(\sum_{j=1}^{k}\beta_j X_{jt}-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+dY_{t-1}+\frac{d(1-d)}{2}Y_{t-2}+...\right)}+\left(e^{-\frac{UCL}{\nu}}-1\right)}. \quad (13)$$

Moreover, the explicit formula for ARL_0 when the exponential parameter $\nu=\nu_0$ is:

$$\mathrm{ARL}_0=1-\frac{\lambda e^{\frac{(1-\lambda)\varphi}{\lambda\nu_0}}\left(e^{-\frac{\mathrm{UCL}}{\lambda\nu_0}}-1\right)}{\lambda e^{-\frac{1}{\nu_0}\left(\sum_{j=1}^{k}\beta_j X_{jt}-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+dY_{t-1}+\frac{d(1-d)}{2}Y_{t-2}+...\right)}+\left(e^{-\frac{UCL}{\nu_0}}-1\right)}. \quad (14)$$

The explicit formula for ARL_1 when $\nu=\nu_1,$ is:

$$\mathrm{ARL}_1=1-\frac{\lambda e^{\frac{(1-\lambda)\varphi}{\lambda\nu_1}}\left(e^{-\frac{\mathrm{UCL}}{\lambda\nu_1}}-1\right)}{\lambda e^{-\frac{1}{\nu_1}\left(\sum_{j=1}^{k}\beta_j X_{jt}-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+dY_{t-1}+\frac{d(1-d)}{2}Y_{t-2}+...\right)}+\left(e^{-\frac{UCL}{\nu_1}}-1\right)}. \quad (15)$$

Using the above-mentioned equations, the Wolfram Mathematica program was written and run using the code of both methods. The ARLs for in-control or non-shifted processes were approximated and analyzed ARL for mean monitoring under long-memory FI-MAX with exponential white noise for different process settings.

4 Performance Evaluation and Comparison

We compared the ARL for a long-memory FI-MAX process with exponential white noise on an EWMA control chart obtained using the proposed NIE method by applying the Gaussian rule with $m=800$ subintervals, as described in Eq. (12), with that using the explicit formulas specified in Eqs. (13) and (14). The optimal values for parameters λ and UCL were obtained by minimizing the out-of-control ARL for process mean shifts of 0.03, 0.05, or 0.10.

It is assumed that the white noise has an exponential distribution $(\varepsilon_t \sim Exp(\nu))$ with the mean parameter $(\nu=\nu_0)$ equal to 1 for the in-control process and $\nu=\nu_1=(1+\delta)\nu_0$ for the out-of-control process (therein, δ = 0.01, 0.02, 0.04, 0.08, 0.10, 0.20, 0.40, or 0.80). Four long-memory FI-MAX models were constructed: FI-MAX(d, 1, 1), FI-MAX(d, 2, 1), FI-MAX(d, 2, 1) and FI-MAX(d, 2, 2) with d =0.1, 0.2, 0.4, $\theta_1=0.1, \theta_2=0.2, \beta_1=0.1$ and $\beta_2=0.2.$

Eq. (12) can be solved via a grid search of various combinations of possible values when the value of the smoothing constant (λ) is specified. Se pre-specified the value of ARL_0 = 370 and calculated the UCL for each process, as reported in Table 1.

The performance metric used to compare the ARL values calculated using the proposed method and explicit formulas is the accuracy percentage, which is defined as follows:

$$\%\,\text{Accuracy} = 100 - \left| \frac{L(\varphi) - \tilde{L}(\varphi)}{L(\varphi)} \right| \times 100\%, \qquad (16)$$

where $L(\varphi)$ and $\tilde{L}(\varphi)$ represent the ARL values using the NIE and explicit formulas methods, respectively. A high accuracy percentage (>95%) means that the ARL value obtained using the proposed method is close to that obtained using the explicit formulas. Note that all of the coefficient parameters for a long-memory FI-MAX were determined, and the optimal smoothing parameter was used to compute the UCL.

The out-of-control ARL (ARL_1) values obtained using both methods for the different process settings are provided in Table 2, Table 3 and Table 4. It can be seen that the ARL_1 values depend on the value of δ: as the value of δ was increased, the ARL_1values decreased, and vice versa. For example, from Table 2, the ARL_1 values using the proposed NIE method for the long-memory $\text{FI-MAX}(d = 0.1, 1, 1)$ process with δ = 0.01, 0.02, 0.04, 0.08, 0.10, 0.20, 0.40, or 0.80 were 303.176, 249.390, 170.717, 83.564, 59.699, 13.599, 2.101,and 1.041, respectively. Importantly, the ARL_1 values calculated using the NIE method obtained accuracy percentages of 100% in every case, meaning that it is highly accurate at detecting shifts in the mean for all of the processes. Intriguingly, we also observed that its detection sensitivity was greater for the long-memory processes with d = 0.4 than d = 0.2 or 0.1. The proposed method also performed well when chart parameter λ = 0.3, 0.1, or 0.05 when $0.01 \le \delta < 0.2,$ and the optimal smoothing constant value was 0.3. Similarly, for $0.2 \le \delta \le 0.8,$ the optimal smoothing constant value was 0.05, as shown in Figure 1 for $\text{FI-MAX}(0.4, q, k).$

Another performance metric used in the study was the standard deviation of the run length (SDRL) [28]. The results in Table 5 indicate that for the optimal smoothing constant value of 0.3, the $SDRL_1$ values for all of the long-memory $\text{FI-MAX}(d, q, k)$ processes evaluated using both methods were consistently less than their ARL_1 counterparts, albeit the outcomes were the

In summary, the proposed NIE method could rapidly detect changes in the mean of FI-MAX processes with underlying exponential white noise on an EWMA control chart as consistently as the established explicit formulas method.

Table 1. The parameter values for long-memory $\text{FI-MAX}(d, q, k)$ processes on an EWMA control chart for ARL_0 = 370.

Long-memory process		Coefficient parameters				λ		
		θ_1	θ_2	β_1	β_2	0.05	0.10	0.30
d = 0.1	FI-MAX(d, 1, 1)	0.1	-	0.1	-	0.0000000869061	0.00372766	0.28380500
	FI-MAX(d, 1, 2)	0.1	-	0.1	0.2	0.0000000711527	0.00304198	0.22572572
	FI-MAX(d, 2, 1)	0.1	0.2	0.1	-	0.0000001061474	0.004571322	0.35982135
	FI-MAX(d, 2, 2)	0.1	0.2	0.1	0.2	0.0000000869061	0.00372766	0.28380500
d = 0.2	FI-MAX(d, 1, 1)	0.1	-	0.1	-	0.0000000744650	0.00318577	0.23765043
	FI-MAX(d, 1, 2)	0.1	-	0.1	0.2	0.0000000609667	0.00260101	0.18995314
	FI-MAX(d, 2, 1)	0.1	0.2	0.1	-	0.0000000909517	0.00390448	0.29930070
	FI-MAX(d, 2, 2)	0.1	0.2	0.1	0.2	0.0000000744650	0.00318577	0.23765043
d = 0.4	FI-MAX(d, 1, 1)	0.1	-	0.1	-	0.0000000576464	0.002457675	0.178574370
	FI-MAX(d, 1, 2)	0.1	-	0.1	0.2	0.0000000471969	0.002007845	0.143620154
	FI-MAX(d, 2, 1)	0.1	0.2	0.1	-	0.0000000704095	0.003009740	0.223070220
	FI-MAX(d, 2, 2)	0.1	0.2	0.1	0.2	0.0000000576464	0.002457672	0.178574370

Table 2. The ARL_1 values were derived by using the proposed NIE and explicit formulas methods for long-memory $\text{FI-MAX}(d,q,k)$ processes on an EWMA control chart when $\lambda = 0.05$.

Long-memory	δ	FI-MAX(d, 1, 1)			FI-MAX(d, 1, 2)			FI-MAX(d, 2, 1)			FI-MAX(d, 2, 2)		
		NIE	Explicit	%Acc	NIE	Explicit	%Acc	NIE	Explicit	%Acc	NIE	Explicit	%Acc
$d = 0.1$	0.01	303.176	303.176	100%	302.578	302.578	100%	303.775	303.775	100%	303.176	303.176	100%
	0.02	249.390	249.390	100%	248.428	248.428	100%	250.377	250.377	100%	249.390	249.390	100%
	0.04	170.717	170.717	100%	169.417	169.417	100%	172.028	172.028	100%	170.717	170.717	100%
	0.08	83.564	83.564	100%	82.349	82.349	100%	84.796	84.796	100%	83.564	83.564	100%
	0.10	59.699	59.699	100%	58.642	58.642	100%	60.777	60.777	100%	59.699	59.699	100%
	0.20	13.599	13.599	100%	13.177	13.177	100%	14.017	14.017	100%	13.590	13.590	100%
	0.40	2.101	2.101	100%	2.040	2.040	100%	2.166	2.166	100%	2.101	2.101	100%
	0.80	1.041	1.041	100%	1.037	1.037	100%	1.045	1.045	100%	1.041	1.041	100%
$d = 0.2$	0.01	302.714	302.714	100%	302.117	302.117	100%	303.312	303.312	100%	302.714	302.714	100%
	0.02	248.649	248.649	100%	247.679	247.679	100%	249.622	249.622	100%	248.649	248.649	100%
	0.04	169.712	169.712	100%	168.419	168.419	100%	171.014	171.014	100%	169.712	169.712	100%
	0.08	82.625	82.625	100%	81.424	81.424	100%	83.843	83.843	100%	82.625	82.625	100%
	0.10	58.881	58.881	100%	57.838	57.838	100%	59.943	59.943	100%	58.881	58.881	100%
	0.20	13.270	13.270	100%	12.868	12.868	100%	13.686	13.686	100%	13.270	13.270	100%
	0.40	2.053	2.053	100%	1.995	1.995	100%	2.115	2.115	100%	2.053	2.053	100%
	0.80	1.038	1.038	100%	1.035	1.035	100%	1.042	1.042	100%	1.038	1.038	100%
$d = 0.4$	0.01	**301.949**	301.949	100%	**301.355**	301.355	100%	**302.547**	302.547	100%	**301.949**	301.949	100%
	0.02	**247.409**	247.409	100%	**246.445**	246.445	100%	**248.377**	248.377	100%	**247.409**	247.409	100%
	0.04	**168.059**	168.059	100%	**166.778**	166.778	100%	**169.349**	169.349	100%	**168.059**	168.059	100%
	0.08	**81.091**	81.091	100%	**79.914**	79.914	100%	**82.287**	82.287	100%	**81.091**	81.091	100%
	0.10	**57.550**	57.550	100%	**56.531**	56.531	100%	**58.587**	58.587	100%	**57.550**	57.550	100%
	0.20	**12.757**	12.757	100%	**12.372**	12.372	100%	**13.156**	13.156	100%	**12.757**	12.757	100%
	0.40	**1.979**	1.979	100%	**1.925**	1.925	100%	**2.037**	2.037	100%	**1.979**	1.979	100%
	0.80	**1.034**	1.034	100%	**1.031**	1.031	100%	**1.037**	1.037	100%	**1.034**	1.034	100%

Table 3. The ARL_1 values derived by using the proposed NIE and explicit formulas methods for long-memory $\text{FI-MAX}(d,q,k)$ processes on an EWMA control chart when $\lambda = 0.10$.

Long-memory	δ	FI-MAX(d, 1, 1)			FI-MAX(d, 1, 2)			FI-MAX(d, 2, 1)			FI-MAX(d, 2, 2)		
		NIE	Explicit	%Acc	NIE	Explicit	%Acc	NIE	Explicit	%Acc	NIE	Explicit	%Acc
$d = 0.1$	0.01	334.508	334.508	100%	333.808	333.808	100%	335.212	335.212	100%	334.508	334.508	100%
	0.02	302.999	302.999	100%	301.745	301.745	100%	304.263	304.263	100%	302.999	302.999	100%
	0.04	249.984	249.984	100%	247.962	247.962	100%	252.031	252.031	100%	249.984	249.984	100%
	0.08	173.729	173.729	100%	171.043	171.043	100%	176.469	176.469	100%	173.729	173.729	100%
	0.10	146.237	146.237	100%	143.473	143.473	100%	149.068	149.068	100%	146.237	146.237	100%
	0.20	67.279	67.279	100%	64.994	64.994	100%	69.658	69.658	100%	67.279	67.279	100%
	0.40	19.990	19.990	100%	18.888	18.888	100%	21.167	21.167	100%	19.990	19.990	100%
	0.80	4.428	4.428	100%	4.125	4.125	100%	4.761	4.761	100%	4.428	4.428	100%
$d = 0.2$	0.01	333.966	333.966	100%	333.269	333.269	100%	334.668	334.668	100%	333.966	333.966	100%
	0.02	302.029	302.029	100%	300.782	300.782	100%	303.286	303.286	100%	302.029	302.029	100%
	0.04	248.419	248.419	100%	246.416	246.416	100%	250.448	250.448	100%	248.419	248.419	100%
	0.08	171.649	171.649	100%	169.003	169.003	100%	174.348	174.348	100%	171.649	171.649	100%
	0.10	144.096	144.096	100%	141.379	141.379	100%	146.875	146.875	100%	144.096	144.096	100%
	0.20	65.506	65.506	100%	63.289	63.289	100%	67.812	67.812	100%	65.506	65.506	100%
	0.40	19.133	19.133	100%	18.084	18.084	100%	20.251	20.251	100%	19.133	19.133	100%
	0.80	4.192	4.192	100%	3.911	3.911	100%	4.501	4.501	100%	4.191	4.191	100%
$d = 0.4$	0.01	**333.074**	333.074	100%	**332.379**	332.379	100%	**333.771**	333.771	100%	**333.074**	333.074	100%
	0.02	**300.435**	300.435	100%	**299.199**	299.199	100%	**301.679**	301.679	100%	**300.435**	300.435	100%
	0.04	**245.859**	245.859	100%	**243.883**	243.883	100%	**247.857**	247.857	100%	**245.859**	245.859	100%
	0.08	**168.272**	168.272	100%	**165.688**	165.688	100%	**170.904**	170.904	100%	**168.271**	168.271	100%
	0.10	**140.631**	140.631	100%	**137.991**	137.991	100%	**143.329**	143.329	100%	**140.631**	140.631	100%
	0.20	**62.684**	62.684	100%	**60.573**	60.573	100%	**64.877**	64.877	100%	**62.683**	62.683	100%
	0.40	**17.802**	17.802	100%	**16.835**	16.835	100%	**18.832**	18.832	100%	**17.802**	17.802	100%
	0.80	**3.838**	3.838	100%	**3.590**	3.590	100%	**4.110**	4.110	100%	**3.838**	3.838	100%

Table 4. The ARL_1 values derived by using the proposed NIE and explicit formulas methods for long-memory $\text{FI-MAX}(d,q,k)$ processes on an EWMA control chart when $\lambda = 0.30$.

Long-memory	δ	FI-MAX(d, 1, 1)			FI-MAX(d, 1, 2)			FI-MAX(d, 2, 1)			FI-MAX(d, 2, 2)		
		NIE	Explicit	%Acc	NIE	Explicit	%Acc	NIE	Explicit	%Acc	NIE	Explicit	%Acc
d = 0.1	0.01	240.321	240.321	100%	228.369	228.369	100%	255.158	255.158	100%	240.321	240.321	100%
	0.02	176.309	176.309	100%	163.478	163.478	100%	193.095	193.095	100%	176.309	176.309	100%
	0.04	113.018	113.018	100%	102.283	102.283	100%	127.776	127.776	100%	113.018	113.018	100%
	0.08	63.276	63.276	100%	56.148	56.148	100%	73.417	73.417	100%	63.276	63.276	100%
	0.10	51.092	51.092	100%	45.105	45.105	100%	59.665	59.665	100%	51.092	51.092	100%
	0.20	24.342	24.342	100%	21.229	21.229	100%	28.823	28.823	100%	24.342	24.342	100%
	0.40	10.568	10.568	100%	9.168	9.168	100%	12.557	12.557	100%	10.568	10.568	100%
	0.80	4.542	4.542	100%	3.975	3.975	100%	5.328	5.328	100%	4.542	4.542	100%
d = 0.2	0.01	230.899	230.899	100%	220.414	220.414	100%	243.389	243.389	100%	230.899	230.899	100%
	0.02	166.147	166.147	100%	155.256	155.256	100%	179.701	179.701	100%	166.147	166.147	100%
	0.04	104.478	104.478	100%	95.641	95.641	100%	115.933	115.933	100%	104.478	104.478	100%
	0.08	57.589	57.589	100%	51.841	51.841	100%	65.247	65.247	100%	57.589	57.589	100%
	0.10	46.312	46.312	100%	41.504	41.504	100%	52.754	52.754	100%	46.312	46.312	100%
	0.20	21.855	21.855	100%	19.367	19.367	100%	25.208	25.208	100%	21.855	21.855	100%
	0.40	9.451	9.451	100%	8.327	8.327	100%	10.956	10.956	100%	9.451	9.451	100%
	0.80	4.091	4.091	100%	3.632	3.632	100%	4.697	4.697	100%	4.091	4.091	100%
d = 0.4	0.01	**217.737**	217.737	100%	**208.879**	208.879	100%	**227.798**	227.798	100%	**217.737**	217.737	100%
	0.02	**152.545**	152.545	100%	**143.766**	143.766	100%	**162.879**	162.879	100%	**152.545**	152.545	100%
	0.04	**93.490**	93.490	100%	**86.656**	86.656	100%	**101.794**	101.794	100%	**93.490**	93.490	100%
	0.08	**50.463**	50.463	100%	**46.142**	46.142	100%	**55.828**	55.828	100%	**50.463**	50.463	100%
	0.10	**40.355**	40.355	100%	**36.761**	36.761	100%	**44.837**	44.837	100%	**40.355**	40.355	100%
	0.20	**18.775**	18.775	100%	**16.934**	16.934	100%	**21.090**	21.090	100%	**18.775**	18.775	100%
	0.40	**8.060**	8.060	100%	**7.228**	7.228	100%	**9.106**	9.106	100%	**8.060**	8.060	100%
	0.80	**3.522**	3.522	100%	**3.182**	3.182	100%	**3.950**	3.950	100%	**3.522**	3.522	100%

Table 5. The ARL_1 values derived from NIE and $SDRL_1$ values derived from [28] for various shift sizes in the mean of long-memory $\text{FI-MAX}(d,q,k)$ processes on an EWMA control chart when $\lambda = 0.30$.

Long-memory	δ	FI-MAX(d, 1, 1)		FI-MAX(d, 1, 2)		FI-MAX(d, 2, 1)		FI-MAX(d, 2, 2)	
		ARL	SDRL	ARL	SDRL	ARL	SDRL	ARL	SDRL
d = 0.1	0.01	240.321	239.820	228.369	227.868	255.158	254.658	240.321	239.820
	0.02	176.309	175.808	163.478	162.977	193.095	192.594	176.309	175.808
	0.04	113.018	112.517	102.283	101.782	127.776	127.275	113.018	112.517
	0.08	63.276	62.774	56.148	55.646	73.417	72.915	63.276	62.774
	0.10	51.092	50.590	45.105	44.602	59.665	59.163	51.092	50.590
	0.20	24.342	23.837	21.229	20.723	28.823	28.319	24.342	23.837
	0.40	10.568	10.056	9.168	8.654	12.557	12.047	10.568	10.056
	0.80	4.542	4.011	3.975	3.439	5.328	4.802	4.542	4.011
d = 0.2	0.01	230.899	230.398	220.414	219.913	243.389	242.888	230.899	230.398
	0.02	166.147	165.646	155.256	154.755	179.701	179.200	166.147	165.646
	0.04	104.478	103.977	95.641	95.140	115.933	115.432	104.478	103.977
	0.08	57.589	57.087	51.841	51.339	65.247	64.745	57.589	57.087
	0.10	46.312	45.809	41.504	41.001	52.754	52.252	46.312	45.809
	0.20	21.855	21.349	19.367	18.860	25.208	24.703	21.855	21.349
	0.40	9.451	8.937	8.327	7.811	10.956	10.444	9.451	8.937
	0.80	4.091	3.556	3.632	3.092	4.697	4.167	4.091	3.556
d = 0.4	0.01	217.737	217.236	208.879	208.378	227.798	227.297	217.737	217.236
	0.02	152.545	152.044	143.766	143.265	162.879	162.378	152.545	152.044
	0.04	93.490	92.989	86.656	86.155	101.794	101.293	93.490	92.989
	0.08	50.463	49.960	46.142	45.639	55.828	55.326	50.463	49.960
	0.10	40.355	39.852	36.761	36.258	44.837	44.334	40.355	39.852
	0.20	18.775	18.268	16.934	16.426	21.090	20.584	18.775	18.268
	0.40	8.060	7.543	7.228	6.709	9.106	8.591	8.060	7.543
	0.80	3.522	2.980	3.182	2.635	3.950	3.414	3.522	2.980

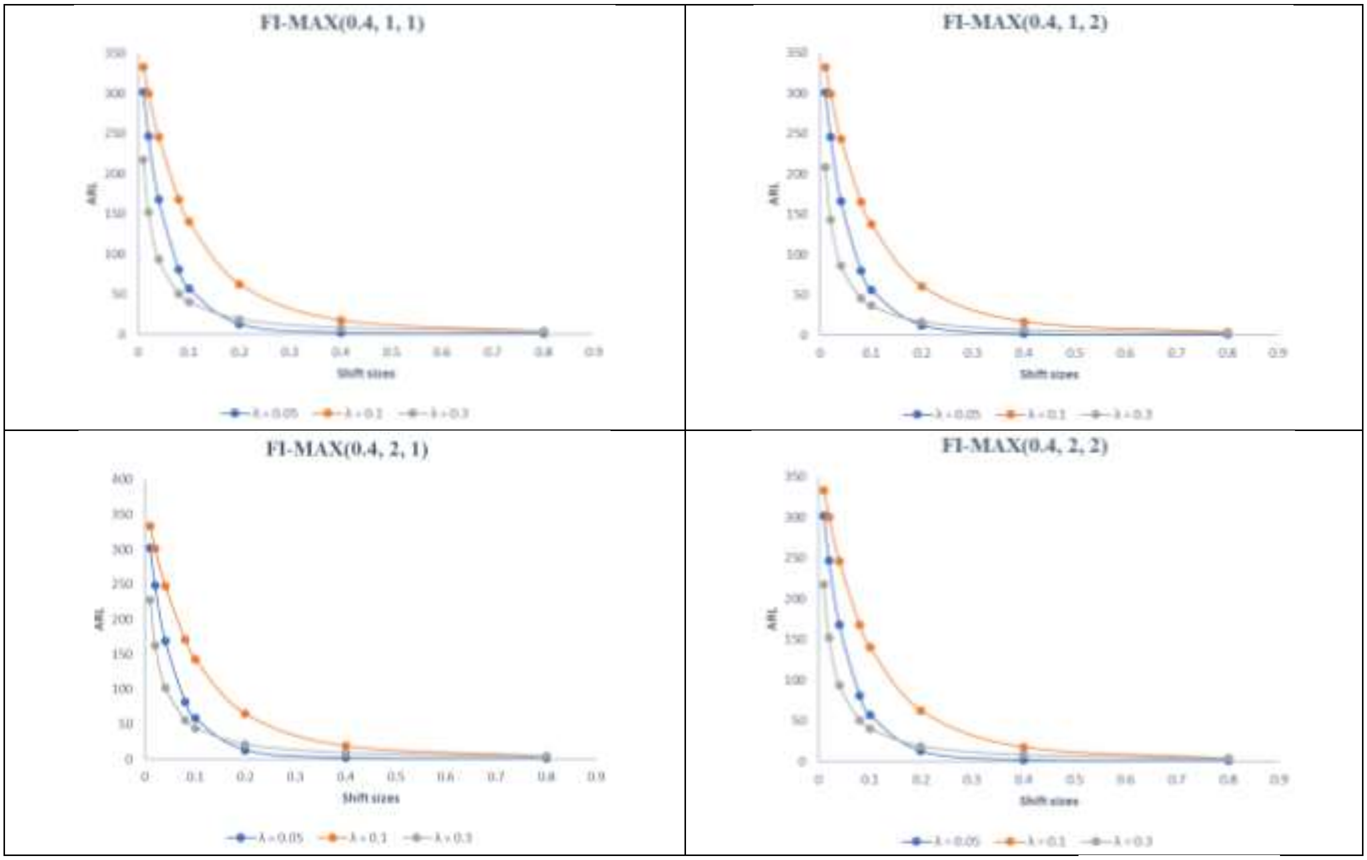

Fig. 1: Graphical displays of the ARL_1 values derived by using the NIE method for long-memory $\text{FI-MAX}(d = 0.4, q, k)$ processes on an EWMA control chart

5 An Empirical Example with Real Data

We demonstrate the applicability and performance of the proposed method and compare it with the explicit formulas for a real-life scenario. To this end, a dataset of the Bank of America Corporation (BAC) stock price and accompanying Bitcoin USD (BTC-USD) prices and EUR/USD exchange rates as exogenous variables were obtained from https://th.investing.com. The dataset consisted of 212 weekly observations from March 8, 2020, to March 24, 2024, for which the best-fitting long-memory FI-MAX process was determined. In Table 6 , estimates of the process coefficients are $\hat{d}$ =0.499999, $\hat{\theta}_1$=0.423979, $\hat{\beta}_1$=0.000141, and $\hat{\beta}_2$=42.22037 for a long-memory $\text{FI-MAX}(0.499999, 1, 2)$ process. The exponential distribution of the white noise was confirmed by using the Kolmogorov-Smirnov test (KS = 1.3340; p−value > 0.05). The exponential parameter $(\nu = \nu_0)$ was 1.2772, $\varepsilon_t \sim Exp(1.2772)$.

Table 6. Parameter estimation for fitting the BAC stock price dataset with BTC-USD prices and EUR/USD exchange rates as exogenous variables to an FI-MAX process

Parameters:	Coefficient	Std. Error	t-Statistic	Prob.
EUR/USD	42.22037	7.789263	5.420330	0.0000*
BTC/USD	0.000141	0.000027	5.216840	0.0000*
d	0.499999	0.000424	1178.181	0.0000*
MA(1)	0.423979	0.064186	6.605506	0.0000*
Testing distribution of the white noise				
Exponential Parameter ($\nu = \nu_0$)				1.2772
Kolmogorov-Smirnov				1.3340
Asymptotic Significance (2-Sided)				0.0570^{ns}

* significance level of 0.05
^{ns} non-significance level of 0.05.

Afterward, the proposed NIE method in Eq. (12) was applied to the long-memory $\text{FI-MAX}(0.499999, 1, 2)$ on an EWMA control chart as follows:

$$\tilde{L}(\varphi) \approx 1 + \frac{1}{\lambda}\sum_{r=1}^{m} w_r \tilde{L}(a_r) f\left(\frac{a_r - (1-\lambda)\varphi}{\lambda} - 0.423979\varepsilon_{t-1} + 0.49999Y_{t-1} + 0.125Y_{t-2} + 0.0625Y_{t-3} + ... + 42.22037X_{1t} + 0.000141X_{2t}\right) \quad (17)$$

where $a_r = \frac{UCL}{m}\left(r - \frac{1}{2}\right)$, $w_r = \frac{UCL}{m}$; $r = 1, 2, ..., m.$ represents a set of constant weights.

Table 7. ARL_1 values derived by using the NIE and explicit formulas methods and SDRL_1 values for the long-memory FI-MAX$(0.499999, 1, 2)$ process on an EWMA control chart

λ	UCL	δ	ARL_1 NIE	ARL_1 Explicit	SDRL_1	%Acc
0.05	1.89×10^{-10}	0.01	284.703	284.703	284.203	100%
		0.02	220.228	220.228	219.727	100%
		0.04	133.831	133.831	133.330	100%
		0.08	52.502	52.502	52.000	100%
		0.10	33.891	33.891	33.387	100%
		0.20	5.353	5.353	4.827	100%
		0.40	1.178	1.178	0.458	100%
		0.80	1.002	1.002	0.045	100%
0.10	4.8×10^{-17}	0.01	245.554	245.554	245.053	100%
		0.02	164.388	164.388	163.887	100%
		0.04	75.647	75.647	75.145	100%
		0.08	17.998	17.998	17.491	100%
		0.10	9.445	9.445	8.931	100%
		0.20	1.363	1.363	0.703	100%
		0.40	1.002	1.002	0.045	100%
		0.80	1.000	1.000	0.000	100%
0.30	5.5×10^{-17}	0.01	242.907	242.907	242.406	100%
		0.02	160.905	160.905	160.404	100%
		0.04	72.558	72.558	72.056	100%
		0.08	16.670	16.670	16.162	100%
		0.10	8.643	8.643	8.128	100%
		0.20	1.302	1.302	0.627	100%
		0.40	1.002	1.002	0.045	100%
		0.80	1.000	1.000	0.000	100%

The proposed NIE method was used to compute ARL_1 values for a fixed $\mathrm{ARL}_0 = 370$, λ = 0.05, 0.10, or 0.30; and UCL values were calculated using Eq. (12). For λ = 0.05, $\mathrm{UCL} = 1.89\times10^{-10}, 4.8\times10^{-17}$, and 5.5×10^{-17}, . Table 7 reports the ARL_1 and SDRL_1 for a range of shifts (δ) in the process mean. The ARL_1 results using both methods are similar, and the high accuracy percentages indicate the excellent efficacy of the proposed NIE results. The ARL_1 results are also consistent with those in Table 2, Table 3 and Table 4. The ARL_1 and SDRL_1 results also yield the same results, showing a decreasing pattern as the shift size increased for this real-life scenario, mirroring the findings in Table 5. Overall, the proposed NIE approach is a highly efficient option.

6 Conclusions

We used the Gauss-Legendre quadrature to approximate the ARL for a long-memory FI-MAX process with exponential white noise running on an EWMA control chart. It performed well in comparison with the established explicit formulas method and is thus a novel method for verifying ARL computations for long-memory FI-MAX scenarios. Its applicability to real-life scenarios involving FI-MAX processes was effectively demonstrated by using BAC stock prices with BTC-USD price and EUR/USD exchange rates as exogenous variables.

In future research, we will expand on the practicability of our approach for FI-MAX processes on other control charts such as CUSUM and for other white noise distributions.

Acknowledgement:
The author would like to express her gratitude to the Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Thailand for support with research grant No. 672170.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the author used Google Gemini in order to study the source and importance of research.. After using this tool/service, the author reviewed and edited the content as needed and took full responsibility for the content of the publication.

References:

[1] Shewhart, W. A., Quality control charts, *Bell System Technical Journal,* Vol.5, No.4, 1926, pp. 593–603.
[2] Page, E.S., Continuous inspection schemes, *Biometrika*, Vol.41, No.1-2, 1954, pp. 100-115.
[3] Roberts, S.W., Control Chart Test Based on Geometric Moving Averages, *Technometrics*, Vol.1, 1959, pp. 239-250.
[4] Steiner, S.H., EWMA control charts with time-varying control limits and fast initial response,

Journal of Quality Technology, Vol.31, 1999, pp. 75–86.

[5] Piyaphon, P., Areepong, Y. and Sukparungsee, S., Exact Expression of Average Run Length of EWMA chart for SARIMA(P, D, Q)L procedure, *International Journal of Applied Mathematics and Statistics*, Vol.52, 2014, pp. 62-73.

[6] Hawkins, D. M. and Wu, Q., The CUSUM and the EWMA Head-to-Head, *Quality Engineering*, Vol.26, No.2, 2014, pp. 215–222.

[7] Sunthornwat, R., Areepong, Y. and Sukparungsee, S., Analytical and numerical solutions of average run length integral equations for an EWMA control chart over a long memory SARFIMA process, *Songklanakarin Journal of Science and Technology*, Vol.40, No.4, 2018, pp. 885-895.

[8] Lucas, J. M. and Saccucci, M. S., Exponentially weighted moving average control schemes: properties and enhancements, *Technometrics*, Vol.32, No.1, 1990, pp. 1-29.

[9] Knoth, S., Accurate ARL Computation for EWMA-S 2 Control Charts, *Statistics and Computing*, Vol.15, No.4, 2005, pp. 341–352.

[10] Knoth, S., Run Length Quantiles of EWMA Control Charts Monitoring Normal Mean or/and Variance, *International Journal of Production Research*, 53(15), 2015, pp. 4629–4647.

[11] Suraiykat, W., Areepong, Y., Sukparungsee, S. and Mititelu, G., On EWMA Procedure for an AR(1) Observations with Exponential White Noise, *International Journal of Pure and Applied Mathematics*, Vol.77, No.1, 2012, pp. 73-83.

[12] Bualuang, D. and Peerajit, W., Performance of the CUSUM Control Chart Using Approximation to ARL for Long-Memory Fractionally Integrated Autoregressive Process with Exogenous Variable, *Applied Science and Engineering Progress*, Vol.16, No.2, 2023, pp. 1-13.

[13] Areepong, Y. and Peerajit, W., Integral equation solutions for the average run length for monitoring shifts in the mean of a generalized seasonal ARFIMAX(P, D, Q, r)$_s$ process running on a CUSUM control chart, *PLoS ONE*, Vol.17, No.2, 2022, pp. 1-25.

[14] Granger C. W. J. and Joyeux, R., An Introduction to Long Memory Time Series Models and Fractional Differencing, *Journal of Time Series Analysis*, Vol.1, No.1, 1980, pp. 15-29.

[15] Hosking J. R. M., Fractional differencing, *Biometrika*, Vol.68, No.1, 1981, pp. 165-176.

[16] Beran, J., *Statistics for long-memory processes*, Chapman & Hall, London, 1994.

[17] Palma, W., *Long-memory time series: theory and methods*, Wiley, New York, 2007.

[18] Beran, J., Feng, Y., Ghosh, S. and Kulik, R., *Long-memory processes—probabilistic properties and statistical methods*, Springer, New York, 2013.

[19] Ebens H., *Realized stock index volatility*, Department of Economics, Johns Hopkins University. 1999.

[20] Ramjee, R., Crato, N. and Ray B.K., Note on Moving Average Forecasts of Long Memory Processes with an Application to Quality Control, *International Journal of Forecasting*, Vol. 18, No.2, 2022, pp. 291-297.

[21] Pan, J.N. and Chen, S.T., Monitoring Long-memory Air Quality Data Using ARFIMA Model, *Environmetrics*, Vol.19, No.2, 2008, pp. 209-219.

[22] Rabyk, L. and Schmid, W., EWMA Control Charts for Detecting Changes in the Mean of a Long-memory Process, *Metrika*, Vol.79, No.3, 2016, pp. 267–301.

[23] Ibazizen, M. and Fellag, H., Bayesian estimation of an AR(1) process with exponential white noise, *Statistics*, Vol.37, No.5, 2003, pp. 365-372.

[24] Suparman, S., A new estimation procedure using a reversible jump MCMC algorithm for AR models of exponential white noise, *International Journal of GEOMATE*, Vol.15, No.49, 2018, pp. 85-91.

[25] Lazariv, T., Okhrin, Y. and Schmid, W., Behavior of EWMA type control charts for small smoothing parameters, *Computational Statistics and Data Analysis*, Vol.89, 2015, pp. 115-125.

[26] Crowder S.V., A Simple Method for Studying Run Length Distributions of Exponentially Weighted Moving Average Charts, *Technometrics*, Vol.29, No.4, 1987, pp. 401-407.

[27] Matheus G. and Dmitry P., *Numerical Mathematics*. Sudbury (Massachusetts) Boston: Jones and Bartlett, 2008.

[28] Fonseca, A., Ferreira, P. H., Nascimento, D. C.,

Fiaccone, R., Correa, C. U., Piña A. G., Louzada, F**.,** Water Particles Monitoring in the Atacama Desert: SPC Approach Based on Proportional Data, *Axioms*, Vol.10, 2021, 10.3390/axioms10030154

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Conceptualization: Wilasinee Peerajit.
Data curation: Wilasinee Peerajit.
Formal analysis: Wilasinee Peerajit.
Funding acquisition: Wilasinee Peerajit.
Investigation: Wilasinee Peerajit.
Methodology: Wilasinee Peerajit.
Software: Wilasinee Peerajit.
Validation: Wilasinee Peerajit.
Writing – original draft: Wilasinee Peerajit.
Writing – review and editing: Wilasinee Peerajit

The authors contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

The author would like to express her gratitude to the Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Thailand for support with research grant No. 672170.

Conflict of Interest

The authors declare no conflict of interest.