Interval State Estimation of Systems with Metzler Polytopic Models

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Abstract: - The paper deals with the design of interval observers for interval-defined strictly Metzler polytopic positive systems. The stability conditions for the proposed structure of the interval observer are formulated using linear matrix inequalities to ensure a positive estimate of the system state. The proposed method makes it possible to calculate time-varying lower and upper estimates of the state vector, assuming that the disturbance is bounded. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

Key-Words: - Metzler systems, parametric constraints, diagonal stabilisation, linear matrix inequalities, applied interval analysis, interval observers.

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1 Introduction

For linear systems with non-negative states, [1], [2], the standards of their description, based on linear equations, must be supplemented with additional parametric constraints, [3], by which the positivity of the evolution of the state variables is defined using the properties of the Metzler matrices, [4], [5]. A suitable unification, considering the principle of diagonal stabilization of such defined positive Metzler systems is the design strategy based only on linear matrix inequalities (LMIs), being proposed in [6].

Unlike systems with fixed parameters, the approach outlined in [7] provides an estimate of the system state for given bounds on the system matrices. A modified approach, using a technique based only on the parametric properties of Metzler matrices in the synthesis of interval observers, is presented in [8]. It is also worth noting that this task can be well formulated for positive Metzler systems using the representation of matrix bounds by the LMI structure, which is implicitly verified in [9]. Analogously, the synthesis of state observers for Takagi-Sugeno fuzzy systems was formulated in [10], also considering the guarantee of the positivity of the system state in polytopic linear systems, [11].

Unlike ordinary state observers, the outputs of the interval observer are the time-varying lower and upper estimates of the state vector, and the states of the system fall into the interval defined in this way, even in the presence of stationary input disturbances, [12], [13]. Due to some system advantages, provided that the description of uncertain Meztler systems has matrix parameter constraints, the so-called principle of cooperative observers is commonly used in the synthesis, which assumes that the resulting matrices of the dynamics of the interval observer will be Metzler and Hurwitz, and in the optimization, as a tuning parameter, is used H_∞ norm of the disturbance transfer function matrix, [14], [15], [16]. A survey on interval observer design using positive system approach is presented in [17].

Focusing on the above mentioned strategies, the target adaptation of the authors' results in strictly Metzler positive systems to the synthesis of positive interval state observers, as well as the interdependence of diagonal stabilization with the representation of interval constraints of this class of systems in the synthesis task, constitute the main topic of this paper. The presented new LMI formulation of the definition of parametric and interval constraints refines the design conditions so that the inequalities are sharp and result in strictly positive observer gain matrices. Since the LMI problem is formulated in this way, the interval observer synthesis conditions include interval constraints and guarantee the H_∞ disturbance input cost in the interval estimation error, and a positive estimate of the lower observer state vector. Since only a set of LMIs is used to define the synthesis conditions of a positive interval state observer, the methodology provides a standard environment for implementation. The uncertainties are modeled using a polytopic framework and the calculated gains of the interval observer are optimized for this class of uncertainties. Although the basic procedures are also applicable under more general assumptions in the design of interval observers for uncertain positive continuous-time linear systems, the proposed solution assumes that the polytopic uncertainties are time invariant, which may be a conservative assumption.

In Section 2, the parametrization of the restrictions of Mezler matrix structures is analyzed with regard to the synthesis of state observers of positive systems, and Section 3, different from the existing papers, presents the basis of the procedures defining the synthesis conditions for strictly Metzler positive interval observers based on LMIs. Section 4 is the extension of the results to the robust fault detection. The the efficiency and validity of the proposed solution is illustrated in Section 5 on a numerical example and, within the given concept, the analysis of the results is generalized in Section 6, providing a scope of the authors further research work in the future.

For sake of convenience, throughout this paper used notations reflect usual conventionality so that $\boldsymbol{x}^{\mathrm{T}}$, $\boldsymbol{X}^{\mathrm{T}}$ denotes the transpose of the vector $\boldsymbol{x}$, and the matrix $\boldsymbol{X}$, respectively, diag $[\cdot]$ marks a (block) diagonal matrix, for a square symmetric matrix $\boldsymbol{X} \prec 0$ means its negative definiteness, $\boldsymbol{I}_n$ labels the n-th order unit matrix, $\boldsymbol{X}^{-1}$, $\rho(\boldsymbol{X})$ signify the inverse and the eigenvalue spectrum of a square matrix $\boldsymbol{X}$, the symbol $*$ denotes a block-symmetric element in LMI matrix variables, $\mathbb{R}$ ($\mathbb{R}_+$) marks the set of (nonnegative) real numbers, $\mathbb{R}^{n\times m}$ ($\mathbb{R}_+^{n\times m}$) refers to the set of (nonnegative) real matrices and $\mathbb{M}_{-+}^{n\times n}$ indicates the set of strictly Metzler matrices.

2 Generalized Metzler Systems

The given task categorises that a set of matrices $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}_+^{n\times d}$ belongs to the polytopic uncertainty domain

$$\mathcal{O} \coloneqq \left\{ \begin{array}{c} \boldsymbol{a} \in \mathcal{Q}, \quad (\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})(\boldsymbol{a}) : \\ (\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})(\boldsymbol{a}) = \sum_{i=1}^{s} a_i (\boldsymbol{A}_i,\boldsymbol{B}_i,\boldsymbol{C}_i,\boldsymbol{D}_i) \end{array} \right\} \tag{1}$$

$$\mathcal{Q} = \{(a_1,...a_s) : \sum_{i=1}^{s} a_i = 1;\ a_i > 0,\ i = 1,...s\} \tag{2}$$

where $\mathcal{Q}$ is the unit simplex, $\boldsymbol{A}_i \in \mathbb{M}_{-+}^{n\times n}$, $\boldsymbol{B}_i \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C}_i \in \mathbb{R}_+^{m\times n}$, $\boldsymbol{D}_i \in \mathbb{R}_+^{n\times d}$ are constant matrices and $a_i,\ i = 1,2,\ldots,s$ are uncertainties.

Since $\boldsymbol{a}$ is restricted to the unit simplex as (2), the matrices $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})(\boldsymbol{a})$ are affine functions of the uncertain parameter vector $\boldsymbol{a} \in \mathbb{R}_+^{n}$ and the system is described by a convex combination of vertex matrices $(\boldsymbol{A}_i,\boldsymbol{B}_i,\boldsymbol{C}_i,\boldsymbol{D}_i,)$, $i = 1,\ldots,s$.

The class of uncertain polytopic systems is characterized by multi-input and multi-output (MIMO) dynamics, represented by the compact form

$$\dot{\boldsymbol{q}}(t) = \sum_{i=1}^{s} \mathrm{a}_i(\boldsymbol{A}_i\boldsymbol{q}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{D}_i\boldsymbol{d}(t)) \tag{3}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{s} \mathrm{a}_i\boldsymbol{C}_i\boldsymbol{q}(t) \tag{4}$$

where $\boldsymbol{q}(t) \in \mathbb{R}_+^{n}$, $\boldsymbol{u}(t) \in \mathbb{R}^{r}$, $\boldsymbol{y}(t) \in \mathbb{R}^{m}$ are vectors of the state, input, and output variables and $\boldsymbol{d}(t) \in \mathbb{R}^{d}$ is the bounded.

A short overview of new trends and starting points in this research area can be found in [18].

To the system (3), (4) can be designed an observer with Luenberger structure, given by the formula

$$\begin{aligned} \dot{\boldsymbol{q}}_e(t) = \sum_{i=1}^{s} a_i(\boldsymbol{A}_i\boldsymbol{q}_e(t) + \boldsymbol{B}_i\boldsymbol{u}(t)) + \\ + \sum_{i=1}^{s} a_i\boldsymbol{J}_i\boldsymbol{C}(\boldsymbol{q}(t) - \boldsymbol{q}_e(t)) \end{aligned} \tag{5}$$

$$\boldsymbol{y}_e(t) = \sum_{i=1}^{s} \mathrm{a}_i\boldsymbol{C}_i\boldsymbol{q}_e(t) \tag{6}$$

where $\boldsymbol{J}_i \in \mathbb{R}_+^{n\times m}$, $i = 1,\ldots,s$, are the observer gain matrices, vector $\boldsymbol{q}_e(t) \in \mathbb{R}_+^{n}$ is the state vector of the observer and $\boldsymbol{y}_e(t) \in \mathbb{R}_+^{m}$ is the estimated system output vector.

Consequently, the observer (5), (6) can be rewritten as:

$$\dot{\boldsymbol{q}}_e(t) = \sum_{i=1}^{s} a_i(\boldsymbol{A}_{ei}\boldsymbol{q}_e(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{J}_i\boldsymbol{C}\boldsymbol{q}(t) \tag{7}$$

$$\boldsymbol{y}_e(t) = \sum_{i=1}^{s} \mathrm{a}_i\boldsymbol{C}_i\boldsymbol{q}_e(t) \tag{8}$$

where

$$\boldsymbol{A}_{ei} = \boldsymbol{A}_i - \boldsymbol{J}_i\boldsymbol{C}_i \tag{9}$$

whilst $\boldsymbol{A}_{ei} \in \mathbb{M}_{-+}^{n\times n}$ has to be strictly Metzler and Hurwitz.

Since positive linear systems are only diagonally stabilizable, [19], it is appropriate to use positive definite diagonal matrix variables in formulating stability conditions and combine them with Metzler parametric constraints.

Since $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$ is strictly Metzler, its description is characterized by negative diagonal elements and strictly positive off diagonal elements, which means formal, [5]

$$a_{ll} < 0, \quad a_{lj} > 0, l \neq j,\ \forall\, l,j \in \langle 1,n \rangle \tag{10}$$

and then, if $\boldsymbol{A} = \{a_{lj}\} \in \mathbb{M}_{-+}^{n\times n}$ is represented in the equivalent rhombic structures, [20]

$$\boldsymbol{A}_{\ominus} = \begin{bmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ a_{31} & a_{32} & a_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \\ & a_{12} & a_{13} & \cdots & a_{1n} \\ & & a_{23} & \cdots & a_{2n} \\ & & & \ddots & \vdots \\ & & & & a_{n-1,n} \end{bmatrix} \tag{11}$$

the set of diagonal matrix inequalities for $h = 0,1,\ldots,n-1$,

$$\begin{cases} \boldsymbol{A}(\kappa+h,\kappa) \prec 0, & h=0 \\ \boldsymbol{A}(\kappa+h,\kappa) \succ 0, & h=1,\ldots,n-1 \\ \boldsymbol{A}(\kappa+h,\kappa) = & \\ \operatorname{diag}\left[a_{1+h,1}\cdots a_{n,n-h}\, a_{1,n-h+1}\cdots a_{h,n}\right] & \end{cases} \tag{12}$$

implying from diagonals of (11), can be used to represent (10).

Evidently, the equivalent conditions hold

$$\begin{cases} \boldsymbol{L}^h \boldsymbol{A}(\kappa+h,\kappa)\boldsymbol{L}^{h\mathrm{T}} \prec 0 & h=0 \\ \boldsymbol{L}^h \boldsymbol{A}(\kappa+h,\kappa)\boldsymbol{L}^{h\mathrm{T}} \succ 0 & h=1,\ldots,n-1 \end{cases} \tag{13}$$

where $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ of the structure

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_{n-1} & \boldsymbol{0} \end{bmatrix} \tag{14}$$

is the circulant form of a permutation matrix, [21].

The Metzler matrix (9) can be parameterized as follows:

Lemma 1 *(see, [20]). Applying for observer dynamics (9) with $\boldsymbol{A}_i \in \mathbb{M}_{-+}^{n\times n}$, $\boldsymbol{C}_i \in \mathbb{R}_{+}^{m\times n}$, $\boldsymbol{J}_i \in \mathbb{R}_{+}^{n\times m}$, the conditions for diagonal parametrisation of $\boldsymbol{A}_{ei} \in \mathbb{M}_{-+}^{n\times n}$ are given by using the defined diagonal matrices $\boldsymbol{A}_i(\kappa+h,\kappa) \in \mathbb{R}_{+}^{n\times n}$, $\boldsymbol{C}_{ik} \in \mathbb{R}_{+}^{n\times n}$, $\boldsymbol{J}_{ik} \in \mathbb{R}_{+}^{n\times n}$, $\boldsymbol{J}_{ikh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}_{ik}\boldsymbol{L}^h \in \mathbb{R}_{+}^{n\times n}$ constructed such that*

$$\boldsymbol{C}_i = \begin{bmatrix} \boldsymbol{c}_{i1}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{c}_{im}^{\mathrm{T}} \end{bmatrix},\ \boldsymbol{C}_{ik} = \operatorname{diag}\left[\boldsymbol{c}_{ik}^{\mathrm{T}}\right] = \operatorname{diag}\left[c_{ik1}\cdots c_{ikn}\right] \tag{15}$$

$$\boldsymbol{J}_i = [\boldsymbol{j}_{i1}\cdots\boldsymbol{j}_{im}],\ \boldsymbol{J}_{ik} = \operatorname{diag}\left[\boldsymbol{j}_{ik}\right] = \operatorname{diag}\left[j_{ik1}\cdots j_{ikn}\right] \tag{16}$$

and the observer system matrices $\boldsymbol{A}_{ei} \in \mathbb{M}_{-+}^{n\times n}$, $i = 1,\ldots,s$, are parameterizable by

$$\boldsymbol{A}_{ei} = \sum_{h=0}^{n-1}\boldsymbol{L}^h\Big(\boldsymbol{A}_i(\kappa+h,\kappa) - \sum_{k=0}^{m}\boldsymbol{J}_{ikh}\boldsymbol{C}_{ik}\Big) \tag{17}$$

Theorem 1 *Matrices $\boldsymbol{A}_{ei} \in \mathbb{R}_{-+}^{n\times n}$ for all $i \in \langle 1,s\rangle$ are strictly Metzler and Hurwitz if for by the system defined strictly Metzler matrices $\boldsymbol{A}_i \in \mathbb{R}_{-+}^{n\times n}$ and nonnegative matrices $\boldsymbol{C}_i \in \mathbb{R}_{+}^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}_{+}^{n\times d}$ there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{V}_{ik} \in \mathbb{R}_{+}^{n\times n}$ and positive scalars $\xi \in \mathbb{R}_+$ such that for $i = 1,\ldots,s$, $h = 1,\ldots,n-1$, $\boldsymbol{l}^{\mathrm{T}} = [1\cdots 1]$*

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{V}_{ik} \succ 0, \quad \xi > 0 \tag{18}$$

$$\boldsymbol{P}\boldsymbol{A}_i(\kappa,\kappa) - \sum_{k=1}^{m}\boldsymbol{V}_{ik}\boldsymbol{C}_{ik} \prec 0 \tag{19}$$

$$\boldsymbol{P}\boldsymbol{L}^h\boldsymbol{A}_i(\kappa+h,\kappa)\boldsymbol{L}^{h\mathrm{T}} - \sum_{k=1}^{m}\boldsymbol{V}_{ik}\boldsymbol{L}^h\boldsymbol{C}_{ik}\boldsymbol{L}^{h\mathrm{T}} \succ 0 \tag{20}$$

$$\begin{bmatrix} \boldsymbol{\Xi}_i & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\xi\boldsymbol{I}_d & * \\ \boldsymbol{C}_i & \boldsymbol{0} & -\xi\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{21}$$

$$\boldsymbol{\Xi}_i = \boldsymbol{P}\boldsymbol{A}_i + \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{m}\boldsymbol{V}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{ik} - \sum_{k=1}^{m}\boldsymbol{C}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{V}_{ik} \tag{22}$$

Confirming feasibility for $i = 1,\ldots,s$, then by computing

$$\boldsymbol{J}_{ik} = \boldsymbol{P}^{-1}\boldsymbol{V}_{ik},\ \boldsymbol{j}_{ik} = \boldsymbol{J}_{ik}\boldsymbol{l},\ \boldsymbol{J}_i = [\boldsymbol{j}_{i1}\cdots\boldsymbol{j}_{im}] \tag{23}$$

the observer parameters are achieved.

Hereafter, $*$ is the symmetric item in a symmetric matrix.

Proof: Because the state observation produces

$$\boldsymbol{e}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_e(t), \quad \boldsymbol{e}_y(t) = \sum_{i=1}^{s}\mathrm{a}_i\boldsymbol{C}_i\boldsymbol{e}(t) \tag{24}$$

whilst $\boldsymbol{q}_e(0) = \boldsymbol{0}$ is freely assignable, it should be noticed that using (24)

$$\dot{\boldsymbol{e}}(t) = \sum_{i=1}^{s}a_i\boldsymbol{A}_{ei}\boldsymbol{e}(t) + \boldsymbol{D}\boldsymbol{d}(t) \tag{25}$$

and it is possible for stable (25) to define a positive function $v(\boldsymbol{e}(t) > 0$

$$v(\boldsymbol{e}(t)) = \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) + \\ + \xi^{-1}\int_0^t(\boldsymbol{e}_y^{\mathrm{T}}(\nu)\boldsymbol{e}_y(\nu) - \xi^2\boldsymbol{d}^{\mathrm{T}}(\nu)\boldsymbol{d}(\nu))\mathrm{d}\nu \tag{26}$$

with a positive definite diagonal matrix (PDDM) $\boldsymbol{P} \in \mathbb{R}_{+}^{n\times n}$ and a positive scalar $\xi \in \mathbb{R}_+$. In this way it is necessary to request that the time derivative

$$\dot{v}(\boldsymbol{e}(t)) = \dot{\boldsymbol{e}}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) + \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{e}}(t) + \\ + \xi^{-1}\boldsymbol{e}_y^{\mathrm{T}}(t)\boldsymbol{e}_y(t) - \xi\boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{d}(t) \tag{27}$$

is negative for all observer error trajectories or, by substituting (25) that negative is the inequality

$$\begin{aligned}\dot{v}(\boldsymbol{e}(t)) = &\sum_{i=1}^{s}a_i\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{A}_{ei}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{ei})\boldsymbol{e}(t) + \\ &+ \sum_{i=1}^{s}a_i(\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{D}\boldsymbol{d}(t) + \boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}(t)) + \\ &+ \xi^{-1}\sum_{i=1}^{s}\sum_{j=1}^{s}a_ia_j\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{C}_j\boldsymbol{e}(t) - \\ &- \xi\boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{d}(t)\end{aligned} \tag{28}$$

In order to derive LMIs with respect to

$$\boldsymbol{e}_d^{\mathrm{T}}(t) = \begin{bmatrix}\boldsymbol{e}^{\mathrm{T}}(t) & \boldsymbol{d}^{\mathrm{T}}(t)\end{bmatrix} \tag{29}$$

it follows that

$$\dot{v}(\boldsymbol{e}_d(t)) = \sum_{i=1}^{s}\sum_{j=1}^{s}a_ia_j\boldsymbol{e}_d^{\mathrm{T}}(t)\boldsymbol{\Omega}_{ij}\boldsymbol{e}_d(t) < 0 \tag{30}$$

where it can be stated

$$\boldsymbol{\Omega}_{ij} = \begin{bmatrix} \boldsymbol{A}_{ei}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{ei} + \xi^{-1}\boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{C}_j & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\xi\boldsymbol{I}_d \end{bmatrix} \prec 0 \tag{31}$$

$$\boldsymbol{\Phi}_i = \begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_{ei} + \boldsymbol{A}_{ei}^{\mathrm{T}}\boldsymbol{P} & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\xi\boldsymbol{I}_d & * \\ \boldsymbol{C}_i & \boldsymbol{0} & -\xi\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{32}$$

respectively, when applying the Schur-complement with respect to a block partitioning of the matrix. Thus,

$$\dot{v}(\boldsymbol{e}_d(t)) = \sum_{i=1}^{s} a_i \boldsymbol{e}_d^{\mathrm{T}}(t)\boldsymbol{\Phi}_i\boldsymbol{e}_d(t) < 0 \tag{33}$$

and the observer design requires finding for each vertex $i = 1, \ldots, s$ such PDDMs $\boldsymbol{P}$, $\boldsymbol{V}_{ik}$ and positive scalar ξ that conform to LMI (21).

To have the diagonal constraints it yields for (9)

$$\begin{aligned} &\boldsymbol{P}(\boldsymbol{A}_i - \boldsymbol{J}_i\boldsymbol{C}_i) + (\boldsymbol{A}_i - \boldsymbol{J}_i\boldsymbol{C}_i)^{\mathrm{T}}\boldsymbol{P} \\ =&\boldsymbol{P}(\boldsymbol{A}_i - \sum_{k=1}^{m}\boldsymbol{j}_{ik}\boldsymbol{c}_{ik}^{\mathrm{T}}) + (\boldsymbol{A}_i - \sum_{k=1}^{m}\boldsymbol{j}_{ik}\boldsymbol{c}_{ik}^{\mathrm{T}})^{\mathrm{T}}\boldsymbol{P} \\ =&\boldsymbol{P}(\boldsymbol{A}_i - \sum_{k=1}^{m}\boldsymbol{J}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{ik}) + (\boldsymbol{A}_i - \sum_{k=1}^{m}\boldsymbol{J}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{ik})^{\mathrm{T}}\boldsymbol{P} \end{aligned} \tag{34}$$

Respecting the basic rule (9) when including system parameters in the LMI structure means that

$$\boldsymbol{P}\boldsymbol{A}_{ei} = \boldsymbol{P}\boldsymbol{A}_i - \boldsymbol{P}\sum_{k=1}^{m}\boldsymbol{J}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{ik} = \boldsymbol{P}\boldsymbol{A}_i - \sum_{k=1}^{m}\boldsymbol{V}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{ik} \tag{35}$$

where

$$\boldsymbol{V}_{ik} = \boldsymbol{P}\boldsymbol{J}_{ik} \tag{36}$$

and applying the last given, then (34) implies (22), whilst (32) gives (21).

Since the observer system matrix $\boldsymbol{A}_{ei} \in \mathbb{M}_{-+}^{n\times n}$ is parameterizable as (17) where $\boldsymbol{J}_{ikh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}_{ik}\boldsymbol{L}^{h}$, pre-multiplying the left side by $\boldsymbol{P}$ and post-multiplying the right side by $\boldsymbol{L}^{h\mathrm{T}}$ then (17), (36) implies

$$\begin{aligned} &\boldsymbol{P}\boldsymbol{L}^{h}\boldsymbol{A}_i(\kappa+h,\kappa)\boldsymbol{L}^{h\mathrm{T}} - \boldsymbol{P}\sum_{k=0}^{m}\boldsymbol{J}_{ikh}\boldsymbol{C}_{ik}\boldsymbol{L}^{h\mathrm{T}} \\ =&\boldsymbol{P}\boldsymbol{L}^{h}\boldsymbol{A}_i(\kappa+h,\kappa)\boldsymbol{L}^{h\mathrm{T}} - \sum_{k=0}^{m}\boldsymbol{V}_{ik}\boldsymbol{L}^{h}\boldsymbol{C}_{ik}\boldsymbol{L}^{h\mathrm{T}} \end{aligned} \tag{37}$$

Thus, one can conclude that (37) implies the Metzler parametric constraints as (18) for $h = 0$ and (18) for $h > 0$ as LMIs, which is the key to close the proof.

3 Interval Observer Design

It can be considered that in (3), (4) the parameters $(\boldsymbol{A}_i, \boldsymbol{C}_i)$ and the initial state of the system $\boldsymbol{q}(0)$ are unknown, but their upper and lower matrix and vector bounds are known, and element-wise holds for all $i \in \langle 1, s\rangle$ and $t \geq 0$

$$\underline{\boldsymbol{A}}_i \leq \boldsymbol{A}_i \leq \overline{\boldsymbol{A}}_i, \quad \underline{\boldsymbol{C}}_i \leq \boldsymbol{C}_i \leq \overline{\boldsymbol{C}}_i \tag{38}$$

$$0 \leq \underline{\boldsymbol{q}}(0) \leq \boldsymbol{q}(0) \leq \overline{\boldsymbol{q}}(0) \tag{39}$$

$$\underline{\boldsymbol{d}} \leq \boldsymbol{d}(t) \leq \overline{\boldsymbol{d}}, \quad \underline{\boldsymbol{d}} = -\overline{\boldsymbol{d}} \tag{40}$$

$\underline{\boldsymbol{A}}_i, \overline{\boldsymbol{A}}_i \in \mathbb{M}_{-+}^{n\times n}$, $\underline{\boldsymbol{C}}_i, \overline{\boldsymbol{C}}_i \in \mathbb{R}_{+}^{m\times n}$, $\underline{\boldsymbol{d}} \in \mathbb{R}^{d}$. Constraint (40) is basic in the literature of interval observers supposing that the disturbances are assumed to be bounded with known bounds.

Considering the interval-given parameters of the system, it is possible to define an interval observer

$$\begin{aligned} \dot{\overline{\boldsymbol{q}}}_e(t) &= \sum_{i=1}^{s} a_i\big(\overline{\boldsymbol{A}}_i\overline{\boldsymbol{q}}_e(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{J}_i(\boldsymbol{y}(t) - \overline{\boldsymbol{y}}_e(t))\big) \\ &= \sum_{i=1}^{s} a_i\big(\overline{\boldsymbol{A}}_{ei}\overline{\boldsymbol{q}}_e(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{J}_i\overline{\boldsymbol{C}}_i\boldsymbol{q}(t)\big) \end{aligned} \tag{41}$$

$$\begin{aligned} \dot{\underline{\boldsymbol{q}}}_e(t) &= \sum_{i=1}^{s} a_i\big(\underline{\boldsymbol{A}}_i\underline{\boldsymbol{q}}_e(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{J}_i(\boldsymbol{y}(t) - \underline{\boldsymbol{y}}_e(t))\big) \\ &= \sum_{i=1}^{s} a_i\big((\underline{\boldsymbol{A}}_{ei}\underline{\boldsymbol{q}}_e(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{J}_i\underline{\boldsymbol{C}}_i\boldsymbol{q}(t) \end{aligned} \tag{42}$$

$$\overline{\boldsymbol{y}}_e(t) = \sum_{i=1}^{s} a_i\overline{\boldsymbol{C}}_i\overline{\boldsymbol{q}}_e(t), \quad \underline{\boldsymbol{y}}_e(t) = \sum_{i=1}^{s} a_i\underline{\boldsymbol{C}}_i\underline{\boldsymbol{q}}_e(t) \tag{43}$$

where for $t \geq 0$, if $\overline{\boldsymbol{q}}_e(0) = \overline{\boldsymbol{q}}(0)$, $\underline{\boldsymbol{q}}_e(0) = \underline{\boldsymbol{q}}(0)$, it is expected that

$$\boldsymbol{0} \leq \underline{\boldsymbol{q}}_e(t) \leq \boldsymbol{q}(t) \leq \overline{\boldsymbol{q}}_e(t) \tag{44}$$

$$\overline{\boldsymbol{A}}_{ei} = \overline{\boldsymbol{A}}_i - \boldsymbol{J}_i\overline{\boldsymbol{C}}_i, \quad \underline{\boldsymbol{A}}_{ei} = \underline{\boldsymbol{A}}_i - \boldsymbol{J}_i\underline{\boldsymbol{C}}_i \tag{45}$$

when considering priori nonnegative matrices $\boldsymbol{B}_i \in \mathbb{R}_{+}^{n\times r}$, $\boldsymbol{J}_i \in \mathbb{R}_{+}^{n\times m}$, $\boldsymbol{D}_i \in \mathbb{R}_{+}^{n\times d}$, $\overline{\boldsymbol{C}}_i$, $\underline{\boldsymbol{C}}_i \in \mathbb{R}_{+}^{m\times n}$ and strictly Metzler $\overline{\boldsymbol{A}}_i$, $\underline{\boldsymbol{A}}_i \in \mathbb{R}_{-+}^{n\times n}$.

The diagonal parametrisation constraints from Lemma 1 can be generalized as follows:

Lemma 2 *Applying for observer dynamics (45) with* $\overline{\boldsymbol{A}}_i$, $\underline{\boldsymbol{A}}_i \in \mathbb{R}_{-+}^{n\times n}$, $\overline{\boldsymbol{C}}_i$, $\underline{\boldsymbol{C}}_i \in \mathbb{R}_{+}^{m\times n}$, $\boldsymbol{J}_i \in \mathbb{R}_{+}^{n\times m}$, *the conditions for diagonal parametrisation of* $\overline{\boldsymbol{A}}_{ei}$, $\underline{\boldsymbol{A}}_{ei} \in \mathbb{M}_{-+}^{n\times n}$ *are given by using the rhombic diagonal matrices* $\overline{\boldsymbol{A}}_i(\kappa+h,\kappa)$, $\underline{\boldsymbol{A}}_i(\kappa+h,\kappa) \in \mathbb{R}_{+}^{n\times n}$ *and diagonal matrices* $\overline{\boldsymbol{C}}_{ik}$, $\underline{\boldsymbol{C}}_{ik} \in \mathbb{R}_{+}^{n\times n}$, $\boldsymbol{J}_{ik} \in \mathbb{R}_{+}^{n\times n}$, $\boldsymbol{J}_{ikh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}_{ik}\boldsymbol{L}^{h} \in \mathbb{R}_{+}^{n\times n}$ *constructed such that*

$$\overline{\boldsymbol{C}}_i = \begin{bmatrix} \overline{\boldsymbol{c}}_{i1}^{\mathrm{T}} \\ \vdots \\ \overline{\boldsymbol{c}}_{im}^{\mathrm{T}} \end{bmatrix}, \ \overline{\boldsymbol{C}}_{ik} = \operatorname{diag}\left[\overline{\boldsymbol{c}}_{ik}^{\mathrm{T}}\right] = \operatorname{diag}\left[\overline{c}_{ik1}\cdots\overline{c}_{ikn}\right] \tag{46}$$

$$\underline{\boldsymbol{C}}_i = \begin{bmatrix} \underline{\boldsymbol{c}}_{i1}^{\mathrm{T}} \\ \vdots \\ \underline{\boldsymbol{c}}_{im}^{\mathrm{T}} \end{bmatrix}, \ \underline{\boldsymbol{C}}_{ik} = \mathrm{diag}\left[\underline{\boldsymbol{c}}_{ik}^{\mathrm{T}}\right] = \mathrm{diag}\left[\underline{c}_{ik1} \cdots \underline{c}_{ikn}\right] \tag{47}$$

$$\boldsymbol{J}_i = [\boldsymbol{j}_{i1} \cdots \boldsymbol{j}_{im}], \boldsymbol{J}_{ik} = \mathrm{diag}\,[\boldsymbol{j}_{ik}] = \mathrm{diag}\,[j_{ik1} \cdots j_{ikn}] \tag{48}$$

and the observer system matrices $\overline{\boldsymbol{A}}_{ei},\ \underline{\boldsymbol{A}}_{ei} \in \mathbb{M}_{-+}^{n\times n}$ *are parameterizable as*

$$\overline{\boldsymbol{A}}_{ei} = \sum_{h=0}^{n-1} \boldsymbol{L}^h (\overline{\boldsymbol{A}}_i(\kappa+h,\kappa) - \sum_{k=0}^{m} \boldsymbol{J}_{ikh} \overline{\boldsymbol{C}}_{ik}) \tag{49}$$

$$\underline{\boldsymbol{A}}_{ei} = \sum_{h=0}^{n-1} \boldsymbol{L}^h (\underline{\boldsymbol{A}}_i(\kappa+h,\kappa) - \sum_{k=0}^{m} \boldsymbol{J}_{ikh} \underline{\boldsymbol{C}}_{ik}) \tag{50}$$

It is no restriction to assume in the diagonal constraints that $\boldsymbol{d} \in \mathbb{R}^d$ acts through the gain $\boldsymbol{D} \in \mathbb{R}^{n\times d}$.

This design task of interval observer can be so formulated as a generalization of Theorem 1.

Theorem 2 *The matrices* $\overline{\boldsymbol{A}}_{ei},\ \underline{\boldsymbol{A}}_{ei} \in \mathbb{R}_{-+}^{n\times n}$ *for all* $i \in \langle 1, s\rangle$ *are strictly Metzler and Hurwitz if for by the system defined strictly Metzler matrices* $\overline{\boldsymbol{A}}_i,\ \underline{\boldsymbol{A}}_i \in \mathbb{R}_{-+}^{n\times n}$ *and non-negative matrices* $\overline{\boldsymbol{C}}_i,\ \underline{\boldsymbol{C}}_i \in \mathbb{R}_{+}^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}_{+}^{n\times d}$ *there exist positive definite diagonal matrices* $\boldsymbol{P},\ \boldsymbol{V}_{ik} \in \mathbb{R}_{+}^{n\times n}$ *and a positive scalar* $\xi \in \mathbb{R}_+$ *such that for all* $i = 1,\ldots,s,\ h = 1,\ldots,n-1,$ $\boldsymbol{l}^{\mathrm{T}} = [1 \cdots 1]$

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{V}_{ik} \succ 0, \qquad \xi > 0 \tag{51}$$

$$\begin{cases} \boldsymbol{P}\overline{\boldsymbol{A}}_i(\kappa,\kappa) - \sum\limits_{k=1}^{m} \boldsymbol{V}_{ik}\overline{\boldsymbol{C}}_{dk} \prec 0 \\ \boldsymbol{P}\underline{\boldsymbol{A}}_i(\kappa,\kappa) - \sum\limits_{k=1}^{m} \boldsymbol{V}_{ik}\underline{\boldsymbol{C}}_{dk} \prec 0 \end{cases} \tag{52}$$

$$\begin{cases} \boldsymbol{P}\boldsymbol{L}^h\overline{\boldsymbol{A}}_i(\kappa+h,\kappa)\boldsymbol{L}^{h\mathrm{T}} - \sum\limits_{k=1}^{m} \boldsymbol{V}_{ik}\boldsymbol{L}^h\overline{\boldsymbol{C}}_{ik}\boldsymbol{L}^{h\mathrm{T}} \succ 0 \\ \boldsymbol{P}\boldsymbol{L}^h\underline{\boldsymbol{A}}_i(\kappa+h,\kappa)\boldsymbol{L}^{h\mathrm{T}} - \sum\limits_{k=1}^{m} \boldsymbol{V}_{ik}\boldsymbol{L}^h\underline{\boldsymbol{C}}_{ik}\boldsymbol{L}^{h\mathrm{T}} \succ 0 \end{cases} \tag{53}$$

$$\begin{bmatrix} \overline{\boldsymbol{\Xi}}_i & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} - \xi\boldsymbol{I}_d & * \\ \overline{\boldsymbol{C}}_i & \boldsymbol{0} & -\xi\boldsymbol{I}_m \end{bmatrix} \prec 0, \ \begin{bmatrix} \underline{\boldsymbol{\Xi}}_i & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} - \xi\boldsymbol{I}_d & * \\ \underline{\boldsymbol{C}}_i & \boldsymbol{0} & -\xi\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{54}$$

$$\begin{cases} \overline{\boldsymbol{\Xi}}_i = \\ \boldsymbol{P}\overline{\boldsymbol{A}}_i + \overline{\boldsymbol{A}}_i^{\mathrm{T}}\boldsymbol{P} - \sum\limits_{k=1}^{m} \boldsymbol{V}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\overline{\boldsymbol{C}}_{ik} - \sum\limits_{k=1}^{m} \overline{\boldsymbol{C}}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{V}_{ik} \\ \underline{\boldsymbol{\Xi}}_i = \\ \boldsymbol{P}\underline{\boldsymbol{A}}_i + \underline{\boldsymbol{A}}_i^{\mathrm{T}}\boldsymbol{P} - \sum\limits_{k=1}^{m} \boldsymbol{V}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\underline{\boldsymbol{C}}_{ik} - \sum\limits_{k=1}^{m} \underline{\boldsymbol{C}}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{V}_{ik} \end{cases} \tag{55}$$

and according to (23), $\boldsymbol{J}_i \in \mathbb{R}_{+}^{n\times m}$ *can be determined.*

The proof is omitted for clarity of notation when extending Theorem 1.

Assuming that each sub-model dynamics of interval observer is strictly Metzler, nontrivial interval observer matrices can be designed so that each is stable using the proposed LMI-based algorithm. The choice ensures that the observer error sets are described by stable dynamics, satisfying stability and nonnegativity properties of the observation errors.

Starting from the initial state $\boldsymbol{g}(0)$ which verifies (39) and taking into account the system uncertainties, then the positive gains of the interval observer are optimized by H_∞ approach to guarantee the attenuation of disturbances effect. That means, the optimal design results are only tuned by the user-specified disturbance attenuation. Since

$$\overline{\boldsymbol{e}}(t) = \overline{\boldsymbol{q}}(t) - \boldsymbol{q}(t), \quad \underline{\boldsymbol{e}}(t) = \boldsymbol{q}(t) - \underline{\boldsymbol{q}}(t) \tag{56}$$

$$\overline{\boldsymbol{e}}(t) - \underline{\boldsymbol{e}}(t) = \overline{\boldsymbol{q}}(t) - \boldsymbol{q}(t) + \boldsymbol{q}(t) - \underline{\boldsymbol{q}}(t) = \overline{\boldsymbol{q}}(t) - \underline{\boldsymbol{q}}(t) \tag{57}$$

then

$$\|\overline{\boldsymbol{q}}(t) - \underline{\boldsymbol{q}}(t)\| \le \|\overline{\boldsymbol{e}}(t)\| + \|\underline{\boldsymbol{e}}(t)\| \tag{58}$$

and the stability of the interval observer (41)-(43) can be simply deduced.

4 Robust Fault Detection

The main application field of interval observers is the fault residual generation in the model-based fault diagnosis. The standard principle means to generate fault residuals in dependency on the system output $\boldsymbol{y}(t)$ and their estimated values $\boldsymbol{y}_e(t)$, which can be generated as

$$\boldsymbol{r}(t) = \boldsymbol{C}\boldsymbol{e}(t) \tag{59}$$

where

$$\boldsymbol{e}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_e(t) \tag{60}$$

is the state variable estimation error. It is obvious that the estimation error in the occurrence of an additive fault cannot be zero while, in a fault-free operation, the residuals are around zero. Nevertheless, when considering a system affected by perturbations and parameter uncertainties, the residuals deviate from zero even in the fault-free scenario.

Considering the interval error dynamics described by (56), the following robust fault residuals are given

$$\underline{\boldsymbol{r}}(t) = \underline{\boldsymbol{C}}\underline{\boldsymbol{e}}(t), \quad \overline{\boldsymbol{r}}(t) = \overline{\boldsymbol{C}}\overline{\boldsymbol{e}}(t) \tag{61}$$

and the fault detection test can be formulated as

$$\boldsymbol{0} \notin [\underline{\boldsymbol{r}}(t), \overline{\boldsymbol{r}}(t)] \tag{62}$$

which give the possibility to apply adaptive thresholds on the residuals, [22].

To enable all followers tracking the trajectory formed by the leader in the multi-agent system in the presence of agent model interval parameters, external disturbances and single actuator faults, the fault has to be detected to determine the faulty agent in the team. Based on the distributed strategy of fault detection and isolation, the residual flags can be generated corresponding to the residual signals of the team agents, assuming that each agent constructs the defined fault pattern in the flag, [23], [24]. Adaptation of the presented intervally defined parametric topic for network agent systems is moved for future research.

5 Illustrative Example

The system in the example is described by the model (3), (4) where

$$\underline{\boldsymbol{A}}_1=\begin{bmatrix}-0.1904 & 1.3580 & 1.0150\\ 0.0406 & -2.7720 & 0.0500\\ 0.0700 & 0.0350 & -2.0370\end{bmatrix},\ \underline{\boldsymbol{C}}_1=\begin{bmatrix}0.9 & 0 & 0\\ 0 & 1.2 & 0\end{bmatrix}$$

$$\underline{\boldsymbol{A}}_2=\begin{bmatrix}-0.2176 & 1.5520 & 1.1600\\ 0.0464 & -3.1680 & 0.0575\\ 0.0800 & 0.0640 & -2.3280\end{bmatrix},\ \underline{\boldsymbol{C}}_2=\begin{bmatrix}0.9 & 0 & 0\\ 0 & 1.2 & 0\end{bmatrix}$$

$$\overline{\boldsymbol{A}}_1=\begin{bmatrix}-0.1806 & 1.4420 & 1.0850\\ 0.0994 & -2.5480 & 0.0540\\ 0.1400 & 0.0460 & -1.7850\end{bmatrix},\ \overline{\boldsymbol{C}}_1=\begin{bmatrix}1.1 & 0 & 0\\ 0 & 1.5 & 0\end{bmatrix}$$

$$\overline{\boldsymbol{A}}_2=\begin{bmatrix}-0.2064 & 1.6480 & 1.2400\\ 0.1136 & -2.9120 & 0.0665\\ 0.1600 & 0.0640 & -2.0400\end{bmatrix},\ \overline{\boldsymbol{C}}_2=\begin{bmatrix}1.1 & 0 & 0\\ 0 & 1.5 & 0\end{bmatrix}$$

For all $i = 1, 2$, it is not difficult to confirm that $\overline{\boldsymbol{A}}_i,\ \underline{\boldsymbol{A}}_i$ are strictly Metzler and Hurwitz, $\underline{\boldsymbol{C}}_i,\ \overline{\boldsymbol{C}}_i$ are nonnegative matrices and $\underline{\boldsymbol{A}}_i \leq \overline{\boldsymbol{A}}_i, \underline{\boldsymbol{C}}_i \leq \overline{\boldsymbol{C}}_i$.

In order to take into account the principle of diagonal stabilization, the diagonal representations from $\underline{\boldsymbol{C}}_i, \overline{\boldsymbol{C}}_i$ are

$$\underline{\boldsymbol{C}}_{11}=\underline{\boldsymbol{C}}_{21}= |1\text{diag}\,[0.9\ 0\ 0],\ \underline{\boldsymbol{C}}_{12}=\underline{\boldsymbol{C}}_{22}=\text{diag}\,[0\ 0.2\ 0]$$

$$\overline{\boldsymbol{C}}_{11}=\overline{\boldsymbol{C}}_{21}=\text{diag}\,[1.1\ 0\ 0],\ \overline{\boldsymbol{C}}_{12}=\overline{\boldsymbol{C}}_{22}=\text{diag}\,[0\ 0.5\ 0]$$

while the set of circular diagonal representations of the matrices $\underline{\boldsymbol{A}}_i,\ \overline{\boldsymbol{A}}_i$ is

$$\underline{\boldsymbol{A}}_1(\kappa,\kappa) = \text{diag}\,[-0.1904\ -2.7720\ -2.0370]$$
$$\underline{\boldsymbol{A}}_1(\kappa{+}1,\kappa) = \text{diag}\,[0.0406\ 0.0350\ 1.0150]$$
$$\underline{\boldsymbol{A}}_1(\kappa{+}2,\kappa) = \text{diag}\,[0.0700\ 1.3580\ 0.0500]$$
$$\underline{\boldsymbol{A}}_2(\kappa,\kappa) = \text{diag}\,[-0.2176\ -3.1680\ -2.3280]$$
$$\underline{\boldsymbol{A}}_2(\kappa{+}1,\kappa) = \text{diag}\,[0.0464\ 0.0640\ 1.1600]$$
$$\underline{\boldsymbol{A}}_2(\kappa{+}2,\kappa) = \text{diag}\,[0.0800\ 1.5520\ 0.0575]$$
$$\overline{\boldsymbol{A}}_1(\kappa,\kappa) = \text{diag}\,[-0.1806\ -2.5480\ -1.7850]$$
$$\overline{\boldsymbol{A}}_1(\kappa{+}1,\kappa) = \text{diag}\,[0.0994\ 0.0460\ 1.0850]$$
$$\overline{\boldsymbol{A}}_1(\kappa{+}2,\kappa) = \text{diag}\,[0.1400\ 1.4420\ 0.0540]$$
$$\overline{\boldsymbol{A}}_2(\kappa,\kappa) = \text{diag}\,[-0.2064\ -2.9120\ -2.0400]$$
$$\overline{\boldsymbol{A}}_2(\kappa{+}1,\kappa) = \text{diag}\,[0.1136\ 0.0640\ 1.2400]$$
$$\overline{\boldsymbol{A}}_2(\kappa{+}2,\kappa) = \text{diag}\,[0.1600\ 1.6480\ 0.0665]$$

The condition for inserting the task into the SeDuMi toolbox, [25], means the construction of a total of $N = 22$ LMIs, of which $N_{mv} = 8$ are used to declare the positivity of the matrix variables, $N_{st} = 4$ are needed to enter stability conditions and $N_{pb} = 10$ is necessary to define the parametric constraints for the structures of the Metzler matrices. Then, a feasible solution results

$$\boldsymbol{P} = \text{diag}\,[4.5394\ \ 2.9283\ \ 4.8781],\quad \xi = 5.6530$$
$$\boldsymbol{V}_{11} = \text{diag}\,[3.9176\ \ 0.0493\ \ 0.1192]$$
$$\boldsymbol{V}_{12} = \text{diag}\,[2.3752\ \ 1.1860\ \ 0.0461]$$
$$\boldsymbol{V}_{21} = \text{diag}\,[3.9063\ 0.0554\ 0.1301]$$
$$\boldsymbol{V}_{22} = \text{diag}\,[2.8408\ 1.1311\ 0.0721]$$

which implies the positive gains

$$\boldsymbol{J}_1 = \begin{bmatrix}0.8630 & 0.5232\\ 0.0168 & 0.4050\\ 0.0244 & 0.0094\end{bmatrix},\ \boldsymbol{J}_2 = \begin{bmatrix}0.8605 & 0.5232\\ 0.0189 & 0.4050\\ 0.0267 & 0.0094\end{bmatrix}$$

to construct Metzler and Hurwitz stable matrices

$$\underline{\boldsymbol{A}}_{e1} = \begin{bmatrix}-0.9671 & 0.7301 & 1.0150\\ 0.0255 & -3.2580 & 0.0500\\ 0.0480 & 0.0237 & -2.0370\end{bmatrix}$$

$$\rho(\underline{\boldsymbol{A}}_{e1}) = \{-0.9148\quad -2.0809\quad -3.2664\}$$

$$\underline{\boldsymbol{A}}_{e2} = \begin{bmatrix}-0.9921 & 0.9241 & 1.1600\\ 0.0294 & -3.6540 & 0.0575\\ 0.0560 & 0.0527 & -2.3280\end{bmatrix}$$

$$\rho(\underline{\boldsymbol{A}}_{e2}) = \{-0.9342\quad -2.3746\quad -3.6653\}$$

$$\overline{\boldsymbol{A}}_{e1} = \begin{bmatrix}-1.1299 & 0.6571 & 1.0850\\ 0.0809 & -3.1555 & 0.0540\\ 0.1131 & 0.0318 & -1.7850\end{bmatrix}$$

$$\rho(\overline{\boldsymbol{A}}_{e1}) = \{-0.9542\quad -1.9349\quad -3.1814\}$$

$$\overline{\boldsymbol{A}}_{e2} = \begin{bmatrix}-1.1530 & 0.8631 & 1.2400\\ 0.0928 & -3.5195 & 0.0665\\ 0.1307 & 0.0498 & -2.0400\end{bmatrix}$$

$$\rho(\overline{\boldsymbol{A}}_{e1}) = \{-0.9658\quad -2.1938\quad -3.5529\}$$

Note that programming the task using the LMI MATLAB© toolbox has the same algorithmic complexity.

Using, for example, the method presented in [26], the algorithm can easily be adapted by structured matrix variables to design interval observers for purely Metzler matrices in which some off-diagonal matrix elements may be zero. In that case, the solution leads to non-negative matrix structures $\boldsymbol{J}_i$, which guarantees that the stable matrices $\underline{\boldsymbol{A}}^{\sigma}_{ei}, \overline{\boldsymbol{A}}^{\sigma}_{ei}$ are also purely Metzlerian. Since in general $\boldsymbol{A}^{\sigma}_{ei} \in \mathbb{R}^{n\times n}_{-+}$ for all $i \in \langle 1, s\rangle$ are Metzler and Hurwitz, the asymptotic stability of the interval observer is guaranteed.

As can be seen from the example, the proposed procedure transforms the synthesis problem into acceptable structural LMIs that are easily accessible by calculation.

6 Concluding Remarks

A key result is that it is possible to construct a set of LMIs respecting given constraints to prescribe positive (non-negative) gains in the design of the system Metzler and Hurwitz matrices of the interval state observer of positive systems. The novelty of the given LMI structure links interval bounds, parameters of Metzler matrices and stability conditions directly in the formulation of the problem and defines the criteria to capture the boundedness and positiveness of the interval estimation error dynamics.

Although there are many studies on interval observer design approaches for estimating continuous-time systems, discrete-time linear time-invariant systems, and bounded time-delay systems, [27], [28], the research on interval observers has not been comprehensive, and there are many unsolved problems in the design of interval observers for systems with input by saturation and linear time-varying systems. The proposed methodology could be extended to classes of systems such as positive switching systems and interconnected positive systems, [29], [30]. Moreover, T-S fuzzy systems with non-measurable premise variables is also a challenging task to become one of the future research points, and also the design of a closed loop interval observer supporting the stabilization of possibly unstable plants by state feedback is an interesting perspective. Special cases of these problems may arise in various contexts associated with fault detection a class of distributed multi-agent systems based on ostensible Metzler agents [31].

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