Some Results on Partially Ordered Sets Involving Permuting n-derivations

LATIFA BEDDA, ABDELKARIM BOUA
Department of Mathematics, Polydisciplinary Faculty of Taza,
Sidi Mohammed Ben Abdellah University, Fez,
MOROCCO

Abstract: The main objective of the present work, as a generalization of derivation, is to give the concept of *permuting n-derivations on partially ordered sets (posets)*. Several associated theorems and fondamental properties involving permuting n-derivations are presented. Moreover, we demonstrate that if D is a permuting n-derivation on poset G with the greatest element 1 and the trace δ, then $\delta(1) = 1$ if and only if δ is an identity on G. Furthermore, we discuss the relations among derivations, ideals and fixed sets in posets.

Key-Words: Partially ordered sets (posets), Upper bound, Lower bound, Directed poset, Derivation, Permuting n-derivations, Lower homomorphism, Fixed set.

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1 Introduction

In mathematics, a partially ordered set, often abbreviated as a *poset*, is a set provided with a binary relation (often denoted as $\leq$) that is reflexive, antisymmetric and transitive. Partially ordered sets arise naturally in various mathematical contexts, including order relations, such as less than or equal to ($\leq$) on numbers, subsets of sets under inclusion ($\subseteq$), and many other situations where there's a notion of 'precedence' or 'ordering' among elements, but not necessarily every pair of elements can be compared.
Hausdorff introduced the first comprehensive theory of partially ordered sets in 1914 in his book " Grundzüge der Mengenlehre ". One significant outcome of Hausdorff's work is the maximal chain theorem, which is equivalent to Zorn's lemma..
The derivation is consequent topic to study, [1], defined the derivation on ring and many mathematicians have developed the derivation theory in rings and prime rings, [2], [3]. Multi- derivations (e.g. bi-derivations, tri-derivations, in general, n-derivations) are studied in prime and semi-prime rings, [4], [5], [6].
In this direction, the concept of derivation on lattice was defined and developed in [7], [8], respectively. In [9], the study defined symmetric bi-derivations of a lattice and proved some results, and in [10], he applied his concepts and theorems to the n-derivation of lattices. Derivations on posets have also been a subject of study. Recently, [11], started studying the derivations in poset, establishing several fundamental properties related to ideals and operations associated with these derivations. On partially ordered sets, the notion of bi- and tri-derivations are provided. and the fundamental Characteristics are studied (see, [12], [13], for more details). Our research was mainly inspired by the work in [11], [12]. This research presents a generalization of derivations by introducing a new concept of *permuting n-derivations* of partially ordered sets. Moreover, we present the examples that demonstrate the existence of this class of applications and we have proved important properties. Additionally, we give the fixed set $Fix_\delta(G) = \{a \in G : \delta(a) = a\}$ and proved that is an ideal of G. The final section is devoted to studying some properties involving permuting n-derivations and their traces.
As in [11], for $p, q \in G$ and $X \subseteq P$, we define

(i) $\downarrow p = \{w \in G : w \leq p\}$.

(ii) $\uparrow q = \{v \in G : q \leq v\}$.

(iii) $L(X) = \{\lambda \in G : \lambda \leq x, \forall\, x \in X\}$ the *Lower cone* of the set X.

(iv) $U(X) = \{\alpha \in G : x \leq \alpha, \forall\, x \in X\}$ the *Upper cone* of X.

As mentioned in [14], we write "$L(U(L(Y))) = L(Y)$" and "$U(L(U(M))) = U(M)$", for all $Y, M \subset G$. If $Y = \{y_1, ..., y_n\}$, we write $L(Y) = L(y_1, y_2, ..., y_n)$ and $U(Y) = U(y_1, ..., y_{n-1}, y_n)$. Further, For $X, Y \subseteq P$, $L(X \cup Y)$ will be represented by $L(X, Y)$ and $U(X \cup Y)$ by $U(X, Y)$. We write also, $\downarrow X = \{w \in P : w \leq y \;\; for \;\; some \;\; y \in X\}$". According to [15], a set X is named a *Lower set* if $X = \downarrow X$. The *directed set* is a nonempty set X that for every finite subset of X, the supremum has existed in X. Given that X is nonempty, it suffices to expect that every pair $\{a, b\}$ of elements in X has the supremum in X. For $\mathcal{J} \subset G$ is said ideal of G if $\mathcal{J}$ is `directed lower` set.

2 Permuting n-Derivations on Posets

Throughout the present work, G represents a partially ordered set, which will be abbreviated as $poset$

Definition 1. [11] *Let G be a poset, a function $D : G \to G$ is called a derivation on G if these two conditions are verified,* $(\forall a, b \in P)$,

(i) $D(L(a, b)) = L(U(L(D(a), b), L(a, D(b))));$

(ii) $L(D(U(a, b))) = L(U(D(a), D(b)));$

A mapping $D : G \times G \to G$ is called symmetric if D(s, t) = D(t, s), $\forall\ s, t \in G$, and a mapping $\delta : G \to G$ given by $\delta(s) = D(s, s), \forall s \in G$ is named a $trace$ of D under which D is symmetric.

This section introduces a new notion called *permuting n-derivations* for a partially ordered set, followed by the examples that demonstrates the existence of this type of application.

Let $n \in \mathbb{N}$ such that $n \geqslant 2$ and $G^n = \underbrace{G \times G \times \ldots \times G}_{n \text{ times}}$. The map $D : G \to G$ is said to be permuting if the equation

$$D(a_1, a_2, \ldots, a_n) = D(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}) \quad (1)$$

holds $\forall\ a_i \in G$ and for every permutation $\pi(i)$, $i = 1, ..., n$.

Definition 2. *Let G be a poset and $D : G^n \to G$ be a map. We say D is n-derivation if D is a derivation for all components, which means:*
(1) $D(L(a_1, w), a_2, \ldots, a_n) =$

$$L(U(L(D(a_1, \ldots, a_n), w), L(a_1, D(w, a_2, \ldots, a_n))))$$

$D(a_1, L(a_2, w), a_3, \ldots, a_n) =$

$$L(U(L(a_2, D(a_1, w, a_3, \ldots, a_n)), L(D(a_1, a_2, \ldots, a_n), w)))$$

$$\ldots$$

$D(a_1, a_2, \ldots, a_{n-1}, L(a_n, w)) =$

$$L(U(L(a_n, D(a_1, \ldots, a_{n-1}, w)), L(D(a_1, a_2, \ldots, a_n), w)))$$

(2) $L(D(U(a_1, w), a_2, \ldots, a_n)) =$

$$L(U(D(a_1, a_2, \ldots, a_n), D(w, a_2, \ldots, a_n)));$$

$$\ldots$$

$L(D(a_1, a_2, a_3, \ldots, a_{n-1}, U(a_n, w)) =$

$$L(U(D(a_1, a_2, \ldots, a_n), D(a_1, \ldots, a_{n-1}, w)));$$

are valid, $(\forall a_i, w \in P)$.

Example 1
Let $D : \mathbb{N}^n \to \mathbb{N}$ be a function defined by
$D(m_1, m_2, \ldots, m_n) = min\{m_1, m_2, \ldots, m_n\}$.
It is simple to confirm that D is a permuting n-derivation on $\mathbb{N}$.

Example 2
Let 0 be the least element of a poset G. A function $D : G^n \to G$ defined by $D(a_1, a_2, \ldots, a_n) = 0$ is a permuting and a n-derivation on G.

In the following, we assume that G is a poset and D is a permuting n-derivation on G.

Proposition 1. *Let 0 be the least element of a poset G and δ be the trace of D. Then*

(i) $D(a_1, a_2, a_3, \ldots, a_n) \leq a_i$, $\forall a_i \in G$;

(ii) $D(a_1, \ldots, a_n) \in L(a_1, a_2, \ldots, a_n), \forall\ a_i \in G$;

(iii) $D(a_1, a_2, ..., a_n) = 0$ *if there exist* $i \in \{1, 2, \ldots, n\}$ *which satisfy* $a_i = 0$;

(iv) *For each i in $\{1, 2, \ldots, n\}$, if $a_i \leq b_i$, then* $D(a_1, ..., a_i, \ldots, a_n) \leq D(a_1, \ldots, b_i, \ldots, a_n)$;

(v) $\delta(a) \leq a$, $\forall\ a \in G$;

(vi) $\delta(0) = 0$;

(vii) $\delta(L(a)) \subset L(\delta(a))$, $\forall\ a \in G$;

(viii) $\forall\ g_1, g_2 \in G$, $g_1 \leq g_2$ *implies* $\delta(g_1) \leq \delta(g_2)$;

(ix) $\delta^2(s) = \delta(s)$, $\forall\ s \in G$;

Proof. (i) From Definition 2 (i), we have

$$D(L(a_1), a_2, \ldots, a_n) = D(L(a_1, a_1), a_2, \ldots, a_n)$$

$$= L(U(L(D(a_1, \ldots, a_n), a_1), L(a_1, D(a_1, a_2, \ldots, a_n))))$$
$$= L(U(L(a_1, D(a_1, a_2, \ldots, a_n)))) = L(D(a_1, a_2, \ldots, a_n), a_1)$$

Then,

$$D(L(a_1), a_2, \ldots, a_n) = L(D(a_1, a_2, \ldots, a_n), a_1) \quad (2)$$

Since $D(a_1, a_2, a_3, \ldots, a_n) \in D(L(a_1), a_2, \ldots, a_n)$, the above result (2) imply that

$$D(a_1, a_2, \ldots, a_n) \in L(D(a_1, a_2, \ldots, a_n), a_1).$$

Therefore, $D(a_1, \ldots, a_n) \leq a_1$.
Similar to above processe, we can see that $D(a_1, a_2, \ldots, a_n) \leq a_i, \forall\ a_i \in G$.

It is evident that (ii) and (iii) are induced by (i).
(iv) Suppose that $a_i \leq b_i$ for $a_i, b_i \in G$. By using Definition 2 (ii), we get
$L(D(a_1, a_2, \ldots, U(b_i), \ldots, a_{n-1}, a_n)) =$
$L(D(a_1, \ldots, U(a_i, b_i), \ldots, a_n)) =$

$$L(U(D(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n), D(a_1, \ldots, b_{i-1}, b_i, \ldots, a_n))). \quad (3)$$

Since $D(a_1,\dots,a_{i-1},a_i,a_{i+1},\dots,a_n) \in L(U(D(a_1,\dots,a_i,\dots,a_n),D(a_1,\dots,b_i,\dots,a_n)))$, equation (3) proves that

$$D(a_1,a_2,\dots,a_i,\dots,a_n) \in L(D(a_1,\dots,U(b_i),a_{i+1},\dots,a_n))$$

Therefore,

$$D(a_1,\dots,a_i,\dots,a_n) \leq D(a_1,\dots,b_i,\dots,a_n)$$

(v) Let $a \in G$, we have $\delta(a) = D(a,a,\dots,a)$. By (i), we get $D(a,\dots,a) \leq a$, $\forall\, a \in G$. Hence, $\delta(a) \leq a, \forall\, a \in G$.
(vi) Since $\delta(0) \leq 0$ by (v), we get $0 \leq \delta(0) \leq 0$. This means that $\delta(0) = 0$.
(vii) Let $a \in G$,
$D(L(a),\dots,L(a)) = D(L(a,a),L(a),\dots,L(a))$
$= \{D(L(a,a),y_2,\dots,y_n) \mid y_i \in G$ and $y_i \leqslant a$, $\forall\, i = 2,...,n\}$
$= \{L(U(L(D(a,y_2,\dots,y_n),a),L(a,D(a,y_2,\dots,y_n))))$ $\mid y_i \in G$ and $y_i \leqslant a, \forall\, i = 2,...,n\}$
$= \{L(U(L(D(a,y_2,\dots,y_n))) \mid y_i \in G$ and $y_i \leqslant a$, $\forall\, i = 2,...,n\}$
$= \{L(D(a,y_2,\dots,y_n)) \mid y_i \in G$ and $y_i \leqslant a$, $\forall\, i = 2,...,n\}$
then

$$D(L(a),...,L(a)) = L(D(a,L(a),...,L(a))). \quad (4)$$

Since $\delta(L(a)) \subset D(L(a),\dots,L(a))$, the Equation (4) implies that $\delta(L(a)) \subset L(D(a,L(a),\dots,L(a)))$, so $\delta(L(a)) \subset L(D(a,a,\dots,a))$. This shows that $\delta(L(a)) \subset L(\delta(a))$, for all $a \in G$.
(viii) Let g_1 and g_2 be two different elements in G which satisfy the condition $g_1 \leq g_2$, then $g_1 \in L(g_2)$, and this implies that $\delta(g_1) \in \delta(L(g_2))$. By using (vii), we can get $\delta(g_1) \in L(\delta(g_2))$, so $\delta(g_1) \leq \delta(g_2)$.
(ix) According to (v) and (viii), we can see $\delta(\delta(a)) \leq \delta(a), \forall\, a \in P$, so

$$\delta^2(a) \leq \delta(a). \quad (5)$$

Let $a \in G$, combining (v) and (viii) we get $\delta^2(a) \in L(a)$ and by using (4), we obtain

$$\delta(L(a)) \subset L(D(a,\delta^2(a),\dots,\delta^2(a))). \quad (6)$$

since $d(a,\delta^2(a),\dots,\delta^2(a)) \leq \delta^2(a)$ by (i), we have

$$L(D(a,\delta^2(a),\dots,\delta^2(a))) \subset L(\delta^2(a)). \quad (7)$$

Adding the equations (6) and (7), we find that $\delta(L(a))$ is included in $L(\delta^2(a))$, $\forall\, a \in G$. Since $\delta(a) \in \delta(L(a))$, we obtain $\delta(a) \in L(\delta^2(a))$, so

$$\delta(a) \leq \delta^2(a), \ \ \forall a \in P. \quad (8)$$

Therefore, (5) and (8) imply that $\delta^2(a) = \delta(a)$, $\forall\, a \in G$.

□

Theorem 1. *Let $D : G^n \to G$ be a permuting mapping on poset G. D is an n-derivation on G if and only if*

(1) $D(L(a_1,w),a_2,\dots,a_n) = L(D(a_1,\dots,a_n),w) = L(a_1,D(w,a_2,\dots,a_n))$;

(2) $L(D(U((a_1,w),a_2,\dots,a_n)) = L(U(D(a_1,\dots,a_n),D(w,a_2,\dots,a_n)))$;

$\forall\, a_i, w \in G$ and $i = 1,\dots,n$.

Proof. Assume that the condition (1) holds. Then,

$D(L(a_1,w),a_2,\dots,a_{n-1},a_n) = L(D(a_1,a_2,\dots,a_n),w)$
$= L(U(L(D(a_1,\dots,a_n),w)))$
$= L(U(L(w,D(a_1,a_2,\dots,a_n))),L(D(a_1,\dots,a_n),w)))$
$= L(U(L(w,D(a_1,\dots,a_{n-1},a_n))),L(a_1,D(w,a_2,\dots,a_n))$.
In addition to the condition (2), we deduce that D is an n-derivation on G.
Inversement, suppose that D is a n-derivation on G. it holds that:
$L(D(a_1,\dots,a_n),w) = L(U(L(D(a_1,\dots,a_n),w))$
$\subset L(U(L(w,D(a_1,\dots,a_n))),L(a_1,D(w,a_2,a_3,\dots,a_n))$
$= D(L(a_1,w),a_2,\dots,a_n)$,
then

$$L(D(a_1,\dots,a_n),w) \subset D(L(a_1,w),a_2,\dots,a_n). \quad (9)$$

Now, let $z \in D(L(a_1,w),a_2,\dots,a_n)$, then there exixts $t \in L(a_1,w)$ satisfying $z = D(t,a_2,\dots,a_n)$. Since $t \in L(a_1,w)$, we get $t \leq a_1$ and by using Proposition 1 (iv) we obtain $D(t,a_2,\dots,a_n) \leq D(a_1,\dots,a_n)$, so $z \leq D(a_1,\dots,a_n)$. From Proposition 1 (i), we can get $D(t,a_2,\dots,a_n) \leq t \leq w$, so $z \leq w$ and this imply that $z \in L(D(a_1,\dots,a_n),w)$. Therefore,

$$D(L(a_1,w),a_2,\dots,a_n) \subset L(D(a_1,\dots,a_n),w). \quad (10)$$

Combining the results (9) and (10), we get $D(L(a_1,w),a_2,\dots,a_n) = L(D(a_1,\dots,a_n),w)$, $\forall\, a_i, w \in G$.

Symmetrically, we can also prove the second equality $D(L(a_1,w),a_2,\dots,a_n) = L(a_1,D(w,a_2,\dots,a_n))$, $\forall\, a_i, w \in G$. □

Lemma 1. *Let G be a poset. If $s \leq t$ and $L(s) = L(t)$, then $s = t$.*

Proof. Assume that $s \leq t$ and $L(s) = L(t)$. It is evident that $t \in L(t)$, $\forall\, t \in G$, then $t \in L(s)$, so $t \leq s$. By hypothesis, we conclude that $s = t$. □

Lemma 2. *Let G be a poset. If δ be the trace of D, then the subsequent claims are valid:*

(1) *If* $D(L(s), s, \ldots, s, s) = L(t)$*, then* $\delta(s) = t$*,* $\forall s, t \in G$*;*

(2) *If* $D(U(s), s, \ldots, s) = U(t)$*, then* $\delta(s) = t$*,* $\forall s, t \in G$*.*

Proof. Let $s, t \in G$ such that

$$D(L(s), s, \ldots, s) = L(t). \quad (11)$$

By using Theorem 1 (1) and Proposition 1 (v), we get

$$\begin{aligned} D(L(s), s, \ldots, s) &= D(L(s, s), s, \ldots, s) \\ &= L(D(s, \ldots, s), s) \\ &= L(D(s, \ldots, s)) \end{aligned}$$

then

$$D(L(s), s, \ldots, s) = L(\delta(s)). \quad (12)$$

For all $s \in P$.
Using (11) and (12), we infer that $L(\delta(s)) = L(t)$. Since $D(s, s, \ldots, s) \in D(L(s), s, \ldots, s) = L(t)$, we find $\delta(s) \leq t$ and with Lemme 1 the result holds. □

Definition 3. *Let G be a poset, a mapping* $\phi : G \to G$ *is known as a L-homomorphism of G if* $U(\phi(L(\lambda, \mu))) = U(L(\phi(\lambda), \phi(\mu)))$*,* $(\forall \lambda, \mu \in G)$*.*

Proposition 2. *Let G be a poset and* $D : G^n \to G$ *be a permuting n-derivation on G.*
Then, $D(L(a_1, w), a_2, a_3, \ldots, a_{n-1}, a_n) = L(D(a_1, \ldots, a_n), D(w, a_2, \ldots, a_n))$*,* $\forall a_i, w \in G$*.*

Proof. Let $z \in D(L(a_1, w), a_2, \ldots, a_n)$ then, $\exists t \in L(a_1, w)$:

$$z = D(t, a_2, \ldots, a_n). \quad (13)$$

Since $t \in L(a_1, w)$, we get $t \leq a_1$ and $t \leq w$. Proposition 1 (iv) implies that

$$z = D(t, a_2, \ldots, a_n) \leq D(a_1, a_2, \ldots, a_{n-1}, a_n)). \quad (14)$$

and

$$z = D(t, a_2, \ldots, a_n) \leq D(w, a_2, ..., a_n). \quad (15)$$

Combining (14) and (15), we get $z \in L(D(a_1, \ldots, a_n), D(w, a_2, \ldots, a_n))$. This shows that, $\forall a_i, w \in G$, we have
$D(L(a_1, w), a_2, a_3, \ldots, a_n) \subset$

$$L(D(a_1, a_2, a_3, \ldots, a_{n-1}, a_n), D(w, a_2, \ldots, a_n))) \quad (16)$$

Moreover, we suppose that $v \in L(D(a_1, a_2, \ldots, a_n), D(w, a_2, \ldots, a_n))$, then $v \leq D(a_1, \ldots, a_n)$ and $v \leq D(w, a_2, \ldots, a_n) \leq w$, so $v \in L(D(a_1, \ldots, a_n), w) = D(L(a_1, w), a_2, \ldots, a_n)$ by application of Theorem 1 (1). Consequently,

$$\begin{aligned} &L(D(a_1, a_2, \ldots, a_n), D(w, a_2, \ldots, a_n)) \subset \\ &\qquad D(L(a_1, w), a_2, \ldots, a_n), \forall a_i, w \in G. \end{aligned} \quad (17)$$

The results (16) and (17) proves the theorem. □

Theorem 2. *Let G be a poset and* 1 *its greatest element and* δ *be the trace of D. Then,*
$\delta(1) = 1 \iff D(a, 1, .., 1, ., 1) = a$*,* $\forall a \in G$*.*

Proof. Suppose that $D(a, 1, 1, 1, ..., 1) = a, \forall a \in G$, then $D(1, ..., 1, 1) = 1$, hence $\delta(1) = 1$.
Conversely, we suppose that $\delta(1) = 1$. Let $a \in G$, by using Theorem 1, we have

$$\begin{aligned} D(L(a), 1, 1, 1, ..., 1) &= D(L(a, 1), 1, ..., 1, 1) \\ &= L(a, D(1, 1, ..., 1)) \\ &= L(a, \delta(1)) \\ &= L(a, 1) \\ &= L(a), \end{aligned}$$

then

$$D(L(a), 1, 1, ..., 1, 1) = L(a). \quad (18)$$

Furthermore,

$$\begin{aligned} D(L(a), 1, 1, ..., 1, .., 1) &= D(L(a, a), 1, ..., 1) \\ &= L(D(a, 1, ..., 1, 1), a) \\ &= L(D(a, 1, 1, 1, ..., 1)), \end{aligned}$$

then,

$$D(L(a), 1, ..., 1) = L(D(a, 1, 1, ..., 1, 1)). \quad (19)$$

Hence, (18) and (19) implies that

$$L(D(a, 1, ..., 1)) = L(a). \quad (20)$$

By using Proposition 1 (i), we get

$$D(a, 1, ..., 1) \leq a. \quad (21)$$

In view of Lemma 1 together (20) and (21), we conclude that $D(a, 1, ..., 1) = a, \forall a \in G$. □

Theorem 3. *Let G be a poset and* δ *be the trace of D on G. We have,*
$\delta(L(a_1, ..., a_n)) \subset L(\delta(a_1), .., \delta(a_n))$*,* $\forall a_i \in G$*.*

Proof. Let $t \in \delta(L(a_1, a_2, ..., a_n))$, then there exists $y \in L(a_1, a_2, a_3, ..., a_n)$ such that $t = \delta(y)$. The relation $y \in L(a_1, a_2, ..., a_{n-1}, a_n)$ implies that $y \leq a_i$, $\forall a_i \in G$, and by using Proposition 1 (viii), we get $\delta(y) \leq \delta(a_i)$, $\forall i = 1, .., n$, then $t = \delta(y) \in L(\delta(a_1), .., \delta(a_n))$. This means that $\delta(L(a_1, a_2, ..., a_n)) \subset L(\delta(a_1), .., \delta(a_n))$, $\forall a_i \in G$. □

Corollary 1. *Let G be a poset and 1 be the greatest element of G and δ be the trace of D. If $a \leq \delta(1)$, then $D(a, 1, ..., 1) = a$, $\forall\, a \in G$.*

Proof. Let $a \in G$, assume that $a \leq \delta(1)$, from Theorem 1 we can get,

$$\begin{aligned} D(L(a), 1, 1, ..., 1) &= D(L(a, 1), 1, ..., 1, 1) \\ &= L(a, D(1, 1, ..., 1)) \\ &= L(a, \delta(1)) \\ &= L(a). \end{aligned}$$

Then,

$$D(L(a), 1, ..., 1) = L(a). \tag{22}$$

In addition,

$$\begin{aligned} D(L(a), 1, 1, 1, ..., 1) &= D(L(a, a), 1, 1, ..., 1, 1, 1) \\ &= L(D(a, 1, ..., 1, 1), a) \\ &= L(D(a, 1, ..., 1)). \end{aligned}$$

Then,

$$D(L(a), 1, ..., 1) = L(D(a, 1, 1, ..., 1)). \tag{23}$$

Therefore, (22) and (23) shows that
$L(D(a, 1, 1, ..., 1)) = L(a)$.
Combining Lemma 1 and Proposition 1 (i), we can get $D(a, 1, ..., 1) = a$, $\forall\, a \in G$. ☐

Proposition 3. *Let G be a poset and 1 its greatest element. Let δ be a the trace of a permuting n-derivation D on G. Then $\delta(1) = 1 \iff \delta = id_D$.*

Proof. It is obvious that if $\delta = id_D$, then $\delta(1) = 1$. Inversely, let $a \in G$. Combining Theorem 1 and Proposition 1 (v) we can get

$$\begin{aligned} D(L(a), a, ..., a) &= D(L(a, a), a, ..., a) \\ &= L(a, D(a, ..., a)) \\ &= L(a, \delta(a)) \\ &= L(\delta(a)) \end{aligned}$$

$$D(L(a), a, ..., a) = L(\delta(a)). \tag{24}$$

Moreover,

$$\begin{aligned} D(L(a), a, ..., a) &= D(L(a, 1), a, ..., a) \\ &= L(a, D(1, a, ..., a, a)) \\ &= L(D(1, a, ..., a)) \end{aligned}$$

$$D(L(a), a, ..., a) = L(D(1, a, .., a)). \tag{25}$$

According (24) and (25) we get

$$L(D(1, a, a, ..., a)) = L(\delta(a)). \tag{26}$$

Proposition 1 (iv) implies that $\delta(a) = D(a, a, ..., a) \leq D(1, a, ..., a)$, which, because of (26) together Lemma 1, Show that $\delta(a) = D(1, a, ..., a)$, $\forall\, a \in G$. Since$D$ is a permuting map, we get $D(1, a, ..., a) = D(a, ..., a, 1)$. Hence,

$$\delta(a) = D(a, ..., a, 1). \tag{27}$$

With the similar process, we show that $\delta(a) = D(a, ..., a, 1, 1)$. In fact, Combining Theorem 1 (1) and Proposition 1 (v) we have

$$\begin{aligned} D(L(a), a, ..., a, 1) &= D(L(a, 1), a, ..., a, 1) \\ &= L(a, D(1, a, a, ..., a, 1)) \\ &= L(D(1, a, a, ..., a, a, 1)) \end{aligned}$$

$$D(L(a), a, a, ..., a, 1) = L(D(1, a, ..., a, a, 1)). \tag{28}$$

Moreover,

$$\begin{aligned} D(L(a), a, a, ..., a, 1) &= D(L(a, a), a, ..., a, 1) \\ &= L(D(a, a, ..., a, 1), a) \\ &= L(D(a, a, ..., a, 1)) \end{aligned}$$

$$D(L(a), a, ..., a, a, 1) = L(D(a, ..., a, 1)). \tag{29}$$

Adding these last tow equations (28) and (29) we see that

$$L(D(1, a, ..., a, 1)) = L(D(a, ..., a, 1)) \tag{30}$$

Proposition 1 (iv) implies that $D(a, a, ..., a, 1) \leq D(1, a, ..., a, 1)$ which, because of (30) together Lemma 1, implies that $D(1, a, ..., a, 1) = D(a, a, ..., a, 1)$. Since$D$ is a permuting map, we have $D(1, a, ..., a, 1) = D(a, ..., a, 1, 1)$. Hence,

$$D(a, ..., a, 1) = D(a, ..., a, 1, 1). \tag{31}$$

Combining (27) and (31), we cleam that

$$\delta(a) = D(a, ..., a, 1, 1). \tag{32}$$

Similarly, we get

$$\begin{aligned} D(L(a), a, ..., a, 1, 1) &= D(L(a, 1), a, a, ..., a, 1, 1) \\ &= L(a, D(1, a, a, a, ..., a, 1, 1)) \\ &= L(D(1, a, ..., a, 1, 1)), \end{aligned}$$

then

$$D(L(a), a, ..., a, 1, 1) = L(D(1, a, ..., a, 1, 1)). \tag{33}$$

Furthermore,

$$\begin{aligned}D(L(a), a, a, ..., a, 1, 1) &= D(L(a, a), a, ..., a, 1, 1)\\ &= L(D(a, ..., a, a, 1, 1), a)\\ &= L(D(a, a, ..., a, 1, 1)),\end{aligned}$$

then

$$D(L(a), a, ..., a, 1, 1) = L(D(a, a, ..., a, 1, 1)). \quad (34)$$

Combining (33) and (34), we get $L(D(1, a, ..., a, 1, 1)) = L(D(a, a, ..., a, 1, 1))$, and by using Proposition 1 (iv) we see that $D(a, a, ..., a, a, 1, 1) \leq D(1, a, ..., a, 1, 1)$ and from Lemma 1, we deduce that

$$\begin{aligned}D(a, ..., a, 1, 1) &= D(1, a, ..., a, ..., a, 1, 1)\\ &= D(a, ..., a, a, 1, 1, 1). \quad (35)\end{aligned}$$

Therefore, (32) and (35) show that

$$\delta(a) = D(a, a, ..., a, a, 1, 1, 1). \quad (36)$$

According to the results (27), (32) and (36), we get $\delta(a) = D(a, ..., a, 1) = D(a, ..., a, 1, 1) = D(a, ..., a, 1, 1, 1), \forall\, a \in G$.
Using the same method of proof, we arrive at the following conclusion
$\delta(a) = D(a, a, ..., a, 1) = D(a, a, ..., a, 1, 1) = D(a, a, ..., a, 1, 1, 1) = ... = D(a, 1, 1, ..., 1), \forall\, a \in G$.
To complete this demonstration, it is enough to show that $D(a, 1, 1, ..., 1) = a$.
From Theorem 2, since $\delta(1) = 1$, we get $D(a, 1, 1, ..., 1) = a, \forall\, a \in G$. This means that $\delta(a) = a, \forall\, a \in G$. Thus, the theorem is proved.. □

Proposition 4. *Considered G be a poset and 0 its least element. Let δ be the trace of D. Denote $Fix_\delta(G) = \{a \in G : \delta(a) = a\}$. Then,*

(1) $0 \in Fix_\delta(G)$.

(2) *If $a \in Fix_\delta(G)$ and $b \leq a$, then $b \in Fix_\delta(G)$.*

(3) *If G is directed, then $\forall\, b_1, b_2 \in Fix_\delta(P)$, $\exists k \in Fix_\delta(G) : b_1 \leq k$ and $b_2 \leq k$.*

Proof. (1) It is clear that since $\delta(0) = 0$.
(2) Let $a, b \in G$. Assume that $a \in Fix_\delta(G)$ and $b \leq a$, then $\delta(a) = a$. By using Theorem 1 (1), we have

$$\begin{aligned}D(L(b), a, ..., a) &= D(L(a, b), a, ..., a, a)\\ &= L(D(a, ..., a), b)\\ &= L(\delta(a), b)\\ &= L(a, b)\\ &= L(b).\end{aligned}$$

Since $b \in L(b)$, it follows that $b \in D(L(b), a, ..., a, a)$. Hence, $\exists t \in L(b)$ provided that $b = D(t, a, ..., a)$, by using Proposition 1 (iv) and (i), we get
$b = D(t, a, ..., a) \leq D(b, a, ..., a) \leq b$, so

$$D(b, a, ..., a) = b. \quad (37)$$

Again,

$$\begin{aligned}D(b, L(b), a, ..., a) &= D(b, L(a, b), a, ..., a)\\ &= L(D(b, a, ..., a), b)\\ &= L(b, b) \ using\ (37)\\ &= L(b).\end{aligned}$$

Since $b \in L(b)$, we get $b \in D(b, L(b), a, ..., a)$. Hence, there exists $t \in L(b)$ such that $b = D(b, t, a, ..., a)$, by using Proposition 1 (iv) and (i), we get
$b = D(b, t, a, ..., a) \leq D(b, b, a, a, ..., a, a) \leq b$, so

$$D(b, b, a, ..., a) = b. \quad (38)$$

Also by using Theorem 1 (1), we have

$$\begin{aligned}D(b, b, L(b), a, a, a, ..., a) &= D(b, b, L(a, b), a, ..., a)\\ &= L(D(b, b, a, a, ..., a, a), b)\\ &= L(b, b) \ by\ ,\ using\ (38)\\ &= L(b).\end{aligned}$$

Since $b \in L(b)$, we find $b \in D(b, b, L(b), a, ..., a)$. Hence, we can find an $t \in L(b)$ which $b = D(b, b, t, a, ..., a)$, by using Proposition 1 (iv) and (i), we get
$b = D(b, b, t, a, ..., a) \leq D(b, b, b, a, a, ..., a) \leq b$, so

$$D(b, b, b, a, ..., a) = b. \quad (39)$$

From the results (37), (38) and (39), we obtain $D(b, a, ..., a) = D(b, b, a, a, ..., a) = D(b, b, b, a, ..., a) = b, \forall\, a, b \in G$.
With the same method, we arrive at
$D(b, a, ..., a) = D(b, b, a, ..., a) = D(b, b, b, a, ..., a) = ... = D(b, b, ..., b, b, a) = b, \forall\, b \in G$. So

$$D(b, b, ..., b, a) = b. \quad (40)$$

Moreover,

$$\begin{aligned}D(b, ..., b, b, L(b)) &= D(b, ..., b, L(a, b))\\ &= L(D(b, ..., b, a), b)\\ &= L(b, b) \ by\ (40)\\ &= L(b).\end{aligned}$$

Then $D(b,...,b,L(b)) = L(b)$, application Lemma 2 (1) yields that $\delta(b) = b$, $\forall\ b \in G$. This shows that $b \in Fix_\delta(G)$.
(3) Let $b_1, b_2 \in G$, since G is directed, $\exists c \in G : b_1 \leq c$ and $b_2 \leq c$. Since $b_1, b_2 \in Fix_\delta(G)$, we get $\delta(b_1) = b_1$ and $\delta(b_2) = b_2$. By Proposition 1 (viii) we can get $b_1 \leq \delta(c)$ and $b_2 \leq \delta(c)$. Put $k = \delta(c)$, by Proposition 1 (ix) we get $\delta(t) = t$, hence $t \in Fix_\delta(G)$. □

Corollary 2. *Let* 0 *be the least element of* G *and* δ *be the trace of* D*, then* $Fix_\delta(G)$ *is an ideal of* G*.*

Proposition 5. *Let* d_1 *and* d_2 *be two permuting* n*-derivations on* G *with traces* δ_1, δ_2*, respectively. Then* $\delta_1 = \delta_2 \iff Fix_{\delta_1}(G) = Fix_{\delta_2}(G)$*.*

Proof. It is obvious that $\delta_1 = \delta_2$ implies $Fix_{\delta_1}(G) = Fix_{\delta_2}(G)$. Conversely, let $Fix_{\delta_1}(G) = Fix_{\delta_2}(G)$ and $a \in G$. By Proposition 1 (ix), we have $\delta_1(a) \in Fix_{\delta_1}(G) = Fix_{\delta_2}(G)$, so

$$\delta_2(\delta_1(a)) = \delta_1(a). \tag{41}$$

Combining (v) and (viii) in Proposition 1, we get

$$\delta_2(\delta_1(a)) \leq \delta_2(a). \tag{42}$$

These last two equations (41) and (42) show that

$$\delta_1(a) \leq \delta_2(a). \tag{43}$$

Similarly, we can get $\delta_1(\delta_2(a)) = \delta_2(a)$ and $\delta_1(\delta_2(a)) \leq \delta_2(a)$. Then

$$\delta_2(a) \leq \delta_2(a).U(\tag{44}$$

Adding these last two arguments (43) and (44), we find that $\delta_2(a) = \delta_2(a)$, $\forall\ a \in G$. So $\delta_1 = \delta_2$. □

3 Some properties of posets involving permuting n-derivations

Theorem 4. *Let* G *be a poset and* δ *be the of* D*. If* 0 *be the least element of* G*, Then* $ker\delta = \{a \in G : \delta(a) = 0\}$ *is a nonempty and a lower set of* G*.*

Proof. By Proposition 1 (vi), we can see that $\delta(0) = 0$ imply $0 \in ker\delta$. Therefore $ker\delta \neq \phi$. Furthermore, if $a \in ker\delta$ and $b \in G$ in which $b \leq a$, since $\delta(b) \leq \delta(a)$ by Proposition 1 (viii) and $\delta(a) = 0$, so $\delta(b) = 0$. Therefore, $b \in ker\delta$ and thus forces the results. □

Proposition 6. *Let* G *be a poset,* 0 *be the least element of* G *and* δ *be the of* D *on* G*.*
If $\mathcal{J}$ *is an ideal of* G*, then* $\delta^{-1}(\mathcal{J})$ *is an ideal of* G*.*

Proof. Assume that $\mathcal{J}$ is an ideal of G, then $0 \in \mathcal{J}$ and so, $\delta(0) = 0 \in I$. Hence, $0 \in \delta^{-1}(\mathcal{J})$, then $\delta^{-1}(\mathcal{J}) \neq \phi$. Suppose that $a \in \delta^{-1}(\mathcal{J})$ and $b \in G$ where $b \leq a$, then $\delta(a) \in \mathcal{J}$ and $\delta(b) \leq \delta(a)$ by Proposition 1 (viii), this imply that $\delta(b) \in \mathcal{J}$ and so $b \in \delta^{-1}(\mathcal{J})$. This means that $\delta^{-1}(\mathcal{J})$ is an ideal of G. □

Proposition 7. *Let* G *be a poset and* δ *the trace of a permuting* n*-derivation* D *on* G*.*
Let I_1 *and* I_2 *be two ideals of* G*, we have*
$I_1 \subseteq I_2 \Rightarrow \delta(I_1) \subseteq \delta(I_2)$*.*

Proof. Let $b \in \delta(I_1)$, then $\exists\ a \in I_1 \subseteq I_2 : \delta(a) = b$. Hence, $b \in \delta(I_2)$. It follows that $\delta(I_1) \subseteq \delta(I_2)$. □

Theorem 5. *Let* G *be a poset and* D_1*,* D_2 *be two permuting* n*-derivations on* G *with traces* δ_1*,* δ_2*, respectively. Then,* $\forall\ a \in G$*,*
$\delta_1(a) \leq \delta_2(a) \iff \delta_2(\delta_1(a)) = \delta_1(a)$*.*

Proof. Assume that $\delta_1(a) \leq \delta_2(a)$, $\forall\ a \in G$, that is, $\delta_1(\delta_1(a)) \leq \delta_2(\delta_1(a))$. By Proposition 1 (ix), $\delta_1(a) = \delta_1(\delta_1(a))$. So

$$\delta_1(a) \leq \delta_2(\delta_1(a)). \tag{45}$$

Moreover, the Proposition 1 (v) gives that

$$\delta_2(\delta_1(a)) \leq \delta_1(a). \tag{46}$$

From the above arguments (45) and (46), we can get $\delta_2(\delta_1(a)) = \delta_1(a)$, $\forall\ a \in G$. Inversely, suppose that $\delta_2(\delta_1(a)) = \delta_1(a)$, $\forall\ a \in G$. By using Proposition 1 (v) and (viii), we obtain $\delta_2(\delta_1(a)) \leq \delta_2(a)$, and by hypothesis, we can get $\delta_1(a) \leq \delta_2(a)$, $\forall\ a \in G$. □

4 Conclusion

This work has provided a comprehensive analysis of derivations and permuting n-derivations in the context of partially ordered sets (posets), which are generalizations of derivations on a poset. We have introduced and studied the concept of permuting n-derivations on posets and presented several characterization theorems and fundamental properties related to permuting n-derivations. Additionally, we have introduced the fixed set of permuting n-derivations in posets and discussed the relationships among derivations, ideals and fixed sets within posets. This study opens up further avenues for research, inviting deeper exploration into the interactions between derivations and poset structures. Our future research on posets will be inspired by our recent work on lattices in [16] which involves generalized derivations. We aim to explore how these concepts can be applied to posets to develop new theories and applications.

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