Super Metric Space and Fixed Point Results

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Abstract: - Banach contraction principle is the beginning of the Metric fixed point theory. This principle gives existence and uniqueness of fixed points and methods for obtaining approximate fixed points. It is the basic tool of finding fixed points of all contraction type maps. It has a constructive proof which makes the theorem worthy because it yields an algorithm for computing a fixed point. Banach fixed point result has been extended by various authors in many directions either by weakening the conditions of contraction mapping or by changing the abstract structure. Several generalizations and extensions of metric spaces have been introduced. Among these, the prominent extensions are b-metric space, fuzzy metric space, partial metric space and a lot more of their combinations. In particular, a new structure namely Super metric space is introduced. In the present paper, we generalize and extend the fixed point results of fixed point theory in literature in the framework of super metric space.

Key-Words: - Super metric space, fixed point, contraction, weakly compatible maps

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1 Introduction and Preliminaries

The well-known Banach's contraction mapping principle states that if $f : X \to X$ is a contraction on X (i.e. $\rho(f\gamma, f\beta) \leq q\rho(\gamma,\beta)$ for some $q < 1$ and all $\gamma, \beta \in X$ and X is complete, then f has a unique fixed in X.

A number of generalizations of this result have appeared. Particularly in 1971, [1], a new generalized contraction was defined as follows:
A mapping $f : X \to X$ is said to be generalized contraction iff for every $\gamma, \beta \in X$ there exist numbers q, r, s and t which may depend on both γ and β, such that

$$\sup[q + r + s + 2t : \gamma, \beta \in X] < 1$$

and

$$\rho(f\gamma, f\beta) \leq q\rho(\gamma,\beta) + r\rho(\gamma, f\gamma) + s\rho(\beta, f\beta) + t\rho(\gamma, f\beta) + \rho(\beta, f\gamma). \tag{1}$$

The idea of generalized contraction was further extended, [2], by defining quasi-contraction. A mapping $f : X \to X$ of a metric space X into itself is said to be a quasi-contraction iff there exists a number q, $0 \leq q < 1$, such that

$$\rho(f\gamma, f\beta) \leq q \max[\rho(\gamma,\beta), \rho(\gamma, f\gamma), \rho(\beta, f\beta), \rho(\gamma, f\beta), \rho(\beta, f\gamma)] \tag{2}$$

holds for every $\gamma, \beta \in X$. The condition (2) implies condition (1) was supported by an example.

Definition 1.1 ([3]). Let X be a non-empty set and $\rho : X \times X \to [0, +\infty)$ be a mapping which satisfies

(ρ_1) $\rho(\gamma,\beta) = 0$ if and only if $\gamma = \beta$ for all $\gamma, \beta \in X$,

(ρ_2) $\rho(\gamma,\beta) = \rho(\beta,\gamma)$ for all $\gamma, \beta, \in X$,

(ρ_3) $\rho(\gamma,\beta) \leq \rho(\gamma,\alpha) + \rho(\alpha,\beta)$ for all $\gamma, \beta, \alpha \in X$. (triangular inequality)

Then, the pair (X, ρ) is called a Euclidean metric space or a metric space.

For the convenience of the reader, let us recall the following results:

Proposition 1.2 ([4]). *Let (X, ρ) be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy $g(X) \subset f(X)$ and there exists a constant λ in $(0, 1)$ such that for every $\gamma, \beta \in X$,*

$$\rho(g\gamma, g\beta) \leq \lambda\rho(f\gamma, f\beta).$$

Then f and g have a unique common fixed point.

Proposition 1.3 ([5]). *Let (X, ρ) be a complete metric space. Let f be a continuous self-map on X, and*

g be any self-map on X, that commutes with f. Further, let $g(X) \subset f(X)$ and there exists a constant λ in $(0,1)$ such that for every $\gamma, \beta \in X$,

$$\rho(g\gamma, g\beta) \leq \lambda M\rho(\gamma, \beta),$$

where

$$M\rho(\gamma,\beta) = \max[\rho(f\gamma, f\beta), \rho(f\gamma, g\gamma), \rho(f\gamma, g\beta), \rho(f\beta, g\beta), \rho(f\beta, g\gamma)].$$

Then f and g have a unique fixed point.

The concept of metric space has been generalized and extended by various authors. Recently in, [6], a new extension of metric space is introduced and it is named as Super metric space, and an analogue result of Banach's contraction principle in super metric space is established.

Definition 1.4 ([6]). Let X be a nonempty set and $m : X \times X \to [0, +\infty)$ be a mapping satisfying

(m_1) if $m(\gamma, \beta) = 0$, then $\gamma = \beta$ for all $\gamma, \beta \in X$,

(m_2) $m(\gamma, \beta) = m(\beta, \gamma)$ for all $\gamma, \beta \in X$,

(m_3) there exists $s \geq 1$ such that for all $\beta \in X$, there exist distinct sequences $\{\gamma_n\}, \{\beta_n\} \subset X$, with $m(\gamma_n, \beta_n) \to 0$ when n tends to infinity, such that

$$\lim_{n\to\infty} \sup m(\beta_n, y) \leq s \lim_{n\to\infty} \sup m(\gamma_n, y).$$

Then, the pair (X, m) is called a super metric space.

Definition 1.5 ([6]). Let (X, m) be a super metric space and let $\{\gamma_n\}$ be a sequence in X. We say

(i) $\{\gamma_n\}$ converges to γ in X if and only if $m(\gamma_n, \gamma) \to 0$, as $n \to \infty$.

(ii) $\{\gamma_n\}$ is a Cauchy sequence in X if and only if $\lim_{n\to\infty} \sup\{m(\gamma_n, \gamma_m) : m > n\} = 0$.

(iii) (X, m) is a complete super metric space if and only if every Cauchy sequence is convergent in X.

Proposition 1.6 ([6]). *Let (X, m) be a complete super metric space and let $T : X \to X$ be a mapping. Suppose that $0 < k < 1$ such that*

$$m(T\gamma, T\beta) \leq km(\gamma, \beta), \quad \text{for all } \gamma, \beta \in X.$$

Then T has a unique fixed point in X.

Proposition 1.7 ([7]). *On a super metric space, the limit of a convergent sequence is unique.*

Proposition 1.8 ([7]). *Let (X, m) be a complete super metric space and $T : X \to X$ be an asymptotically regular mapping. If there exists $k \in [0, 1)$, such that*

$$m(T\gamma, T\beta) \leq k \max\left[m(\gamma,\beta), \frac{(m(\gamma, T\beta) + m(\beta, T\gamma))}{2s}, \frac{(m(\gamma, T\gamma)\, m(\gamma, T\beta) + m(\beta, T\beta)\, m(\beta, T\gamma))}{(m(\gamma, T\beta) + m(\beta, T\gamma) + 1)}\right].$$

Then T has a unique fixed point.

Proposition 1.9 ([7]). *Let (X, m) be a complete super metric space and let $T : X \to X$ be a mapping such that there exists $k \in (0, 1)$ and*

$$m(T\gamma, T\beta) \leq k \left[\max\left\{m(\gamma, \beta), \frac{m(\gamma, T\gamma)\, m(\beta, T\beta)}{m(\gamma, \beta) + 1}\right\}\right].$$

Then, T has a unique fixed point.

Definition 1.10 ([8]). A pair (f, g) of self mappings of metric space (X, d) is said to be weakly compatible if the mappings commute at all of their coincidence points, that is, $f\gamma = g\gamma$ for some $\gamma \in X$ implies $fg\gamma = gf\gamma$.

Definition 1.11 ([9]). Let f and g be self-maps of a set X. If $w = f\gamma = g\gamma$ for some γ in X, then γ is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Proposition 1.12 ([9]). *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = f\gamma = g\gamma$, then w is the unique common fixed point of f and g.*

In [10], a new concept of the Φ-map was introduced as the following: Let Φ be the set of all functions φ such that $\varphi : [0, +\infty) \to [0, +\infty)$ is a non decreasing function satisfying:

$$\lim_{n\to\infty} \varphi^n(t) = 0, \quad \text{for all } t \in (0, +\infty).$$

If $\varphi \in \Phi$, then φ is called a Φ-map. Furthermore, if φ is a Φ-map, then

(i) $\varphi(t) < t$ for all $t \in (0, \infty)$,

(ii) $\varphi(0) = 0$.

From now on, unless otherwise stated, φ is meant the Φ-map.

Recently in, [11], using the notion of Φ-map a generalization of Proposition 1.8 is proved in the setting of Super Metric Space as the following:

Proposition 1.13 ([11]). *Let (X,m) be a complete super metric space. Suppose that the mappings $f,g : X \to X$ satisfy*

$$m(f\gamma, f\beta) \leq k\Bigg[\max\Bigg\{m(g\gamma, g\beta), \frac{m(g\gamma, f\beta)+m(g\beta, f\gamma)}{2s}, \frac{m(g\gamma, f\gamma)m(g\gamma, f\beta)+m(g\beta, f\beta)m(g\beta, f\gamma)}{m(g\gamma, f\beta)+m(g\beta, f\gamma)+1}\Bigg\}\Bigg]$$

for all $\gamma,\beta \in X$. If $f(X)\subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Our aim is to generalize and extend the fixed point results of, [2], [4], [5], in the framework of Super metric space. Further, the fixed point results of, [6], [7], [11] are generalized.

2 Main Results

Theorem 2.1. *Let (X,m) be a complete super metric space and the mappings $f,g : X \to X$ satisfy*

$$m(f\gamma, f\beta) \leq \varphi \max\Bigg[m(g\gamma, g\beta), m(g\gamma, f\gamma), \frac{m(g\gamma, f\beta)}{2s}, m(g\beta, f\beta), m(g\beta, f\gamma)\Bigg] \quad (3)$$

for all $\gamma,\beta \in X$. If $f(X)\subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Proof. Let $\gamma_0 \in X$ be an arbitrary point of X. Since $f(X)\subset g(X)$, there exists $\gamma_1 \in X$ such that $g\gamma_1 = f\gamma_0$. In this way, we can construct two distinct sequences $\{f\gamma_n\}$ and $\{g\gamma_n\}$ such that $g\gamma_{n+1} = f\gamma_n$ for all $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, we have $g\gamma_n = g\gamma_{n+1}$, then f and g have a point of coincidence. On the contrary, let $g\gamma_n \neq g\gamma_{n+1}$ for all $n \in \mathbb{N}$.

Thus, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} m(g\gamma_n, g\gamma_{n+1}) &= m(f\gamma_{n-1}, f\gamma_n) \\ &\leq \varphi \max\Bigg[m(g\gamma_{n-1}, g\gamma_n), m(g\gamma_{n-1}, f\gamma_{n-1}), \frac{m(g\gamma_{n-1}, f\gamma_n)}{2s}, m(g\gamma_n, f\gamma_n), m(g\gamma_n, f\gamma_{n-1})\Bigg] \\ &= \varphi \max\Bigg[m(g\gamma_{n-1}, g\gamma_n), m(g\gamma_{n-1}, g\gamma_n), \frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s}, m(g\gamma_n, g\gamma_{n+1}), m(g\gamma_n, g\gamma_n)\Bigg] \\ &= \varphi \max\Bigg[m(g\gamma_{n-1}, g\gamma_n), \frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s}, m(g\gamma_n, g\gamma_{n+1})\Bigg]. \end{aligned}$$

If

$$\max\Bigg[m(g\gamma_{n-1}, g\gamma_n), \frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s}, m(g\gamma_n, g\gamma_{n+1})\Bigg] = m(g\gamma_n, g\gamma_{n+1}),$$

then

$$\begin{aligned} m(g\gamma_n, g\gamma_{n+1}) &\leq \varphi m(g\gamma_n, g\gamma_{n+1}) \\ &< m(g\gamma_n, g\gamma_{n+1}), \end{aligned}$$

which is not possible.

Further, if

$$m(g\gamma_n, g\gamma_{n+1}) \leq \frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s},$$

then using (m_3)

$$\begin{aligned} \lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{n+1}) &\leq \frac{1}{2s}\lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_{n+1}) \\ &\leq \frac{s}{2s}\lim_{n\to\infty} \sup m(f\gamma_{n-1}, g\gamma_{n+1}) \\ &= \frac{1}{2}\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{n+1}), \end{aligned}$$

which is again a contradiction. Therefore,

$$\begin{aligned} m(g\gamma_n, g\gamma_{n+1}) &= m(f\gamma_{n-1}, f\gamma_n) \\ &\leq \varphi m(g\gamma_{n-1}, g\gamma_n) \\ &\leq \varphi^2 m(g\gamma_{n-2}, g\gamma_{n-1}) \\ &\vdots \\ &\leq \varphi^n\, m(g\gamma_0, g\gamma_1). \end{aligned} \quad (4)$$

Our aim is to prove that $\{g\gamma_n\}$ is Cauchy sequence. Let $\epsilon > 0$.

Since $\lim_{n\to\infty} \varphi^n\, m(g\gamma_0, g\gamma_1) = 0$, there exists $N \in \mathbb{N}$ such that

$$\varphi^n\, [m(g\gamma_0, g\gamma_1)] < \epsilon \quad \text{for all } n \geq N.$$

Therefore, using (4) for all $n \geq \mathrm{N}$

$$m(g\gamma_n, g\gamma_{n+1}) < \epsilon. \tag{5}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$m(g\gamma_n, g\gamma_m) < \epsilon \quad \text{for all } m \geq n \geq N. \tag{6}$$

Now from (5), we get that the result is true for $m = n+1$. If $\gamma_n = \gamma_m$, (6) is trivially true.

Without loss of generality, we can take $\gamma_n \neq \gamma_m$. Suppose (6) is true for $m = k$ i.e.

$$\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_k) = 0.$$

Therefore, by using (3) for $m = k+1$ we have

$$\begin{aligned} &m(g\gamma_n, g\gamma_{k+1}) \\ &= m(f\gamma_{n-1}, f\gamma_k) \\ &\leq \varphi \max \Big[m(g\gamma_{n-1}, g\gamma_k), m(g\gamma_{n-1}, f\gamma_{n-1}), \\ &\qquad \frac{m(g\gamma_{n-1}, f\gamma_k)}{2s}, m(g\gamma_k, f\gamma_k), \\ &\qquad m(g\gamma_k, f\gamma_{n-1}) \Big] \\ &= \varphi \max \Big[m(g\gamma_{n-1}, g\gamma_k), m(g\gamma_{n-1}, g\gamma_n), \\ &\qquad \frac{m(g\gamma_{n-1}, g\gamma_{k+1})}{2s}, m(g\gamma_k, g\gamma_{k+1}), \\ &\qquad m(g\gamma_k, g\gamma_n) \Big]. \end{aligned}$$

Let

$$\begin{aligned} \Omega = \Big[&m(g\gamma_{n-1}, g\gamma_k), m(g\gamma_{n-1}, gx_n), \\ &\frac{m(g\gamma_{n-1}, g\gamma_{k+1})}{2s}, m(g\gamma_k, g\gamma_{k+1}), \\ &m(g\gamma_k, g\gamma_n) \Big]. \end{aligned}$$

If $\max \Omega = m(g\gamma_{n-1}, g\gamma_k)$, then

$$\begin{aligned} m(g\gamma_n, g\gamma_{k+1}) &= m(f\gamma_{n-1}, f\gamma_k) \\ &\leq \varphi m(g\gamma_{n-1}, g\gamma_k). \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) \\ &\quad \leq \varphi \lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_k). \end{aligned}$$

Using (m_3), we get

$$\begin{aligned} &\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) \\ &\leq s\varphi \lim_{n\to\infty} \sup m(f\gamma_{n-1}, g\gamma_k) \\ &= s\varphi \lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_k) \\ &= 0. \end{aligned}$$

Hence, by induction $\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) = 0$, since $\varphi(t) < t$ and $s \geq 1$ is finite.
If $\max \Omega = m(g\gamma_{n-1}, g\gamma_n)$, then

$$\begin{aligned} &\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) \\ &\quad \leq \varphi \lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_n) \\ &\quad < \lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_n) \\ &\quad \leq s \lim_{n\to\infty} \sup m(f\gamma_{n-1}, g\gamma_n) \quad (\text{by } m_3) \\ &\quad = s \lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_n) \\ &\quad = 0. \end{aligned}$$

If $\max \Omega = \frac{m(g\gamma_{n-1}, g\gamma_{k+1})}{2s}$, then

$$\begin{aligned} &\lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_{k+1}) \\ &\quad \leq \frac{1}{2s}\varphi \lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_{k+1}) \\ &\quad < \frac{1}{2s} \lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_{k+1}) \\ &\quad \leq \frac{s}{2s} \lim_{n\to\infty} \sup m(f\gamma_{n-1}, g\gamma_{k+1}) \quad (\text{by } m_3) \\ &\quad = \frac{1}{2} \lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}), \end{aligned}$$

which is a contradiction.
If $\max \Omega = m(g\gamma_k, g\gamma_{k+1})$, then

$$\begin{aligned} &\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) \\ &\quad \leq \varphi \lim_{n\to\infty} \sup m(g\gamma_k, g\gamma_{k+1}) \\ &\quad < \lim_{n\to\infty} \sup m(g\gamma_k, g\gamma_{k+1}) \\ &\quad \leq s \lim_{n\to\infty} \sup m(f\gamma_k, g\gamma_{k+1}) \quad (\text{by } m_3) \\ &\quad \leq s \lim_{n\to\infty} \sup m(g\gamma_{k+1}, g\gamma_{k+1}) \\ &\quad = 0. \end{aligned}$$

If $\max \Omega = m(g\gamma_k, g\gamma_n) = m(g\gamma_n, g\gamma_k)$, then the result is clear.
Hence, by induction $\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) = 0$.
It follows $\{g\gamma_n\}$ is a Cauchy sequence. Since we have assumed $g(X)$ to be complete, $\{g\gamma_n\}$ converges to a point, say $q \in g(X)$. So $gp = q = \lim_{n\to\infty} g\gamma_n$, for a point p of X. Now we will prove $gp = fp$.
We have,

$$m(gp, fp) = \lim_{n\to\infty} m(g\gamma_n, fp)$$

$$= \lim_{n\to\infty} m(f\gamma_{n-1}, fp).$$

Consider $m(f\gamma_{n-1}, fp)$ and applying (3), we obtain

$$\begin{aligned} &m(f\gamma_{n-1}, fp) \\ &\leq \varphi \max\Big[m(g\gamma_{n-1}, gp), m(g\gamma_{n-1}, f\gamma_{n-1}), \\ &\qquad \frac{m(g\gamma_{n-1}, fp)}{2s}, m(gp, fp), \\ &\qquad m(gp, f\gamma_{n-1})\Big] \\ &< \max\Big[m(g\gamma_{n-1}, gp), m(g\gamma_{n-1}, f\gamma_{n-1}), \\ &\qquad \frac{m(g\gamma_{n-1}, fp)}{2s}, m(gp, fp), \\ &\qquad m(gp, f\gamma_{n-1})\Big] \\ &= \max\Big[m(g\gamma_{n-1}, gp), m(g\gamma_{n-1}, g\gamma_n), \\ &\qquad \frac{m(g\gamma_{n-1}, fp)}{2s}, m(gp, fp), \\ &\qquad m(gp, g\gamma_n)\Big]. \end{aligned}$$

Taking $n \to \infty$, gives

$$\begin{aligned} m(gp, fp) &< \max\Big[m(gp, fp), m(gp, gp), \\ &\qquad \frac{m(gp, fp)}{2s}), m(gp, fp), \\ &\qquad m(gp, gp)\Big] \\ &= m(gp, fp), \end{aligned}$$

which is a contradiction.
Therefore $gp = fp$. We will now show that f and g have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. By applying (3), it follows that

$$\begin{aligned} m(gp, gq) &= m(fp, fq) \\ &\leq \varphi \max\Big[m(gp, gq), m(gp, fp), \\ &\qquad \frac{m(gp, fq)}{2s}, m(gq, fq), \\ &\qquad m(gq, fp)\Big] \\ &\leq \varphi m(gp, gq) < m(gp, gq), \end{aligned}$$

which is a contradiction. Hence we have $gp = gq$.

This implies that f and g have a unique point of coincidence. By Proposition 1.12, we conclude that f and g have a unique common fixed point.
This complete the proof of theorem. □

Remark 2.2. Let $g = I_X$, be Identity map on X in Theorem 2.1, we get a generalization and extension of Proposition 1.6.

Proof. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = kt$. Therefore, φ is a non decreasing function and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in (0, +\infty)$. It follows that the contractive conditions of Theorem 2.1 are now satisfied. This completes the proof. □

Remark 2.3. Taking $g = I_X$, the Identity map on X in Theorem 2.1, one can deduce an extended analogue of Proposition 1.2 in super metric space.

Example 2.4. Let $X = [1, 3]$ and define

$$m(\gamma, \beta) = \begin{cases} \gamma\beta, & \gamma \neq \beta, \\ 0, & \gamma = \beta. \end{cases}$$

It has been shown in [8] that (X, m) is a super metric space. Further, let $\varphi = \frac{1}{2}$. Now consider $f, g : X \to X$ as follows

$$f\gamma = \begin{cases} 2, & \gamma \neq 3, \\ \frac{3}{2}, & \gamma = 3 \end{cases} \quad \text{and} \quad g\gamma = 4 - \gamma.$$

Here $g(X) = [1, 3]$, $f(X) \subset g(X)$ and $g(X)$ is complete space.
We obtain that f and g satisfy the contractive conditions of Theorem 2.1. Indeed for $\gamma \neq 3$, $\beta = 3$ and $s = 6$, we obtain

$$m\left(f\gamma, f\beta\right) = m\left(2, \frac{3}{2}\right) = 2 \times \frac{3}{2} = 3.$$

We calculate the right hand side of Theorem 2.1.

(i) $\varphi[m(g\gamma, g\beta)] = \frac{1}{2}m(g\gamma, 1) = \frac{1}{2}g\gamma$, where $g\gamma \in (1, 3]$.

(ii) $\varphi[m(g\gamma, f\gamma)] = \frac{1}{2}m(g\gamma, 2) = \frac{1}{2}2g\gamma$, where $g\gamma \in (1, 3]$.

(iii) $\frac{\varphi[m(g\gamma, f\beta)]}{2s} = \frac{1}{2}\frac{m(g\gamma, \frac{3}{2})}{s} = \frac{1}{2}(\frac{3g\gamma}{2s}) \leq \frac{1}{2}(\frac{3}{2}g\gamma)$, using (m_3) and where $g\gamma \in (1, 3]$.

(iv) $\varphi[m(g\gamma, f\beta)] = \varphi m(1, \frac{3}{2}) = \frac{1}{2}(3)$.

(v) $\varphi[m(g\beta, f\gamma)] = \varphi m(1, 2) = \frac{1}{2}(4)$.

The other cases are straightforward. Now for $\gamma = 2$, $f\gamma = g\gamma$ and $fg\gamma = gf\gamma$. So, 2 is the unique point of coincidence of f and g. Thus all the conditions of Theorem 2.1 are satisfied. Therefore, 2 is the unique common fixed point by Theorem 2.1.

Theorem 2.5. *Let (X, m) be a complete super metric space. Suppose that the mappings $f, g : X \to X$ satisfy*

$$m(f\gamma, f\beta) \leq \varphi\Bigg[\max\Bigg\{m(g\gamma, g\beta), \frac{m(g\gamma, f\beta) + m(g\beta, f\gamma)}{2s}, \frac{\begin{pmatrix} m(g\gamma, f\gamma)m(g\gamma, f\beta) \\ +m(g\beta, f\beta)m(g\beta, f\gamma) \end{pmatrix}}{m(g\gamma, f\beta) + m(g\beta, f\gamma) + 1}\Bigg\}\Bigg] \tag{7}$$

for all $\gamma, \beta \in X$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Proof. Let $\gamma_0 \in X$ be an arbitrary point. Since $f(X) \subset g(X)$, there exists $\gamma_1 \in X$ such that $g\gamma_1 = f\gamma_0$. Inductively, we can construct two distinct sequences $\{f\gamma_n\}$ and $\{g\gamma_n\}$ such that $g\gamma_{n+1} = f\gamma_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $g\gamma_n = g\gamma_{n+1}$, then f and g have a point of coincidence. Thus, we can suppose that $g\gamma_n \neq g\gamma_{n+1}$, for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&m(g\gamma_n, g\gamma_{n+1}) \\
&= m(f\gamma_{n-1}, f\gamma_n) \\
&\leq \varphi\Bigg[\max\Bigg\{m(g\gamma_{n-1}, g\gamma_n), \frac{m(g\gamma_{n-1}, f\gamma_n) + m(g\gamma_n, f\gamma_{n-1})}{2s}, \\
&\quad \frac{\begin{pmatrix} m(g\gamma_{n-1}, f\gamma_{n-1})\, m(g\gamma_{n-1}, f\gamma_n) \\ +m(g\gamma_n, f\gamma_n)\, m(g\gamma_n, f\gamma_{n-1}) \end{pmatrix}}{m(g\gamma_{n-1}, f\gamma_n) + m(g\gamma_n, f\gamma_{n-1}) + 1}\Bigg\}\Bigg] \\
&= \varphi\Bigg[\max\Bigg\{m(g\gamma_{n-1}, g\gamma_n), \frac{m(g\gamma_{n-1}, g\gamma_{n+1}) + m(g\gamma_n, g\gamma_n)}{2s}, \\
&\quad \frac{\begin{pmatrix} m(g\gamma_{n-1}, g\gamma_n)\, m(g\gamma_{n-1}, g\gamma_{n+1}) \\ +m(g\gamma_n, g\gamma_{n+1})\, m(g\gamma_n, g\gamma_n) \end{pmatrix}}{m(g\gamma_{n-1}, g\gamma_{n+1}) + m(g\gamma_n, f\gamma_{n-1}) + 1}\Bigg\}\Bigg] \\
&\leq \varphi\left[\max\left\{m(g\gamma_{n-1}, g\gamma_n), \frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s}\right\}\right].
\end{aligned}$$

If

$$\max\left[m(g\gamma_{n-1}, g\gamma_n), \frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s}\right] = \frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s},$$

then

$$\begin{aligned}
m(g\gamma_n, g\gamma_{n+1}) &\leq \varphi\left[\frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s}\right] \\
&< \frac{m(g\gamma_{n-1}, g\gamma_{n+1})}{2s}.
\end{aligned}$$

Taking limit as $n \to \infty$ on both sides implies that

$$\begin{aligned}
&\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{n+1}) \\
&\leq \frac{1}{2s} \lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_{n+1}) \\
&\leq \frac{s}{2s} \lim_{n\to\infty} \sup m(f\gamma_{n-1}, g\gamma_{n+1}) \quad \text{(by } m_3\text{)} \\
&= \frac{1}{2} \lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{n+1}),
\end{aligned}$$

giving a contradiction.
Therefore,

$$m(g\gamma_n, g\gamma_{n+1}) \leq \varphi\, m(g\gamma_{n-1}, g\gamma_n)$$

That is, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
m(g\gamma_n, g\gamma_{n+1}) &= m(f\gamma_{n-1}, f\gamma_n) \\
&\leq \varphi\, m(g\gamma_{n-1}, g\gamma_n) \\
&\leq \varphi^2\, m(g\gamma_{n-2}, g\gamma_{n-1}) \\
&\;\;\vdots \\
&\leq \varphi^n\, m(g\gamma_0, g\gamma_1).
\end{aligned}$$

We will show that $\{g\gamma_n\}$ is a Cauchy sequence.
Since $\lim_{n\to\infty} \varphi^n\, m(g\gamma_0, g\gamma_1) = 0$, there exists $N \in \mathbb{N}$, such that

$$\varphi^n\, m(g\gamma_0, g\gamma_1) < \epsilon \quad \text{for all } n \geq \text{N}.$$

This implies that

$$m(g\gamma_n, g\gamma_{n+1}) < \epsilon \quad \text{for all } n \geq N. \tag{8}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$m(g\gamma_n, g\gamma_m) < \epsilon \quad \text{for all } m \geq n \geq N \tag{9}$$

by induction on m. From (8), the result is true for $m = n+1$. Suppose that (9) holds for $m = k$. Therefore, for $m = k+1$, we have

$$\begin{aligned}
&m(g\gamma_n, g\gamma_{k+1}) \\
&= m(f\gamma_{n-1}, f\gamma_k) \\
&\leq \varphi\Bigg[\max\Bigg\{m(g\gamma_{n-1}, g\gamma_k), \\
&\quad \frac{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1})}{2s},
\end{aligned}$$

$$\left.\left.\frac{\begin{pmatrix} m(g\gamma_{n-1}, f\gamma_{n-1})\, m(g\gamma_{n-1}, f\gamma_k) \\ +m(g\gamma_k, f\gamma_k)\, m(g\gamma_k, f\gamma_{n-1}) \end{pmatrix}}{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1}) + 1}\right\}\right].$$

Denote

$$A = \Bigg[m(g\gamma_{n-1}, g\gamma_k), \frac{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1})}{2s}, \frac{\begin{pmatrix} m(g\gamma_{n-1}, f\gamma_{n-1})\, m(g\gamma_{n-1}, f\gamma_k) \\ +m(g\gamma_k, f\gamma_k)\, m(g\gamma_k, f\gamma_{n-1}) \end{pmatrix}}{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1}) + 1}\Bigg].$$

If $\max A = [m(g\gamma_{n-1}, g\gamma_k)]$, then

$$\begin{aligned} m(g\gamma_n, g\gamma_{k+1}) &\le \varphi m(g\gamma_{n-1}, g\gamma_k) \\ &< m(g\gamma_{n-1}, g\gamma_k). \end{aligned}$$

Using (m_3),

$$\begin{aligned} &\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) \\ &\quad \le s \lim_{n\to\infty} \sup m(fx_{n-1}, g\gamma_k) \\ &\quad = s \lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_k) \\ &\quad = 0. \end{aligned}$$

Hence

$$m(g\gamma_n, g\gamma_{k+1}) < \epsilon. \tag{10}$$

If $\max A = \frac{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1})}{2s}$, then

$$\begin{aligned} &m(g\gamma_n, g\gamma_{k+1}) \\ &\quad \le \varphi\left[\frac{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1})}{2s}\right] \\ &\quad < \frac{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1})}{2s} \\ &\quad = \frac{m(g\gamma_{n-1}, g\gamma_{k+1}) + m(g\gamma_k, g\gamma_n)}{2s}. \end{aligned}$$

Taking $n \to \infty$ and using (m_3),

$$\begin{aligned} &\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) \\ &\quad < \frac{1}{2s} \lim_{n\to\infty} \sup m(g\gamma_{n-1}, g\gamma_{k+1}) \\ &\qquad + \frac{1}{2s} \lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_k) \\ &\quad \le \frac{1}{2} \lim_{n\to\infty} \sup m(f\gamma_{n-1}, g\gamma_{k+1}) \\ &\quad = \frac{1}{2} \lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) \end{aligned}$$

which gives a contradiction.

If

$$\max A = \left[\frac{m(g\gamma_{n-1}, f\gamma_{n-1})\, m(g\gamma_{n-1}, f\gamma_k)}{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1}) + 1} + \frac{m(g\gamma_k, f\gamma_k)\, m(g\gamma_k, f\gamma_{n-1})}{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1}) + 1}\right],$$

then

$$\begin{aligned} &m(g\gamma_n, g\gamma_{k+1}) \\ &\quad \le \varphi\left[\frac{\begin{pmatrix} m(g\gamma_{n-1}, f\gamma_{n-1})\, m(g\gamma_{n-1}, f\gamma_k) \\ +m(g\gamma_k, f\gamma_k)\, m(g\gamma_k, f\gamma_{n-1}) \end{pmatrix}}{m(g\gamma_{n-1}, f\gamma_k) + m(g\gamma_k, f\gamma_{n-1}) + 1}\right] \\ &\quad = \varphi\left[\frac{\begin{pmatrix} m(g\gamma_{n-1}, g\gamma_n)\, m(g\gamma_{n-1}, g\gamma_{k+1}) \\ +m(g\gamma_k, g\gamma_{k+1})\, m(g\gamma_k, g\gamma_n) \end{pmatrix}}{m(g\gamma_{n-1}, g\gamma_{k+1}) + m(g\gamma_k, g\gamma_n) + 1}\right] \\ &\quad = \varphi\left[\frac{\begin{pmatrix} m(g\gamma_{n-1}, g\gamma_n)\, m(g\gamma_{n-1}, g\gamma_{k+1}) \\ +m(g\gamma_k, g\gamma_{k+1})\, m(g\gamma_k, g\gamma_n) \end{pmatrix}}{m(g\gamma_{n-1}, g\gamma_{k+1}) + m(g\gamma_k, g\gamma_n) + 1}\right] \\ &\quad = \varphi\left[\frac{m(g\gamma_{n-1}, g\gamma_n)\, m(g\gamma_{n-1}, g\gamma_{k+1})}{m(g\gamma_{n-1}, g\gamma_{k+1})\, m(g\gamma_k, g\gamma_n) + 1} + \frac{m(g\gamma_k, g\gamma_{k+1})\, m(g\gamma_k, g\gamma_n)}{m(g\gamma_{n-1}, g\gamma_{k+1})\, m(g\gamma_k, g\gamma_n) + 1}\right] \\ &\quad \le \varphi\Big[m(g\gamma_{n-1}, g\gamma_n) + m(g\gamma_k, g\gamma_{k+1})\Big]. \end{aligned}$$

Taking $n \to \infty$ and using (m_3), we have

$$\begin{aligned} &\lim_{n\to\infty} \sup m(g\gamma_n, g\gamma_{k+1}) \\ &\quad \le \varphi \lim_{n\to\infty} \sup[m(g\gamma_{n-1}, g\gamma_n) + m(g\gamma_k, g\gamma_{k+1})] \\ &\quad \le s \lim_{n\to\infty} \sup[m(f\gamma_{n-1}, g\gamma_n) + m(f\gamma_k, g\gamma_{k+1})] \\ &\quad = s \lim_{n\to\infty} \sup[m(g\gamma_n, g\gamma_n) + m(g\gamma_{k+1}, g\gamma_{k+1})] \\ &\quad = 0, \quad \text{since } s \ge 1 \text{ is finite.} \end{aligned}$$

Therefore,

$$m(g\gamma_n, g\gamma_{k+1}) < \epsilon. \tag{11}$$

Thus (10) holds for all $m \ge n \ge N$. It follows that $\{g\gamma_n\}$ is a Cauchy sequence. By the completeness of $g(X)$, we obtain that $\{g\gamma_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q = \lim_{n\to\infty} g\gamma_n$. We will show that $gp = fp$. Suppose that $gp \ne fp$.
Now,

$$\begin{aligned} m(gp, fp) &= \lim_{n\to\infty} m(g\gamma_n, fp) \\ &= \lim_{n\to\infty} m(f\gamma_{n-1}, fp). \end{aligned}$$

Consider $m(f\gamma_{n-1}, fp)$ and applying (7), we obtain

$$\begin{aligned} &m(f\gamma_{n-1}, fp) \\ &\le \varphi\Bigg[\max\Bigg\{m(g\gamma_{n-1}, gp), \\ &\frac{m(g\gamma_{n-1}, fp) + m(gp, f\gamma_{n-1})}{2s}, \\ &\frac{\begin{pmatrix} m(g\gamma_{n-1}, f\gamma_{n-1})\, m(g\gamma_{n-1}, fp) \\ +m(gp, fp)\, m(gp, f\gamma_{n-1}) \end{pmatrix}}{m(g\gamma_{n-1}, fp) + m(gp, f\gamma_{n-1}) + 1}\Bigg\}\Bigg] \\ &= \varphi\Bigg[\max\Bigg\{m(g\gamma_{n-1}, gp), \\ &\frac{m(g\gamma_{n-1}, fp) + m(gp, g\gamma_n)}{2s}, \\ &\frac{\begin{pmatrix} m(g\gamma_{n-1}, g\gamma_n)\, m(g\gamma_{n-1}, fp) \\ +m(gp, fp)\, m(gp, g\gamma_n) \end{pmatrix}}{m(g\gamma_{n-1}, fp) + m(gp, g\gamma_n) + 1}\Bigg\}\Bigg]. \end{aligned}$$

Taking limit as $n \to \infty$

$$\begin{aligned} &m(gp, fp) \\ &\le \varphi\Bigg[\max\Bigg\{m(gp, gp), \frac{m(gp, fp) + m(gp, gp)}{2s}, \\ &\frac{\begin{pmatrix} m(gp, gp)\, m(gp, fp) \\ +m(gp, fp)\, m(gp, gp) \end{pmatrix}}{m(gp, fp) + m(gp, gp) + 1}\Bigg\}\Bigg] \\ &= \varphi\left[\frac{m(gp, fp)}{2s}\right] < \left[\frac{m(gp, fp)}{2s}\right], \end{aligned}$$

giving a contradiction, since $s \ge 1$. So, $gp = fp$.
We now show that f and g have a unique point of coincidence. Let $fq = gq$ for some $q \in X$.
Assume that $gp \ne gq$. By applying (7), it follows that

$$\begin{aligned} &m(gp, gq) \\ &= m(fp, fq) \\ &\le \varphi\Bigg[\max\Bigg\{m(gp, gq), \frac{m(gq, fq) + m(gq, fp)}{2s}, \\ &\frac{\begin{pmatrix} m(gp, fp)\, m(gp, fq) \\ +m(gq, fq)\, m(gq, fp) \end{pmatrix}}{m(gp, fq)\, m(gp, fp) + 1}\Bigg\}\Bigg] \\ &= \varphi\Bigg[\max\Bigg\{m(gp, gq), \frac{m(gq, gq) + m(gq, gp)}{2s}, \\ &\frac{\begin{pmatrix} m(gp, gp)\, m(gp, gq) \\ +m(gq, gq)\, m(gq, gp) \end{pmatrix}}{m(gp, gq)\, m(gp, gp) + 1}\Bigg\}\Bigg] \\ &= \varphi\left[\max\left\{m(gp, gq), \frac{m(gp, gq)}{s}\right\}\right] \\ &= \varphi\, m(gp, gq). \end{aligned}$$

Therefore,

$$m(gp, gq) \le \varphi\, m(gp, gq) < m(gp, gq)$$

which leads to a contradiction. Hence $gp = gq$.

This implies that f and g have a unique point of coincidence. By Proposition 1.12, we can conclude that f and g have a unique common fixed point. □

Remark 2.6. If we take $g = I_X$, the Identity map on X in Theorem 2.5, we get a generalization and extension of Proposition 1.13.

Proof. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = kt$. Therefore, φ is a non decreasing function and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in (0, +\infty)$. It follows that the contractive conditions of Theorem 2.5 are now satisfied. This completes the proof. □

3 Conclusion

We have studied the results of, [1, 2], [4], and [5], in Generalized Metric space. Further, the results of, [6], and [7], have also been studied. We have generalized and extended the above results in the framework of Super Metric Space.

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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.