Numerical Comparisons Between Some Recent Modifications of Fractional Euler Methods

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Abstract: - For dealing with systems of Fractional Differential Equations (FDEs), the primary goal of this work is to offer several graphical comparisons performed with the use of specific recent adjustments of the so-called Fractional Euler Method (FEM). These numerical adjustments are the Modified FEM (or MFEM) and Improved Modified FEM (or IMFEM) coupled with the FEM itself. This would reveal the best approximate solution to the fractional-order system consisting of FDEs in comparison with its exact solution.

Key-Words: -Caputooperator; FDEs; FEM; Caputo fractional differentiator; Riemann-Liouville fractional integrator, Fractional differential equations; Fractional Euler method; Modified Fractional Euler method; Improved fractional Euler method.

Received: October 22, 2023. Revised: May 21, 2024. Accepted: July 11, 2024. Published: September 2, 2024.

1 Introduction

Fractional calculus allows for procedures such as the half-derivative by extending the ideas of integrals and derivatives to non-integer orders. In situations where classical calculus is inadequate, this area of mathematical analysis offers methods for modeling and problem-solving in a variety of domains, such as finance, engineering, and physics. Fractional calculus provides a more comprehensive framework that allows for the capture of complicated dynamics and memory effects in systems, leading to more precise descriptions and forecasts, see [1, 2, 3, 4, 5, 6].

Fractional differential equations are regarded very helpful tools used to formulate memory-like or fractal features in many physical models. These equations have progressively been utilized for modeling a lot of engineering, physics, biology, viscoelasticity, and fluid dynamics problems, [7], [8], [9], [10], [11]. Numerous approaches have recently been established for handling FDEs in their linear and nonlinear cases including finite difference approach, differential transform approach, variational iteration approach, homotopy perturbation approach, Adomain decomposition approach, predictor corrector approach, and many others, [12], [13], [14], [15], [16], [17]. In the same regard, there have been many nonlinear phenomena, which have been formulated with the use of FDEs, like fractional-order Stiff's, Duffing's, Rössler's, and Chuah's systems.

This work is devoted to deal with a system of FDEs using three recent numerical approaches; FEM, MFEM, and IMFEM. Such a system can be represented by

$$\begin{aligned} D^{\delta}x_1(s) &= \Theta_1(s, x_1(s), x_2(s), \cdots, x_n(s)),\\ D^{\delta}x_2(s) &= \Theta_2(s, x_1(s), x_2(s), \cdots, x_n(s)),\\ &\vdots\\ D^{\delta}x_n(s) &= \Theta_n(s, x_1(s), x_2(s), \cdots, x_n(s)), \end{aligned} \tag{1}$$

with initial conditions

$$x_1(0) = v_1, x_2(0) = v_2, \cdots, x_n(0) = v_n.$$

The following is how our paper is arrangement: Part 2 remembers some necessary facts and preliminaries connected to the fractional calculus. Part 3 is devoted to recollect the latest numerical versions of the FEM, the MFEM and IMFEM, to handle the systems of FDEs. In Part 4, the collected numerical approaches are implemented to deal with fractional-order systems. In Part 5, a certain numerical example is provided for verifying the capability of the studied approaches, succeeded by Part 6 that summarizes the conclusion of this paper.

2 Fundamental facts

In this paper, we aim to implement certain numerical approaches for approximating solutions to the following Fractional Initial Value Problem (FIVP):

$$D_*^{\delta}z(s) = \varphi(s, z(s)), \tag{2}$$

with initial condition

$$z(0) = z_0, \tag{3}$$

where $\delta \in (0, 1]$. To this end, we remember some fundamental concepts and facts connected with the fractional calculus.

Definition 1 *[18] For $0 < \delta \leq 1$ and $x > 0$, the Riemann-Liouville integrator can be expressed by*

$$J^{\delta}\vartheta(s) = \frac{1}{\Gamma(\delta)} \int_0^s (s-u)^{\delta-1}\vartheta(u).du. \quad (4)$$

The subsequent properties are satisfied by the aforesaid operator, [18]:

- $J^{\delta}J^{\beta}\vartheta(s) = J^{\beta}J^{\delta}\vartheta(s), \delta, \beta > 0.$
- $J^{\delta}J^{\beta}\vartheta(s) = J^{\delta+\beta}\vartheta(s), \delta, \beta > 0.$
- $J^{\delta}x^{\varphi} = \dfrac{\Gamma(\delta+\varphi)}{\Gamma(\delta+\varphi+1)}x^{\varphi+\delta}, \varphi > -1.$

Definition 2 *[18] The Caputo differentiator can be expressed by*

$$D_*^{\delta}\vartheta(s) = J^{m-\delta}D^m\vartheta(s) \quad (5)$$
$$= \frac{1}{\Gamma(m-\delta)} \int_0^x (s-u)^{(m-\delta-1)}\vartheta^{(m)}(u)du, \quad (6)$$

where $m-1 < \delta \leq m$ for which $\vartheta \in C^m[0,b]$ and $m \in \mathbb{N}$.

Lemma 1 *[18] Consider, for $m \in \mathbb{N}$, we have $m-1 < \delta \leq m$ and $\vartheta \in C^m[0,b]$. Then*

$$D_*^{\delta}J^{\delta}\vartheta(s) = \vartheta(s), \quad (7)$$

for $s > 0$. and

$$J^{\delta}D_*^{\delta}\vartheta(s) = \vartheta(s) - \sum_{k=1}^{m-1}\vartheta^k(0^+)\frac{s^k}{k!}. \quad (8)$$

Theorem 1 *[19] For $\delta \in (0,1]$ and $k = 0,1,2,\cdots,n+1$, consider $D_*^{k\delta}\vartheta(s) \in C(0,b]$, the function ϑ might be expanded around the node s_0 in the following manner:*

$$\vartheta(s) = \sum_{i=0}^{n}\frac{(s-s_0)^{i\delta}}{\Gamma(i\delta+1)}D_*^{i\delta}\vartheta(s_0) + \frac{(s-s_0)^{(n+1)\delta}}{\Gamma((n+1)\delta+1)}D_*^{(n+1)\delta}\vartheta(\xi), \quad (9)$$

for which $0 < \xi < s$, $\forall x \in (0,b]$.

For further clarification, one might rewrite the previous equality in the following manner:

$$\vartheta(s) = \vartheta(s_0) + \frac{(s-s_0)^{\delta}D_*^{\delta}\vartheta(s_0)}{\Gamma(\delta+1)} + \frac{(s-s_0)^{2\delta}D_*^{2\delta}\vartheta(s_0)}{\Gamma(2\delta+1)} + \cdots + \frac{(s-s_0)^{n\delta}D_*^{n\delta}\vartheta(s_0)}{\Gamma(n\delta+1)} + \frac{(s-s_0)^{(n+1)\delta}D_*^{(n+1)\delta}\vartheta(\xi)}{\Gamma((n+1)\delta+1)}. \quad (10)$$

3 Novel modifications of FEM

The main aim of this part is to recollect some current numerical approaches (FEM, MFEM and IMFEM) that can be employed to handle FIVPs. For this purpose, we discretize the interval $[0,b]$ as $0 = s_0 < s_1 = s_0+h < s_2 = s_0+2h < \cdots < s_n = s_0+nh = b$ for which $s_i = s_0+ih$ and $h = \frac{b-a}{n}$ are respectively called mesh points and step size, for $i = 1,2,\ldots,n$. In this regard, a useful generalized version of the traditional Euler method, called later on the FEM, was proposed in [19], with the help of the first three terms of Theorem 1 to deal with the FIVP (2-3). Such a method consists of the following formula:

$$\chi_0 = z_0$$
$$\chi_{i+1} = \chi_i + \frac{h^{\delta}}{\Gamma(\delta+1)}\varphi(s_i,\chi_i), \quad (11)$$

for $i = 0,1,\cdots,n-1$. It should be noted that χ_i indicates an approximate solution of problem (2-3).

In more recent time, the researchers in [12], have effectively evolved a novel adjustment of the FEM, named by the MFEM, for handling FIVP (2-3). Such a formula has the following expression:

$$\chi_0 = z_0$$
$$\chi_{i+1} = \chi_i + \frac{h^{\delta}}{\Gamma(\delta+1)} \times \varphi\left(s_i + \frac{h^{\delta}}{2\Gamma(\delta+1)}, \chi_i + \frac{h^{\delta}}{2\Gamma(\delta+1)}\varphi(s_i,\chi_i)\right), \quad (12)$$

for $i = 1,2,\cdots,n-1$.

In a similar manner, a further novel numerical adjustment, called IMFEM, has recently been established in [20], for handling FIVP (2-3). Such a formula has the following expression:

$$\chi_0 = z_0$$
$$\chi_{i+1} = \chi_i + \frac{h^{\delta}}{\Gamma(\delta+1)} \times \varphi\left(s_i + \frac{h^{\delta}}{2\Gamma(\delta+1)}, \chi_i + \frac{h^{\delta}}{2\Gamma(\delta+1)} \times \varphi\left(s_i, \chi_i + \frac{h^{\delta}}{\Gamma(\delta+1)} \times \varphi\left(s_i, \chi_i + \frac{h^{\delta}}{\Gamma(\delta+1)}\varphi(s_i,\chi_i)\right)\right)\right) \quad (13)$$

for $i = 0,1,2,\cdots,n-1$.

4 Dealing with fractional-order systems

In order to provide a numerical solution to system (1) with the use of the MFEM, one might consider the first equation and consequently implement (12) in the following manner:

$$\begin{aligned} x_1(s_{i+1}) &= x_1(s_i) + \frac{h^\delta}{\Gamma(\delta+1)}\Theta_1 \\ &\times \Bigg[s_i + \frac{h^\delta}{2\Gamma(\delta+1)}, x_1(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_1\left(s_i, \mathbf{x}(s)\right), x_2(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\Theta_1\left(s_i, \mathbf{x}(s)\right), x_n(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_1\left(s_i, \mathbf{x}(s)\right) \Bigg]. \end{aligned} \tag{14}$$

for $i = 0, 1, 2, \cdots, n-1$, where $\mathbf{x}(s) = (x_1(s), x_2(s), \cdots, x_n(s))$. To solve the remaining states numerically, the above technique might be repeated similarly to get their corresponding approximate solutions. Eventually, the subsequent recursive formula that provides an approximate solution to system (1) can be deduced:

$$\begin{aligned} x_1(s_{i+1}) &= x_1(s_i) + \frac{h^\delta}{\Gamma(\delta+1)} \\ &\times \Theta_i \Bigg[\Theta_i + \frac{h^\delta}{2\Gamma(\delta+1)}, x_1(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_1\left(s_i, \mathbf{x}(s)\right), x_2(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_i\left(s_i, \mathbf{x}(s)\right), \cdots, x_n(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_1\left(s_i, \mathbf{x}(s)\right) \Bigg] \\ &\vdots \\ x_n(s_{i+1}) &= x_n(s_i) + \frac{h^\delta}{\Gamma(\delta+1)} \\ &\times \Theta_n \Bigg[s_i + \frac{h^\delta}{2\Gamma(\delta+1)}, x_1(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_n\left(s_i, \mathbf{x}(s)\right), x_2(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_n\left(s_i, \mathbf{x}(s)\right), \cdots, x_n(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_n\left(s_i, \mathbf{x}(s)\right) \Bigg]. \end{aligned} \tag{15}$$

On the other side, with the aim of solving system (1) with the help of using IMFEM (13), one might track the same identical way mentioned earlier to get the desired IMFEM's solution.

$$\begin{aligned} x_1(s_{i+1}) &= x_1(s_i) + \frac{h^\delta}{\Gamma(\delta+1)} \\ &\times \Theta_i \Bigg[s_i + \frac{h^\delta}{2\Gamma(\delta+1)}, x_1(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_1\left(s_i, \mathbf{x}(s)\right), x_2(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_i\left(s_i, \mathbf{x}(s)\right), \cdots, x_n(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_i\left(s_i, \mathbf{x}(s)\right), x_2(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_i\left(s_i, \mathbf{x}(s)\right), \cdots, x_n(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_i\left(s_i, \mathbf{x}(s)\right), \cdots, x_n(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_i\left(s_i, \mathbf{x}(s)\right) \Bigg] \\ &\vdots \\ x_n(s_{i+1}) &= x_n(s_i) + \frac{h^\delta}{\Gamma(\delta+1)} \\ &\times \Theta_n \Bigg[s_i + \frac{h^\delta}{2\Gamma(\delta+1)}, x_1(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_n\left(s_i, \mathbf{x}(s)\right), x_2(s_i) + \frac{h^\delta}{2\Gamma(\delta+1)} \\ &\times \Theta_n\left(s_i, \mathbf{x}(s)\right), \cdots, x_n(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_n\left(s_i, \mathbf{x}(s)\right), \cdots, x_n(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_n\left(s_i, \mathbf{x}(s)\right), \cdots, x_n(s_i) \\ &+ \frac{h^\delta}{2\Gamma(\delta+1)}\Theta_n\left(s_i, \mathbf{x}(s)\right) \Bigg]. \end{aligned} \tag{16}$$

In fact, the previous formulas can be valid for each of $x_2(s_{i+1}), s_3(s_{i+1}), \cdots, x_n(s_{i+1})$, that is when the first term $x_1(s_i)$ of such formula is replacing by $x_2(s_i), x_3(s_i), \cdots, x_n(s_i)$ coupled with replacing Θ_1 by $\Theta_2, \Theta_3, \cdots, \Theta_n$, respectively, for $i = 0, 1, 2, \cdots, n$.

In conclusion, formula (15) implies a numerical solution to fractional-order system (1) achieved with the use of MFEM, whereas formula (16) implies another numerical solution to the same system achieved with the use of IMFEM.

5 Numerical findings

This section is devoted to provide an illustrative example to verify the validity of the both numerical procedures discussed previously in Part 4. To this end, we take into consideration the following fractional-order system:

$$\begin{aligned} D^\delta x(t) &= x(t) - 2y(t) + 2t, \\ D^\delta y(t) &= 2x(t) - 0.9y(t) - 3, \end{aligned} \tag{17}$$

with initial conditions:

$$x(0) = 1, \ y(0) = 0. \tag{18}$$

The following form represents the exact solutions of problem (17-18) for $\delta = 1$:

$$\begin{aligned} x(t) = \frac{1}{56699}\Bigg\{ &- 90683 e^{\frac{t}{20}} \cos\left(\frac{\sqrt{1239}t}{20}\right) \\ &+ 147382 \cos\left(\frac{\sqrt{1239}t}{20}\right)^2 \Bigg\} \\ &+ \frac{1}{56699}\Bigg\{ 32922t \cos\left(\frac{\sqrt{1239}t}{20}\right)^2 \\ &+ 457\sqrt{1239} e^{\frac{t}{20}} \sin\left(\frac{\sqrt{1239}t}{20}\right) \Bigg\} \\ &+ \frac{1}{56699}\Bigg\{ 147382 \sin\left(\frac{\sqrt{1239}t}{20}\right)^2 \\ &+ 32922t \sin\left(\frac{\sqrt{1239}t}{20}\right)^2 \Bigg\}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} y(t) = \frac{-10}{56699}\Bigg\{ &5723 e^{\frac{t}{20}} \cos\left(\frac{\sqrt{1239}t}{20}\right) \\ &- 5723 \cos\left(\frac{\sqrt{1239}t}{20}\right)^2 \Bigg\} \\ &+ \frac{-10}{56699}\Bigg\{ - 7316t \cos\left(\frac{\sqrt{1239}t}{20}\right)^2 \\ &+ 205\sqrt{1239} e^{\frac{t}{20}} \sin\left(\frac{\sqrt{1239}t}{20}\right) \Bigg\} \\ &+ \frac{-10}{56699}\Bigg\{ - 5723 \sin\left(\frac{\sqrt{1239}t}{20}\right)^2 \\ &- 7316t \sin\left(\frac{\sqrt{1239}t}{20}\right)^2 \Bigg\}. \end{aligned} \tag{20}$$

With assuming that $\delta = 1$, $h = 0.1$ and using formulae (15) and (16), we depict approximate solutions of system (17-17) in Figure 1 and Figure 2 gotten by MFEM and IMFEM. These figures have also a comparison between such solutions and the exact solutions together with the FEM's solutions.

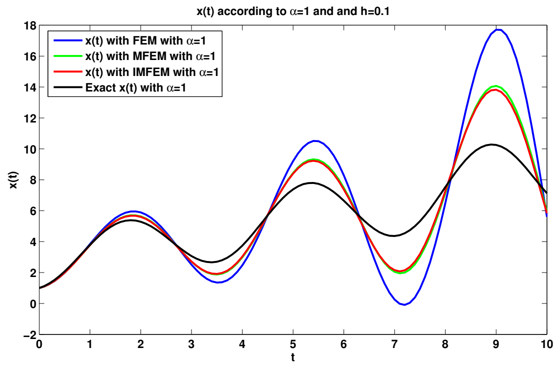

Fig. 1: The solution $x(t)$ obtained with the use of FEM, MFEM and IMFEM when $\delta = 1$ and $h = 0.1$

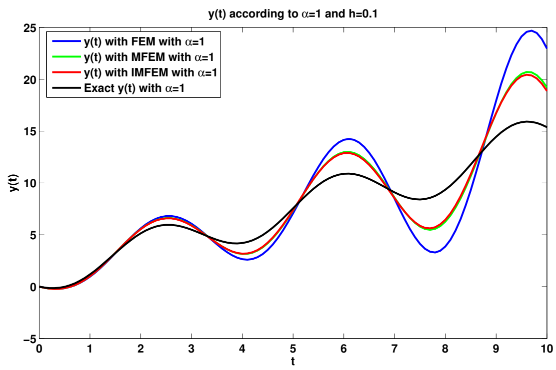

Fig. 2: The solution $y(t)$ obtained with the use of FEM, MFEM and IMFEM when $\delta = 1$ and $h = 0.1$

It should be noted, based on the previous figures, that it is difficult to distinguish which method is the best. To this aim, we re-plot in Figure 3 and Figure 4 the MFEM and IMFEM's solutions and compared them with the FEM's solution coupled with the exact solutions twice again, but this time we take $h = 0.01$.

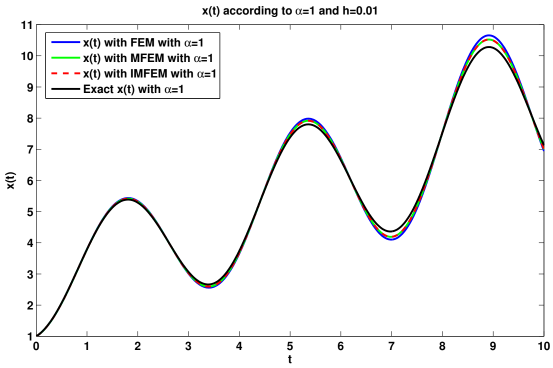

Fig. 3: The solution $x \quad (t)$ obtained with the use of FEM, MFEM and IMFEM when $\delta = 1$ and $h = 0.01$

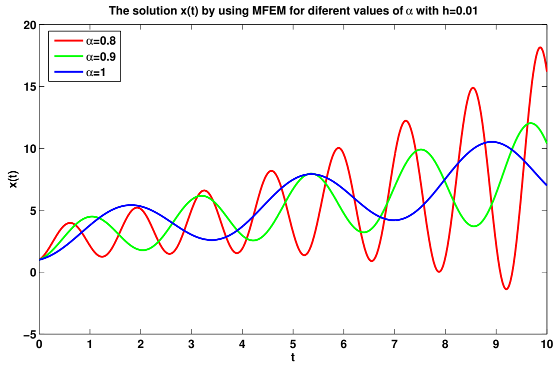

Fig. 5: The solution $x \quad (t)$ obtained with the use of MFEM when $\delta = 0.8, 0.9, 1$ and $h = 0.01$

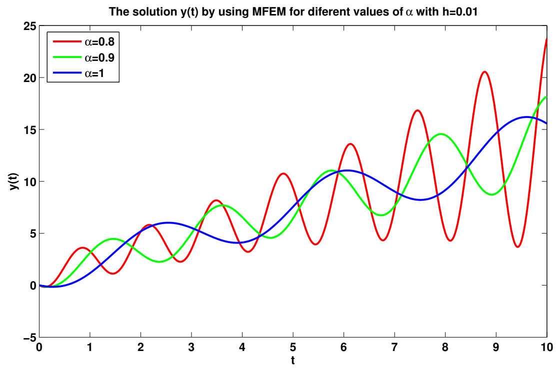

Fig. 6: The solution $y \quad (t)$ obtained with the use of MFEM when $\delta = 0.8, 0.9, 1$ and $h = 0.01$

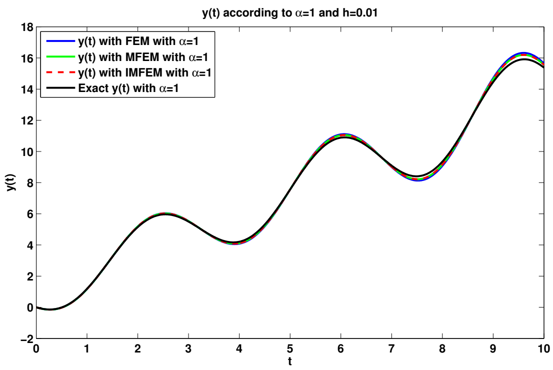

Fig. 4: The solution $y \quad (t)$ obtained with the use of FEM, MFEM and IMFEM when $\delta = 1$ and $h = 0.01$

On the basis of the above simulations, it might be observed that the MFEM's and IMFEM's solutions are closer to the exact solutions of system (17-17) than the other solution generated by the FEM. However, to see how numerical solutions of the considered system look like for different fractional-order values, we plot the MFEM's solution in Figure 5 and the IMFEM's solution in Figure 6 with $\delta = 0.8, 0.9, 1$ and $h = 0.01$.

In fact, the usefulness of the previous numerical simulations lies in illustrating the system's dynamic behavior and how it changes over time.

6 Conclusion

For the purpose of handling the fractional-order systems, this work has performed several graphical comparisons between certain recent adjustments of the FEM, called the MFEM and IMFEM. It has appeared that the IMFEM can generate high accuracy solutions to the fractional-order systems in comparison with the other studied approaches, succeeded by the MFEM and finally FEM.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Iqbal M. Batiha has implemented the Algorithm 1.1 and 1.2 in MATLAB. Amjed Zraiqat was responsible for the theoretical framework of this part. Shameseddin Alshorm was responsible for writing the original draft.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
No funding was received for conducting this study.

Conflicts of Interest
The authors have no conflicts of interest to declare that are relevant to the content of this article.