Confidence Intervals for the Mean and Difference of Means of Birnbaum-Saunders Distributions with Application to Wind Speed Data

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Abstract: - This paper proposes the confidence intervals for the mean and difference of means of Birnbaum-Saunders (BirSau) distributions based on the Bootstrap confidence interval (BCI), Percentile bootstrap confidence interval (PBCI), Generalized confidence interval (GCI), Bayesian credible interval (BayCrI) and the highest posterior density (HPD). The simulation study used R statistical software to evaluate the coverage probabilities and average lengths. The concerning results of the mean suggest that HPD is the recommended method for constructing confidence intervals in the BirSau distributions, except for small sample sizes where the GCI method proves more efficient. For the difference of means, PBCI emerges as the preferred way to construct confidence intervals, except in some cases where small sample sizes with the HPD method are more efficient. Moreover, the average lengths of these proposed confidence intervals decreased as both sample size and shape parameters increased. To illustrate the effectiveness of the suggested confidence intervals, we applied them to wind speed datasets collected in Ayutthaya and Ratchaburi provinces, Thailand.

Key-Words: - Confidence interval, Birnbaum-Saunders distribution, Mean, Bootstrap confidence interval, Generalized confidence interval, Bayesian credible interval, Wind speed.

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1 Introduction

In 1969, [1], initially concentrated on creating a model to describe the lifespan of material samples during fatigue and establishing the corresponding fatigue-life distribution. In this scenario, the proposed fatigue-life distribution introduced by [1], was formulated using a model that outlines the entire period until accumulated damage, arising from the formation and expansion of the primary crack, surpasses a defined threshold, leading to the material failure. In addition to being utilized in the fatigue of materials, the Birnbaum-Saunders (BirSau) distribution has also found application in other fields, such as [2], [3], examined the incidence of chronic cardiac diseases and diverse forms of cancer arising from the cumulative harm inflicted by various risk factors, ultimately resulting in degradation and giving rise to a fatigue process. The findings study of [4], constructing a framework to handle intangibles within the software execution process gives rise to cumulative damage that degrades its performance and ultimately culminates in failure. The study [5], [6], found that the disruption in the renewal process causes the death of small-diameter trees at chest height. According to the study, [7], [8], [9], employed the BirSau distribution to evaluate air quality, accounting for the buildup of pollutants in the air. Research by [10], also applied the BirSau distribution to explore wind energy flow patterns and climatic conditions. Due to the widespread application of the BirSau distribution, particularly in environmental contexts, this study is interested in exploring the BirSau distribution.

Wind energy is a clean and renewable energy source derived from nature, free from pollutants. Currently, Thailand is placing more importance and interest in developing renewable energy. Wind energy has been utilized to reduce the combustion of fossil fuels for electricity production, thereby mitigating the problem of global warming caused by the consequent carbon dioxide emission, [11]. Due to this reason, we are interested in utilizing wind speed data in this study. However, the inherent natural variability in wind speed introduces uncertainty. Therefore, we focus on estimating the mean wind speed and the difference in mean wind speeds using the BirSau distribution.

Several researchers have contributed to developing parameter estimation methods for BirSau distribution. In 1969, [12], introduced the maximum likelihood estimators (MLEs) for the shape (α) and scale (β) parameters. In a subsequent study [13], demonstrated the asymptotic joint distribution of these MLEs, establishing their asymptotic independence. Findings by [14], devised modified moment estimators (MMEs) and a straightforward bias correction technique to enhance the MLEs and MMEs. The work of [15], investigated the asymptotic confidence ellipses of parameters for the BirSau distribution. Research by [16], contributed to the field by presenting percentile bootstrap and generalized pivotal processes that create confidence intervals (CIs) for the α and β parameters of the BirSau distribution. The study of [17], formulated a bootstrap method for forecasting intervals related to the BirSau distribution in a different approach and based on the research by [18], compared CIs for a population mean obtained using the dependent bootstrap procedure to those generated using the independent bootstrap procedure. As shown, the study by [19], developed a high-order likelihood asymptotic-based for the parameters. Research by [20], focused on determining CIs for fundamental reliability measures through generalized interval estimation. Results from [21], extended the discourse by considering Bayesian inference for the parameters of the BirSau distribution. They based their methodology on inverse-gamma priors and computed Bayesian estimates. In recent research, the parameters of the BirSau distributions have been estimated using environmental data. As per the study by [22], [23], contributed by presenting CIs for variance, the difference of variances, and the coefficients of variation of PM 2.5 concentration data when the data have BirSau distributions. Lastly, [24], proposed a multivariate generalization of BirSau distribution based on the multivariate skew-normal distribution, presenting distributional properties and an EM algorithm for parameter estimation.

This paper focuses on the mean of a random variable or expected value in statistical inference. It represents the long-term average value of random variables obtained by integrating the product of the variable with its probability distribution. Since the mean is the most widely used statistical measure, our interest lies in constructing CIs to estimate the population mean and the difference of means between two populations. CIs for the mean and the difference between the two means have applications in various fields. For instance, in medicine [25], compared outpatient costs before and after a Medicaid policy change in Indiana, United States. In environmental science [26], analyzed monthly rainfall totals in Bloemfontein and Kimberley in South Africa. Several studies have delved into CIs for means, offering valuable insights into statistical analysis. The study by [27], suggested CIs for both the mean and coefficient of variation (CV) in a two-parameter exponential distribution. Furthermore, [28], introduced the concept of generalized inference and the method of variance estimates recovery to constructing the CIs, applicable to the common mean of several gamma distributions. Research by [29], proposed the robust CI estimation for the mean of Poisson distribution. The investigation into parameter estimation for the BirSau distribution and interval estimation for the parameter mean showed that prior studies have not delved into creating CIs for both the mean and the difference between the means of BirSau distributions. Consequently, we propose the introduction of CIs for both the mean and the difference between the two means of BirSau distributions. Therefore, this study aims to compare the efficiency of methods for estimating the CIs of the mean and the difference between the means when the population follows a BirSau distribution. The methods utilized include the bootstrap confidence interval (BCI), percentile bootstrap confidence interval (PBCI), generalized confidence interval (GCI), Bayesian credible interval (BayCrI), and the highest posterior density interval (HPD). To demonstrate the effectiveness of these proposed methodologies, we have also applied them to wind speed data from Ayutthaya and Ratchaburi provinces in Thailand, collected between February and April 2022.

2 The CI for the Mean of a BirSau Distribution

A random variable X is said to follow the two-parameter BirSau distribution with parameters α and β, where $x>0$, $\alpha>0$, and $\beta>0$. This distribution is represented as $X \sim BirSau(\alpha,\beta)$. The probability density function is given by:

$$f(x,\alpha,\beta)=\frac{1}{2\alpha\beta\sqrt{2\pi}}\left\{\left(\frac{\beta}{x}\right)^{\frac{1}{2}}+\left(\frac{\beta}{x}\right)^{\frac{3}{2}}\right\}exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right]. \quad (1)$$

The expected value and variance of X are expressed as $E(X)=\beta(1+\frac{1}{2}\alpha^2)$ and $Var(X)=(\alpha\beta)^2\left(1+\frac{5}{4}\alpha^2\right)$, respectively. Therefore, in this study, the focus is on the parameter mean, denoted as ω, and is defined as:

$$\omega = \beta\left(1+\frac{1}{2}\alpha^2\right). \quad (2)$$

2.1 BCI

The bootstrap method, introduced by [30], utilizes resampling techniques to mitigate bias in MLE. In the context of the BirSau distribution, this method facilitates the estimation of parameters α and β through a procedure developed by [31]. This procedure explicitly addresses correcting biases in the MLE of distribution parameters by leveraging bootstrap techniques.

Let $\mathbf{x}=(x_1,x_2,...,x_n)^T$ represent a random sample of size n from $BirSau(\alpha,\beta)$. The MLEs for α and β are $\hat{\beta}=r(\mathbf{x})$ and $\hat{\alpha}=s(\mathbf{x})$, respectively. Next, let B represent a bootstrap sample that is created independently from the initial sample x, with $(\mathbf{x}^{*1},\mathbf{x}^{*2},...,\mathbf{x}^{*B})$. The respective bootstrap replications of $\hat{\alpha}$ and $\hat{\beta}$ are indicated as $(\hat{\alpha}^{*1},\hat{\alpha}^{*2},...,\hat{\alpha}^{*B})$ and $(\hat{\beta}^{*1},\hat{\beta}^{*2},..., \hat{\beta}^{*B})$, where $\hat{\alpha}^{*b}=s(\mathbf{x}^{*y})$ and $\hat{\beta}^{*y}=r(\mathbf{x}^{*y})$, for $y=1,2,...,B$. The approximate bootstrap estimator are calculated by the mean $\hat{\alpha}^{*(\cdot)}=1/B\sum_{y=1}^{B}\hat{\alpha}_y^*$ and $\hat{\beta}^{*(\cdot)}=1/B\sum_{y=1}^{B}\hat{\beta}_y^*$.

The estimates of bootstrap bias based on B replications of $\hat{\alpha}$ and $\hat{\beta}$ are

$$\hat{B}(\hat{\alpha},\alpha)=\hat{\alpha}^{*(\cdot)}-s(\mathbf{x}) \text{ and } \hat{B}(\hat{\beta},\beta)=\hat{\beta}^{*(\cdot)}-r(\mathbf{x}). \quad (3)$$

The correct estimates for $\hat{\alpha}^*$ and $\hat{\beta}^*$ using the idea of constant-bias-correction (CBC) estimates proposed by [32], can be obtained as follows:

$$\tilde{\alpha}_y=\hat{\alpha}_y^*-2\hat{B}(\hat{\alpha},\alpha) \text{ and } \tilde{\beta}_y=\hat{\beta}_y^*-2\hat{B}(\hat{\beta},\beta). \quad (4)$$

The percentile bootstrap estimates for $\hat{\alpha}^*$ and $\hat{\beta}^*$ are:

$$\tilde{\alpha}_y^*=2\hat{\alpha}-\tilde{\alpha}_y \text{ and } \tilde{\beta}_y^*=2\hat{\beta}-\tilde{\beta}_y. \quad (5)$$

Consequently, it can construct the bootstrap estimator of the mean as:

$$\hat{\omega}_y=\tilde{\beta}_y(1+\frac{\tilde{\alpha}_y^2}{2}). \quad (6)$$

and the percentile bootstrap estimator of the mean can be found as:

$$\overset{*}{\omega}_y=\overset{*}{\beta}_y(1+\frac{\alpha_y^{*2}}{2}). \quad (7)$$

Hence, the approximated $100(1-\xi)\%$ CI for ω based on BCI and PBCI, it becomes:

$$CI_{BCI}=[\hat{\omega}(\xi/2),\hat{\omega}(1-\xi/2)]. \quad (8)$$

$$CI_{PBCI}=[\hat{\omega}^*(\xi/2),\hat{\omega}^*(1-\xi/2)]. \quad (9)$$

where $\hat{\omega}(\xi/2)$ and $\hat{\omega}^*(\xi/2)$ denote the $100(\xi/2)$-th percentile of bootstrap and percentile bootstrap distribution of $\hat{\omega}$ and $\hat{\omega}^*$, respectively.

2.2 GCI

In 1993, [33], introduced a method to construct the GCI by applying the Generalized Pivotal Quantity (GPQ) concept. Following, [22], the GPQ for β was established by [34], as follows:

$$J_\beta := J_\beta(T;\boldsymbol{X})=\begin{cases}\max(\beta_1,\beta_2), & J\le 0\\ \min(\beta_1,\beta_2), & J>0.\end{cases} \quad (10)$$

where β_1 and β_2 are the two solutions of the following equation:

$$U\beta^2-2V\beta+W=0, \quad (11)$$

where $U=[(n-1)H^2-(1/n)IJ^2]$, $V=[(n-1)KH-(1-KH)J^2]$, $W=(n-1)K^2-(1/n)LJ^2$, $K=n^{-1}\sum_{i=1}^{n}\sqrt{X_i}$, $H=n^{-1}\sum_{i=1}^{n}1/\sqrt{X_i}$, $L=\sum_{i=1}^{n}(\sqrt{X_i}-K)^2$, $I=\sum_{i=1}^{n}(1/\sqrt{X_i}-H)^2$ and $J\sim t(n-1)$.

The GPQ for α was subsequently developed by [20], then the GPQ for α becomes

$$J_\alpha := J_\alpha(\mathbf{x};t,J)=\left[\frac{s_2J_\beta^2-2nJ_\beta+s_1}{J_\beta t}\right]^{1/2}, \quad (12)$$

where $s_1=\sum_{i=1}^{n}x_i$ and $s_2=\sum_{i=1}^{n}\frac{1}{x_i}$ and $t\sim\chi_n^2$. By substituting J_α and J_β into Equation (5), the GPQ of the mean as:

$$J_\omega=J_\beta(1+\frac{J_\alpha^2}{2}). \quad (13)$$

Hence, the approximated $100(1-\xi)\%$ CI for ω based on GCI, it becomes

$$CI_{GCI}=[J_\omega(\xi/2),J_\omega(1-\xi/2)], \quad (14)$$

where $J_\omega(\xi/2)$ denotes the $100(\xi/2)$-th percentile of J_ω.

2.3 BayCrI

The study conducted by [21], use specific priors with known values to ensure the accuracy of the resulting posteriors. They assume an inverse gamma

(IG) distribution for β, marked as $IG(\beta | d_1, e_1)$, and do the same for α^2 as $IG(\alpha^2 | d_2, e_2)$.

The posterior distribution of β given the data and the posterior distribution of α given β and the data are outlined as follows:

$$\pi(\beta | \mathbf{x}) \propto \beta^{-(n+d_1+1)} \exp\left(\frac{-e_1}{\beta}\right) \prod_{i=1}^{n} \left[\left(\frac{\beta}{x_i}\right)^{1/2} + \left(\frac{\beta}{x_i}\right)^{3/2} \right] \times \left[\sum_{i=1}^{n} \frac{1}{2} \left(\frac{x_i}{\beta} + \frac{\beta}{x_i} - 2 \right) + e_2 \right]^{-(n+1)/2-d_2}. \tag{15}$$

$$\pi(\alpha^2 | \mathbf{x}, \beta) \propto IG\left(\frac{n}{2} + d_2, \frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i}{\beta} + \frac{\beta}{x_i} - 2 \right) + e_2 \right). \tag{16}$$

The sample from Equations (15) and (16) are obtained using Markov Chain Monte Carlo techniques. Research by [21], produced posterior β samples using the extended ratio-of-uniforms technique will be discussed in more detail in the next section. Alternatively, obtaining the posterior samples of α^2 is straightforward using the *LearnBayes* package in the R software suite. Consequently, α equals the square root of α^2.

2.3.1 The Generalized Ratio-of-uniforms Method

According to the study by [35], an effective sampling approach for posterior simulation from Equation (15) using the generalized ratio-of-uniforms method was created. A summary of the algorithm is presented as follows: Supposed a pair of random variables (r,s) follows a uniform distribution over the specified region.

$$K(t) = \left\{ (r,s) : 0 < r \le \left[\pi\left(\frac{s}{r^t} | \mathbf{x} \right) \right]^{1/(t+1)} \right\},\ t \ge 0, \tag{17}$$

where t is constant and $\pi(\cdot | x)$ is given by using Equation (15). Therefore, the density of $\beta = s / r^t$ is $\pi(\beta | \mathbf{x}) / \int \pi(\beta | \mathbf{x}) \mathrm{d}\beta$.

To generate random samples uniformly distributed within the region $K(t)$, random variables (r,s) are generated with a uniform distribution across the one-dimensional rectangle $[0, a(t)] \times [b^-(t), b^+(t)]$, where:

$$a(t) = \sup_{\beta > 0} \{ [\pi(\beta | \mathbf{x})]^{1/(t+1)} \} \tag{18}$$

$$b^-(t) = \inf_{\beta > 0} \{ \beta [\pi(\beta | \mathbf{x})]^{t/(t+1)} \} \tag{19}$$

and

$$b^+(t) = \sup_{\beta > 0} \{ \beta [\pi(\beta | \mathbf{x})]^{t/(t+1)} \} \tag{20}$$

According to research by [21], both $a(t)$ and $b(t^+)$ assume finite values with $b(t^-)$ equating to zero. Consequently, the potential variate $\beta = s / r^t$ is deemed acceptable if $r \le [\pi(\beta | \mathbf{x})]^{1/(t+1)}$; should this not be the case, the process is reiterated. By executing these steps, the BayCrI for the mean of the BirSau distribution can be derived

(1) Indicate the values of d_1, e_1, d_2, e_2 and t then calculate $a(t)$ and $b^+(t)$.

(2) i th iteration:

a. Generate r and s from $Unif(0, a(t))$ and $Unif(0, b^+(t))$, respectively, then compute $\rho = s / r^t$.

b. Set $\beta_{(i)} = \rho$ if $r \le [\pi(\rho | \mathbf{x})]^{1/(t+1)}$ if the value of ρ is acceptable; if not, repeat the previous step.

c. Create α_i^2 using $IG(\frac{n}{2} + d_2, \frac{1}{2} \sum_{j=1}^{n} (\frac{x_i}{\beta_{(i)}} + \frac{\beta_{(i)}}{x_j} - 2) + e_2)$ then set $\alpha_{(i)} = \sqrt{\alpha_i^2}$.

(3) Compute the Bayesian estimator of the mean by

$$\omega_{(i)}^* = \beta_{(i)} (1 + \frac{\alpha_{(i)}^2}{2}). \tag{21}$$

(4) Go through steps (2) and (3) M times.

(5) Compute the approximated $100(1-\xi)\%$ BayCrI for the ω as

$$CI_{BayCrI} = [\omega^*(\xi/2), \omega^*(1-\xi/2)], \tag{22}$$

where $\omega^*(\xi/2)$ denotes the $100(\xi/2)$-th percentile of ω^*.

To create the HPD interval for the mean, we utilized the hdi function provided by the *HDInterval* package in the R software suite. This step was performed after obtaining the Bayesian mean estimate in step 4.

3 The CIs for the Difference between the Means of BirSau Distributions

The concepts of GCI, BCI, PBCI, BayCrI, and HPD presented in the previous section are expanded upon in this section, focusing on new CI to determine the difference between the two means. In a statistical model, the difference between the means results from subtracting or comparing two means. Usually, this is done to compare two quantitative datasets when it is necessary to fit the BirSau distribution closely. Let $X = (X_1, X_2, ..., X_n)$ and $Z = (Z_1, Z_2, ..., Z_m)$

be independent random samples from the BirSau distribution, with sample sizes of n and m, respectively (referred to as $BirSau(\alpha,\beta)$, $BirSau(\alpha_z,\beta_z)$). Consequently, the mean of Z becomes

$$\omega_z = \beta_z\left(1+\frac{1}{2}\alpha_z^2\right). \tag{23}$$

Given the independence of X and Z, the difference between the means (represented by δ) can be expressed as

$$\delta = \omega - \omega_z = \beta\left(1+\frac{1}{2}\alpha^2\right) - \beta_z\left(1+\frac{1}{2}\alpha_z^2\right). \tag{24}$$

3.1 BCI

Let $\mathbf{z} = (z_1, z_2, ..., z_m)^T$ be a random sample of size m from generated by $BirSau(\alpha_z,\beta_z)$. The MLEs of α_z and β_z, denoted as $\hat{\alpha}_z = s(\mathbf{z})$ and $\hat{\beta}_z = r(\mathbf{z})$, respectively. Given that $\left(\mathbf{z}^{*1},\mathbf{z}^{*2},...,\mathbf{z}^{*B}\right)$ represents bootstrap samples generated independently from the original sample z. The corresponding bootstrap replications for α are represented as $\hat{\alpha}_z^{*1},\hat{\alpha}_z^{*2},\ldots,\hat{\alpha}_z^{*B}$, and for β, they are denoted as $\hat{\beta}_z^{*1},\hat{\beta}_z^{*2},\ldots,\hat{\beta}_z^{*B}$. where $\hat{\alpha}_z^{*y} = s(\mathbf{z}^{*y})$ and $\hat{\beta}_z^{*y} = r(\mathbf{z}^{*y})$, $y = 1,2,...,B$. The approximate bootstrap estimators are calculated by $\hat{\alpha}_z^{*(.)} = (1/B)\sum_{y=1}^{B}\hat{\alpha}_z^{*y}$ and $\hat{\beta}_z^{*(.)} = (1/B)\sum_{y=1}^{B}\hat{\beta}_z^{*y}$.

The bootstrap bias estimate based on B replications of $\hat{\alpha}_z$ and $\hat{\beta}_z$ are obtained as:

$$\hat{B}(\hat{\alpha}_z,\alpha_z) = \hat{\alpha}_z^{*(.)} - r(\mathbf{z}) \text{ and } \hat{B}\left(\hat{\beta}_z,\beta_z\right) = \hat{\beta}_z^{*(.)} - s(\mathbf{z}). \tag{25}$$

Therefore, the corrected estimate for $\hat{\alpha}_z^*$ and $\hat{\beta}_z^*$ can be written as:

$$\tilde{\alpha}_{z,y} = \hat{\alpha}_{z,y}^* - 2\hat{B}(\hat{\alpha}_{z,y},\alpha_{z,y}) \text{ and } \tilde{\beta}_{z,y} = \hat{\beta}_{z,y}^* - 2\hat{B}(\hat{\beta}_{z,y},\hat{\beta}_{z,y}). \tag{26}$$

The percentile bootstrap estimators for $\hat{\alpha}_z^*$ and $\hat{\beta}_z^*$ are:

$$\tilde{\alpha}_{z,y}^* = 2\hat{\alpha} - \tilde{\alpha}_{z,y} \text{ and } \tilde{\beta}_{z,y}^* = 2\hat{\beta} - \tilde{\beta}_{z,y}. \tag{27}$$

Therefore, the bootstrap estimator of the difference of means can be obtained as:

$$\hat{\delta}_y = \tilde{\beta}_y\left(1+\frac{1}{2}\tilde{\alpha}_y^2\right) - \tilde{\beta}_{z,y}\left(1+\frac{1}{2}\tilde{\alpha}_{z,y}^2\right). \tag{28}$$

and the percentile bootstrap estimator of the difference of means can be obtained as:

$$\hat{\delta}_y^* = \tilde{\beta}_y^*\left(1+\frac{1}{2}\tilde{\alpha}_y^{*2}\right) - \tilde{\beta}_{z,y}^*\left(1+\frac{1}{2}\tilde{\alpha}_{z,y}^{*2}\right). \tag{29}$$

Thus, the approximated $100(1-\xi)\%$ CI for δ based on BCI and PBCI, it becomes:

$$CI_{BCI}^d = [\hat{\delta}(\xi/2), \hat{\delta}(1-\xi/2)]. \tag{30}$$

$$CI_{PBCI}^d = [\hat{\delta}^*(\xi/2), \hat{\delta}^*(1-\xi/2)]. \tag{31}$$

where $\hat{\delta}(\xi/2)$ and $\hat{\delta}^*(\xi/2)$ denote the $100(\xi/2)$-th percentile of bootstrap and percentile bootstrap distribution of $\hat{\delta}$ and $\hat{\delta}^*$, respectively.

3.2 GCI

The GPQ of β_z can be defined by utilizing the random variable Z as follows:

$$J_{\beta_z} := J_{\beta_z}(J_z;\mathbf{Z}) = \begin{cases} max(\beta_{z,1},\beta_{z,2}), & if\ J_z \le 0 \\ min(\beta_{z,1},\beta_{z,2}), & if\ J_z > 0 \end{cases}, \tag{32}$$

where $J_z \sim t(m-1)$, and $\beta_{z,1}$ and $\beta_{z,2}$ are the two solutions for:

$$U_z\beta_z{}^2 - 2V_z\beta_z + W_z = 0, \tag{33}$$

where $U_z = [(m-1)H_z{}^2 - (1/m)I_zJ_z^2]$, $V_z = [(m-1)K_zH_z - (1-K_zH_z)J_z^2]$, $W_z = (m-1)K_z^2 - (1/m)L_zJ_z^2$, $K_z = m^{-1}\sum_{i=1}^{m}\sqrt{Z_i}$, $H_z = m^{-1}\sum_{i=1}^{m}1/\sqrt{Z_i}$, $L_z = \sum_{i=1}^{m}(\sqrt{Z_i}-K_z)^2$ and $I_z = \sum_{i=1}^{m}(1/\sqrt{Z_i}-H_z)^2$.

For α_z, the GPQ is provided by

$$J_{\alpha_y} := J_{\alpha_z}(\mathbf{Z};t_z,J_z) = \left[\frac{s_{z,2}J_{\beta_z}{}^2 - 2mJ_{\beta_z} + s_{z,1}}{J_{\beta_z}t_z}\right]^{1/2}, \tag{34}$$

where $s_{z,1} = \sum_{i=1}^{m}z_i$, $s_{z,2} = \sum_{i=1}^{m}\frac{1}{z_i}$ and $t_z \sim \chi^2(m)$.

Therefore, the GPQ for the difference of means can be obtained as:

$$J_\delta = J_\beta(1+\frac{1}{2}J_\alpha^2) - J_{\beta_z}(1+\frac{1}{2}J_{\alpha_z}^2). \tag{35}$$

Therefore, the approximated $100(1-\xi)\%$ CI for δ based on GCI, it becomes:

$$CI_{GCI}^d = \left[J_\delta(\xi/2), J_\delta(1-\xi/2)\right], \tag{36}$$

where $J_\delta(\xi/2)$ denotes the $100(\xi/2)$-th percentile of J_δ.

3.3 BayCrI

In the Bayesian method, for $Z \sim BirSau(\alpha_z, \beta_z)$ the IG priors of β_z and α_z^2, are denoted as $IG(\beta_z \mid f_1, g_1)$ and $IG(\alpha_z^2 \mid f_2, g_2)$, respectively. Thus, the marginal distribution of β_z becomes:

$$\pi(\beta_z \mid \mathbf{z}) \propto \beta_z^{-(m+f_1+1)} exp\left(-\frac{g_1}{\beta_y}\right) \prod_{i=1}^{m}\left[\left(\frac{\beta_z}{z_i}\right)^{\frac{1}{2}} + \left(\frac{\beta_z}{z_i}\right)^{\frac{3}{2}}\right]$$
$$\times\left[\sum_{i=1}^{m}\frac{1}{2}\left(\frac{z_i}{\beta_z}+\frac{\beta_z}{z_i}-2\right)+f_2\right]^{-\frac{(m+1)}{2}-g_2}. \quad (37)$$

The posterior conditional distribution of α_z^2 given β_z becomes:

$$\pi(\alpha_z^2 \mid z, \beta_z) \propto IG\left(\frac{m}{2}+f_2, \frac{1}{2}\sum_{i=1}^{m}\left(\frac{z_i}{\beta_z}+\frac{\beta_z}{z_i}-2\right)+g_2\right). \quad (38)$$

The procedure for utilizing BayCrI to estimate the difference between the means can be condensed into the following steps: Initially, determine the values of f_1, g_1, f_2, g_2 and t_z, with the condition that, as a constant. Subsequently, calculate $a(t_z)$ and $b^+(t_z)$, where $a(t_z)$ and $b^+(t_z)$ are defined as follows:

$$a(t_z) = \sup_{\beta_z>0}\left\{\left[\pi(\beta_z \mid z)\right]^{1/(t_z+1)}\right\}, \quad (39)$$

$$b^+(t_z) = \sup_{\beta_z>0}\left\{\beta_z\left[\pi(\beta_z \mid z)\right]^{t_z/(t_z+1)}\right\}. \quad (40)$$

Second, generate r_z and s_z from $r_z \sim Unif(0, a(t_z))$ and $s_z \sim Unif(0, b^+(t_z))$, then compute $\rho_z = s_z / r_z^{t_z}$. If $r_z \le \left[\pi(\rho_z \mid z)\right]^{1/(t_z+1)}$, set $\beta_{z,i} = \rho_z$; otherwise, generate r_z and s_z again. Next, generate $\alpha_{z,i}^2 \sim IG\left(\frac{m}{2}+f_2, \frac{1}{2}\sum_{j=1}^{m}\left(\frac{z_j}{\beta_{(z)}}+\frac{\beta_{(z)}}{z_j}-2\right)+g_2\right)$ and we can find the $\alpha_{z,i} = \sqrt{\alpha_{z,i}^2}$. Hence, the Bayesian estimator of the difference between the means is denoted as δ^*, is given by

$$\delta_i^* = \beta_i\left(1+\frac{1}{2}\alpha_i^2\right) - \beta_{z,i}\left(1+\frac{1}{2}\alpha_{z,i}^2\right), \; i = 1,2,\ldots,M, \quad (41)$$

where M is the number of iterations, and the last calculates the $100(1-\xi)\%$ CI for δ by applying:

$$CI_{BayCrI}^{d} = \left[\delta^*(\xi/2), \delta^*(1-\xi/2)\right], \quad (42)$$

when $\delta^*(\xi/2)$ denotes the $100(\xi/2)$-th percentile of δ^*. The confidence of δ was determined using the R package *HDInterval* for the HPD interval calculation.

4 Simulation Studies

Five approaches were examined in a Monte Carlo simulation using the R statistical software: GCI, BCI, PBCI, BayCrI, and HPD. The purpose of the simulation was to create new CIs for the mean and the difference between the means of two BirSau distributions. The coverage probabilities (CPs) and average lengths (ALs) of the five suggested approaches were compared to evaluate them. Two crucial factors were considered when selecting a preferred method: the CPs should be at least or close to the nominal confidence level of 0.95, and the shortest AL. The simulation settings consist of the number of replications of 5,000, with 5,000 pivotal quantities for GCI, B = 500 for BCI and PBCI, and M = 1,000 for BayCrI and HPD interval. For a single mean of BirSau, the sample size was set n = 10, 20, 30, 50 or 100 with shape parameters α = 0.10, 0.25, 0.50, 0.75 or 1.00. The sample sizes for the difference between the means of the two BirSau distributions, however, were set as (n,m) = (10,10), (20,20), (30,30), (50,50), (100,100), (10,20), (20,30), (30,50) or (50,100) with shape parameter (α, α_2) = (0.25,0.25), (0.25,0.50), (0.25,0.75), (0.25,1.00), (0.50,0.50), (0.50,0.75), (0.50,1.00), (0.75,0.75), (0.75,1.00) or (1.00,1.00). The values for the scale parameters β and β_2 were fixed at 1 for all cases. In the case of BayCrI and HPD, we examined the parameter $t = t_z$ = 2 and the suggested hyperparameter $d_1, d_2, e_1, e_2, f_1, f_2, g_1, g_2 = 10^{-4}$ as proposed by [21].

For the single mean of a BirSau distribution, according to the simulation results shown in Table 1 (Appendix), when dealing with small sample sizes (n =10), the GCI method performs the best in CP and AL. However, for another medium (n =30,50) and large sample sizes (n =100), we observed that GCI, BayCrI, and HPD gave CPs greater than or close to the nominal confidence level of 0.95. Among these, HPD provided the shortest AL. In contrast, although PBCI had the shortest ALs, its CPs were the lowest and under the nominal confidence level of 0.95, but it improved as n increased. When considering the ALs of the other methods, they exhibited similar trends. Additionally, the ALs of all five methods tended to decrease as the sample size increased, as shown in Figure 1.

When considering differences between the means of BirSau distributions, we generated data from two independent BirSau distributions. The CPs and ALs of the 95% CI for the difference between the means, with equal and unequal sample sizes, are listed in Table 2 and Table 3 in Appendix, respectively. For equal sample sizes, we consistently observed that the CPs of the GCI, BayCrI, and HPD methods were greater than or close to the nominal confidence level of 0.95. Notably, the HPD method produced the shortest AL, except for one instance with a large sample size where PBCI outperformed in terms of both CP and AL. In the case of unequal sample sizes, the results indicated that the GCI, BayCrI, HPD, and PBCI methods all provided CPs greater than or very close to the nominal confidence level of 0.95. Regarding AL, HPD resulted in the shortest AL for small sample sizes, while PBCI was associated with the shortest AL for other cases. Moreover, the performances of all five methods in terms of AL improved as sample sizes (n,m) increased, as shown in Figure 2.

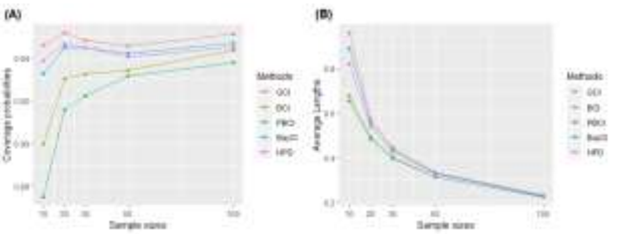

Fig. 1: Comparison of the CPs and ALs for estimating the 95% CI for the mean of BirSau distribution at $\alpha = 0.5$

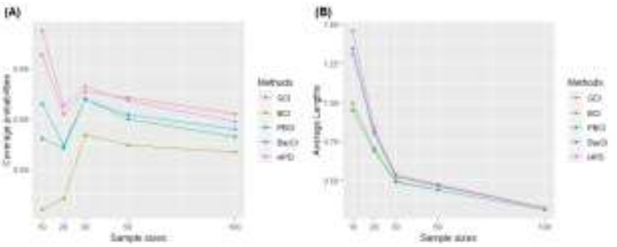

Fig. 2: Comparison of the CPs and ALs for estimating the 95% CI for the difference between the means of BirSau distributions with equal sample sizes at $(\alpha,\alpha_2) = (0.5, 0.5)$

5 An Empirical Application

Wind energy is an eco-friendly and sustainable power source, untainted by carbon emissions or pollution, [11]. According to [36], the BirSau distribution is the optimal method for estimating wind speed distribution. We employed datasets containing daily wind speed records from Ayutthaya and Ratchaburi provinces, Thailand, [37], to demonstrate the efficiency of CIs for the mean and difference of the means of BirSau distributions obtained through methods such as GCI, BCI, PBCI, BayCrI, and HPD. These data sets were gathered from February to April 2022 as detailed in Table 4 (Appendix).

As the data comprises positive values, it is feasible to fit it into various distributions such as Cauchy, logistic, exponential, Weibull, normal, or BirSau distributions. Therefore, we tested the distributions of positive wind speed datasets using the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). The data presented in Table 5 (Appendix) indicates that the wind speed datasets from Ayutthaya and Ratchaburi province conform to a BirSau distribution, as supported by the smallest values of AIC and BIC.

Table 6 (Appendix) presents the basic statistics computed for the daily wind speed data. The mean and difference of the means are accompanied by two-sided CIs derived from the GCI, BCI, PBCI, BayCrI, and HPD, as outlined in Table 7 (Appendix), respectively. In terms of the mean, as per the simulation results, PBCI consistently yielded the shortest ALs. It is worth mentioning, however, that, similar to the simulation results, the CP of PBCI was lower and fell below 0.95, while both GCI and HPD exhibited CPs greater than or close to the nominal confidence level of 0.95. Overall, HPD stands out as the most suitable method for constructing a CI for the mean of the BirSau distribution.

Regarding the two-sided CIs for the difference of means, the results align with the simulation results. For large sample sizes, GCI, BayCrI, HPD, and PBCI consistently achieved CPs greater than or close to the nominal confidence level of 0.95, with PBCI offering the shortest AL. Therefore, we recommend using the PBCI method for constructing CIs for the difference of means in wind speed data collected between February and April 2022 in Ayutthaya and Ratchaburi, Thailand.

6 Discussion

Wind energy is a reusable, environmentally friendly resource. Transforming the speed of wind into kinetic energy can generate electricity. Nevertheless, wind speed can vary beyond typical fluctuations, leading to unpredictability. The research work conducted by [38], highly recommended how these factors are essential in the business strategy and how to manage this sustainable resource. For this reason, estimating CI values for the mean and differences between average wind speeds in different areas is essential. Suppose we know the average wind speed estimate and whether there is a consistent wind speed in each area. It will help in deciding on the area to have a wind farm. This is because the wind speed used to produce electricity is consistent and is required to produce electricity efficiently.

The study by [22], utilized the Highest Posterior Density (HPD) method for both the variance and the difference in variances of the BirSau distribution to enhance the CIs. In some cases, the mean might be considered a better method than the variance. In this study, we were focusing on estimating the CIs of the mean and mean difference for the BirSau distributions. The results suggest that for the mean, HPD was the most consistent with the best CP and AL. It provides the same findings as the CIs for the variance, and the difference between two variances, [22] and CV [23], of BirSau distributions. For the difference between the means, the results suggest that the PBCI is the best approach for constructing CIs, which is consistent with the estimated simultaneous CIs for pairwise comparisons of the means of delta-lognormal distributions, [39].

7 Conclusions

In this study, the CIs for the parameters mean and difference between two means of the BirSau distributions are proposed. The results suggest that the HPD method effectively estimates the CI of the mean of the BirSau distribution. The GCI approach is more effective than other methods with small sample sizes. Regarding the difference of means in BirSau distributions, the data shows that the PBCI is a preferred method for constructing CIs. Once again, the exception is noticeable for smaller data sets. Having CPs above or close to 0.95 with the shortest ALs proves the HPD method is more effective. The methods illustrated using real wind speed datasets. It was found that the results corresponded with the simulation results. This research investigates how to estimate CIs, for the parameters in the BirSau distribution, which works well for positive data. However, if the dataset includes both zeros and positive values it might be better to look into the delta BirSau distribution. In future work, we would like to extend our work to estimating the CIs for parameters in either the BirSau distribution covering zeros well or the delta-BirSau distribution.

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Conflict of Interest

The authors have no conflicts of interest to declare.

APPENDIX

Table 1. The CPs and ALs of the 95% CIs for the mean of a BirSau distribution

n	α	CP (AL)				
		GCI	**BCI**	**PBCI**	**BayCrI**	**HPD**
10	0.1	**0.9498**	0.9010	0.8988	0.9362	0.9318
		(**0.1409**)	(0.1139)	(0.1136)	(0.1314)	(0.1297)
	0.25	**0.9478**	0.8994	0.8960	0.9360	0.9350
		(**0.3841**)	(0.3000)	(0.2960)	(0.3582)	(0.3476)
	0.5	**0.9460**	0.8998	0.8748	0.9326	0.9386
		(**0.9601**)	(0.6782)	(0.6534)	(0.8885)	(0.8203)
	0.75	**0.9482**	0.8964	0.8644	0.9364	0.9366
		(**1.9315**)	(1.2010)	(1.1361)	(1.7485)	(1.5230)
	1	**0.9486**	0.8988	0.8536	0.9366	0.9444
		(**3.5720**)	(1.8871)	(1.7530)	(3.0451)	(2.5058)
20	0.1	0.9440	0.9196	0.9226	0.9348	0.9340
		(0.0933)	(0.0842)	(0.0840)	(0.0903)	(0.0894)
	0.25	**0.9484**	0.9222	0.9210	0.9404	0.9372
		(**0.2452**)	(0.2185)	(0.2170)	(0.2373)	(0.2335)
	0.5	**0.9518**	0.9302	0.9158	**0.9448**	**0.9466**
		(0.5741)	(0.4917)	(0.4823)	(0.5553)	(**0.5348**)
	0.75	**0.9470**	0.9260	0.9058	0.9434	0.9414
		(**1.0477**)	(0.8503)	(0.8296)	(1.0031)	(0.9402)
	1	0.9442	0.9272	0.8998	0.9406	**0.9456**
		(1.7447)	(1.3235)	(1.2947)	(1.6330)	(**1.4923**)
30	0.1	**0.9510**	0.9312	0.9336	**0.9452**	0.9426
		(0.0748)	(0.0698)	(0.0698)	(**0.0733**)	(0.0726)
	0.25	**0.9482**	0.9276	0.9262	0.9416	0.9362
		(**0.1957**)	(0.1809)	(0.1801)	(0.1915)	(0.1889)
	0.5	**0.9482**	0.9324	0.9224	**0.9448**	**0.9448**
		(0.4454)	(0.4016)	(0.3972)	(0.4351)	(**0.4237**)
	0.75	0.9438	0.9258	0.9150	0.9394	0.9398
		(0.7929)	(0.6901)	(0.6796)	(0.7687)	(0.7360)
	1	**0.9476**	0.9354	0.9166	**0.9456**	**0.9480**
		(1.2953)	(1.0804)	(1.0649)	(1.2363)	(**1.1632**)
50	0.1	**0.9498**	0.9370	0.9358	0.9436	0.9404
		(**0.0569**)	(0.0545)	(0.0545)	(0.0562)	(0.0557)
	0.25	**0.9544**	0.9432	0.9414	**0.9500**	**0.9500**
		(0.1477)	(0.1407)	(0.1403)	(0.1455)	(**0.1439**)
	0.5	**0.9456**	0.9342	0.9314	0.9420	0.9404
		(**0.3331**)	(0.3125)	(0.3104)	(0.3281)	(0.3220)
	0.75	**0.9506**	0.9344	0.9284	**0.9474**	**0.9470**
		(0.5884)	(0.5395)	(0.5347)	(0.5759)	(**0.5595**)
	1	0.9426	0.9328	0.9216	0.9418	0.9424
		(0.9373)	(0.8327)	(0.8258)	(0.9039)	(0.8688)
100	0.1	**0.9474**	0.9414	0.9398	0.9444	0.9414
		(**0.0398**)	(0.0388)	(0.0388)	(0.0394)	(0.0391)
	0.25	**0.9506**	**0.9464**	0.9442	**0.9500**	**0.9462**
		(0.1031)	(**0.1002**)	(0.1001)	(0.1022)	(0.1013)
	0.5	**0.9510**	0.9434	0.9378	**0.9470**	**0.9450**
		(0.2299)	(0.2216)	(0.2208)	(0.2273)	(**0.2244**)
	0.75	**0.9504**	**0.9446**	0.9394	**0.9464**	**0.9474**
		(0.3998)	(**0.3801**)	(0.3782)	(0.3934)	(0.3862)
	1	**0.9454**	0.9328	0.9300	0.9430	0.9404
		(**0.6262**)	(0.5841)	(0.5812)	(0.6085)	(0.5943)

Notes: Bold represents values that satisfy criteria and the best-performing method

Table 2. The CPs and ALs of the 95% CIs for the difference between the means of BirSau distributions with equal sample sizes $(n = n_2)$.

(n, n_2)	(α, α_2)	CP (AL)				
		GCI	**BCI**	**PBCI**	**BayCrI**	**HPD**
(10,10)	(0.25,0.25)	**0.9590**	0.9170	0.9244	**0.9464**	**0.9522**
		(0.5548)	(0.4304)	(0.4240)	(0.5158)	(**0.5086**)
	(0.25,0.50)	**0.9556**	0.9148	0.9068	**0.9460**	**0.9528**
		(1.0491)	(0.7468)	(0.7216)	(0.9716)	(**0.9217**)
	(0.25,0.75)	**0.9500**	0.9006	0.8796	0.9356	**0.9448**
		(2.0096)	(1.2534)	(1.1876)	(1.8220)	(**1.6156**)
	(0.25,1.00)	**0.9490**	0.8966	0.8640	0.9396	**0.9482**
		(3.6056)	(1.9102)	(1.7739)	(3.0600)	(**2.5581**)
	(0.50,0.50)	**0.9626**	0.9320	**0.9460**	**0.9530**	**0.9674**
		(1.4558)	(0.9956)	(**0.9510**)	(1.3422)	(1.3062)
	(0.50,0.75)	**0.9620**	0.9258	0.9326	**0.9500**	**0.9690**

(n, n_2)	(α, α_2)	CP (AL)				
		GCI	**BCI**	**PBCI**	**BayCrI**	**HPD**
		(2.3069)	(1.4255)	(1.3424)	(2.0865)	**(1.9603)**
	(0.50,1.00)	**0.9598**	0.9186	0.9236	**0.9482**	**0.9650**
		(2.3086)	(1.4261)	(1.3423)	(2.0850)	**(1.9601)**
	(0.75,0.75)	**0.9604**	0.9360	**0.9578**	**0.9512**	**0.9728**
		(3.0699)	(1.7875)	**(1.6613)**	(2.7530)	(2.6378)
	(0.75,1.00)	**0.9548**	0.9264	0.9416	**0.9450**	**0.9704**
		(4.4769)	(2.3392)	(2.1378)	(3.8568)	**(3.6018)**
	(1.00,1.00)	**0.9564**	0.9390	**0.9648**	0.9438	**0.9764**
		(5.8327)	(2.8448)	**(2.5647)**	(4.8971)	(4.6398)
(20,20)	(0.25,0.25)	**0.9556**	0.9358	0.9400	**0.9510**	**0.9490**
		(0.3506)	(0.3125)	(0.3098)	(0.3395)	**(0.3362)**
	(0.25,0.50)	**0.9516**	0.9302	0.9290	0.9422	**0.9474**
		(0.6296)	(0.5410)	(0.5317)	(0.6091)	**(0.5938)**
	(0.25,0.75)	**0.9500**	0.9300	0.9160	0.9442	**0.9478**
		(1.0847)	(0.8827)	(0.8617)	(1.0408)	**(0.9854)**
	(0.25,1.00)	**0.9510**	0.9276	0.9008	0.9442	**0.9480**
		(1.7581)	(1.3371)	(1.3080)	(1.6437)	**(1.5124)**
	(0.50,0.50)	**0.9510**	0.9342	0.9442	**0.9448**	**0.9526**
		(0.8364)	(0.7079)	(0.6918)	(0.8082)	**(0.7971)**
	(0.50,0.75)	**0.9534**	0.9344	0.9366	**0.9468**	**0.9574**
		(1.2329)	(0.9978)	(0.9721)	(1.1811)	**(1.1469)**
	(0.50,1.00)	**0.9506**	0.9330	0.9224	0.9442	**0.9548**
		(1.8769)	(1.4270)	(1.3858)	(1.7577)	**(1.6560)**
	(0.75,0.75)	**0.9518**	0.9380	**0.9532**	0.9444	**0.9606**
		(1.5709)	(1.2401)	**(1.2000)**	(1.5014)	(1.4734)
	(0.75,1.00)	**0.9484**	0.9342	0.9438	0.9404	**0.9572**
		(2.1472)	(1.6205)	(1.5646)	(2.0200)	**(1.9559)**
	(1.00,1.00)	**0.9508**	0.9428	**0.9620**	**0.9460**	**0.9654**
		(2.6561)	(1.9363)	**(1.8649)**	(2.4707)	(2.4134)
(30,30)	(0.25,0.25)	**0.9522**	0.9378	0.9406	**0.9476**	0.9446
		(0.2767)	(0.2563)	(0.2551)	**(0.2707)**	(0.2682)
	(0.25,0.50)	**0.9476**	0.9324	0.9306	0.9412	**0.9454**
		(0.4892)	(0.4429)	(0.4375)	(0.4781)	**(0.4692)**
	(0.25,0.75)	0.9398	0.9294	0.9210	0.9370	0.9384
		(0.8262)	(0.7203)	(0.7096)	(0.8024)	(0.7726)
	(0.25,1.00)	**0.9492**	0.9378	0.9280	**0.9452**	**0.9520**
		(1.3142)	(1.0980)	(1.0831)	(1.2579)	**(1.1882)**
	(0.50,0.50)	**0.9554**	**0.9468**	**0.9540**	**0.9538**	**0.9564**
		(0.5397)	(0.4966)	**(0.4920)**	(0.5299)	(0.5248)
	(0.50,0.75)	**0.9532**	0.9432	0.9438	**0.9492**	**0.9524**
		(0.7805)	(0.7028)	(0.6930)	(0.7624)	**(0.7491)**
	(0.50,1.00)	**0.9490**	0.9326	0.9272	0.9434	**0.9468**
		(1.1407)	(0.9938)	(0.9805)	(1.1004)	**(1.0648)**
	(0.75,0.75)	**0.9516**	0.9444	**0.9530**	**0.9472**	**0.9580**
		(0.9700)	(0.8626)	**(0.8479)**	(0.9442)	(0.9333)
	(0.75,1.00)	**0.9532**	0.9436	**0.9468**	**0.9478**	**0.9574**
		(1.2964)	(1.1226)	**(1.1046)**	(1.2485)	(1.2257)
	(1.00,1.00)	**0.9524**	**0.9466**	**0.9594**	**0.9516**	**0.9618**
		(1.5691)	(1.3373)	**(1.3118)**	(1.5059)	(1.4860)
(50,50)	(0.25,0.25)	**0.9484**	0.9406	0.9370	**0.9460**	**0.9454**
		(0.2096)	(0.1995)	(0.1992)	(0.2066)	**(0.2048)**
	(0.25,0.50)	**0.9538**	**0.9462**	**0.9454**	**0.9528**	**0.9520**
		(0.3662)	(0.3442)	**(0.3417)**	(0.3606)	(0.3556)
	(0.25,0.75)	**0.9526**	0.9438	0.9352	**0.9488**	**0.9510**
		(0.6082)	(0.5592)	(0.5531)	(0.5952)	**(0.5805)**
	(0.25,1.00)	**0.9492**	0.9396	0.9316	**0.9456**	**0.9466**
		(0.9483)	(0.8445)	(0.8365)	(0.9151)	**(0.8821)**
	(0.50,0.50)	**0.9542**	**0.9448**	**0.9500**	**0.9508**	**0.9526**
		(0.4754)	(0.4452)	**(0.4405)**	(0.4681)	(0.4637)
	(0.50,0.75)	**0.9510**	0.9412	0.9394	**0.9476**	**0.9494**
		(0.6820)	(0.6257)	(0.6191)	(0.6678)	**(0.6577)**
	(0.50,1.00)	**0.9548**	0.9430	0.9366	**0.9484**	**0.9488**
		(0.9984)	(0.8917)	(0.8830)	(0.9670)	**(0.9405)**
	(0.75,0.75)	**0.9554**	**0.9490**	**0.9542**	**0.9532**	**0.9574**
		(0.8456)	(0.7684)	**(0.7582)**	(0.8266)	(0.8181)
	(0.75,1.00)	**0.9486**	0.9414	**0.9458**	**0.9466**	**0.9552**
		(1.1273)	(1.0015)	**(0.9882)**	(1.0899)	(1.0725)
	(1.00,1.00)	**0.9512**	**0.9510**	**0.9526**	**0.9472**	**0.9602**
		(1.3599)	(1.1904)	**(1.1732)**	(1.3075)	(1.2923)
(100,100)	(0.25,0.25)	**0.9516**	0.9444	**0.9468**	**0.9506**	**0.9488**
		(0.1458)	(0.1419)	**(0.1416)**	(0.1445)	(0.1433)
	(0.25,0.50)	**0.9452**	0.9384	0.9396	0.9434	0.9410
		(0.2522)	(0.2436)	(0.2427)	(0.2496)	(0.2469)
	(0.25,0.75)	**0.9512**	**0.9452**	0.9382	**0.9480**	**0.9486**
		(0.4134)	**(0.3936)**	(0.3915)	(0.4068)	(0.4003)
	(0.25,1.00)	**0.9542**	**0.9452**	0.9386	**0.9506**	**0.9508**
		(0.6339)	**(0.5914)**	(0.5885)	(0.6158)	(0.6024)
	(0.50,0.50)	**0.9510**	0.9434	**0.9464**	**0.9480**	**0.9494**
		(0.3271)	(0.3151)	**(0.3134)**	(0.3234)	(0.3207)
	(0.50,0.75)	**0.9530**	**0.9472**	**0.9476**	**0.9504**	**0.9504**
		(0.4633)	(0.4407)	**(0.4389)**	(0.4563)	(0.4512)
	(0.50,1.00)	**0.9578**	**0.9482**	**0.9486**	**0.9528**	**0.9542**
		(0.6696)	(0.6266)	**(0.6231)**	(0.6530)	(0.6416)
	(0.75,0.75)	**0.9458**	0.9428	**0.9448**	0.9438	**0.9480**

(n,n_2)	(α,α_2)	CP (AL)				
		GCI	BCI	PBCI	BayCrI	HPD
		(0.5727)	(0.5421)	(**0.5385**)	(0.5636)	(0.5586)
	(0.75,1.00)	**0.9502**	**0.9462**	**0.9498**	**0.9480**	**0.9510**
		(0.7526)	(0.7021)	(**0.6978**)	(0.7337)	(0.7251)
	(1.00,1.00)	**0.9522**	**0.9498**	**0.9536**	**0.9516**	**0.9554**
		(0.9008)	(0.8328)	(**0.8260**)	(0.8747)	(0.8667)

Notes: Bold represents values that satisfy criteria and the best-performing method

Table 3. The CPs and ALs of the 95% CIs for the difference between the means of BirSau distributions with unequal sample sizes $(n \neq n_2)$.

(n,n_2)	(α,α_2)	CP (AL)				
		GCI	BCI	PBCI	BayCrI	HPD
(10,20)	(0.25,0.25)	**0.9622**	0.9262	0.9304	**0.9512**	**0.9540**
		(0.4582)	(0.3732)	(0.3689)	(0.4318)	(**0.4246**)
	(0.25,0.50)	**0.9592**	0.9314	0.9320	**0.9490**	**0.9540**
		(0.7037)	(0.5811)	(0.5706)	(0.6720)	(**0.6604**)
	(0.25,0.75)	**0.9504**	0.9224	0.9162	**0.9446**	**0.9510**
		(1.1405)	(0.9138)	(0.8911)	(1.0870)	(**1.0413**)
	(0.25,1.00)	**0.9476**	0.9320	0.9126	0.9426	**0.9484**
		(1.8044)	(1.3666)	(1.3327)	(1.6900)	(**1.5707**)
	(0.50,0.50)	**0.9572**	0.9242	0.9298	**0.9452**	**0.9588**
		(1.1619)	(0.8565)	(0.8259)	(1.0868)	(**1.0473**)
	(0.50,0.75)	**0.9660**	0.9420	**0.9560**	**0.9564**	**0.9702**
		(1.5135)	(1.1224)	(**1.0808**)	(1.4236)	(1.3907)
	(0.50,1.00)	**0.9572**	0.9334	0.9350	**0.9522**	**0.9644**
		(2.1095)	(1.5225)	(1.4714)	(1.9592)	(**1.8899**)
	(0.75,0.75)	**0.9526**	0.9266	0.9412	0.9406	**0.9606**
		(2.3624)	(1.5271)	(1.4422)	(2.1522)	(**2.0269**)
	(0.75,1.00)	**0.9584**	0.9424	**0.9608**	**0.9492**	**0.9698**
		(2.8777)	(1.8691)	(**1.7684**)	(2.6158)	(2.5222)
	(1.00,1.00)	**0.9564**	0.9330	**0.9482**	**0.9458**	**0.9706**
		(4.3828)	(2.4446)	(**2.2580**)	(3.8109)	(3.5095)
(20,30)	(0.25,0.25)	**0.9566**	0.9392	0.9418	**0.9498**	**0.9498**
		(0.3157)	(0.2857)	(0.2839)	(0.3069)	(**0.3038**)
	(0.25,0.50)	**0.9572**	0.9390	0.9382	**0.9532**	**0.9522**
		(0.5130)	(0.4610)	(0.4548)	(0.5005)	(**0.4926**)
	(0.25,0.75)	**0.9484**	0.9328	0.9250	**0.9446**	**0.9486**
		(0.8379)	(0.7282)	(0.7177)	(0.8122)	(**0.7850**)
	(0.25,1.00)	**0.9464**	0.9298	0.9150	0.9412	0.9418
		(**1.3119**)	(1.0921)	(1.0776)	(1.2510)	(1.1844)
	(0.50,0.50)	**0.9532**	0.9390	**0.9490**	**0.9486**	**0.9558**
		(0.7392)	(0.6411)	(**0.6286**)	(0.7174)	(0.7069)
	(0.50,0.75)	**0.9566**	0.9436	**0.9504**	**0.9506**	**0.9602**
		(1.0113)	(0.8640)	(**0.8457**)	(0.9785)	(0.9628)
	(0.50,1.00)	**0.9512**	0.9342	0.9344	0.9422	**0.9532**
		(1.4436)	(1.1953)	(1.1729)	(1.3796)	(**1.3351**)
	(0.75,0.75)	**0.9534**	0.9420	**0.9548**	**0.9482**	**0.9630**
		(1.3756)	(1.1252)	(**1.0918**)	(1.3219)	(1.2947)
	(0.75,1.00)	**0.9460**	0.9366	**0.9508**	**0.9426**	**0.9576**
		(1.7480)	(1.4022)	(**1.3660**)	(1.6633)	(1.6329)
	(1.00,1.00)	**0.9562**	0.9438	**0.9542**	**0.9470**	**0.9632**
		(2.2937)	(1.7479)	(**1.6926**)	(2.1532)	(2.0993)
(30,50)	(0.25,0.25)	**0.9536**	0.9386	0.9404	**0.9490**	**0.9472**
		(0.2456)	(0.2301)	(0.2292)	(0.2410)	(**0.2387**)
	(0.25,0.50)	**0.9462**	0.9336	0.9374	0.9428	0.9426
		(**0.3894**)	(0.3637)	(0.3610)	(0.3826)	(0.3780)
	(0.25,0.75)	**0.9542**	**0.9456**	0.9382	**0.9514**	**0.9508**
		(0.6229)	(**0.5706**)	(0.5660)	(0.6091)	(0.5954)
	(0.25,1.00)	0.9426	0.9324	0.9236	0.9440	0.9428
		(0.9534)	(0.8497)	(0.8418)	(0.9202)	(0.8886)
	(0.50,0.50)	**0.9516**	0.9392	**0.9462**	**0.9480**	**0.9526**
		(0.5635)	(0.5139)	(**0.5068**)	(0.5518)	(0.5452)
	(0.50,0.75)	**0.9510**	**0.9460**	**0.9494**	**0.9468**	**0.9542**
		(0.7507)	(0.6787)	(**0.6697**)	(0.7333)	(0.7246)
	(0.50,1.00)	**0.9486**	0.9382	0.9398	0.9440	**0.9504**
		(1.0473)	(0.9284)	(0.9166)	(1.0126)	(**0.9907**)
	(0.75,0.75)	**0.9506**	0.9408	**0.9474**	**0.9460**	**0.9552**
		(1.0207)	(0.8944)	(**0.8785**)	(0.9929)	(0.9771)
	(0.75,1.00)	**0.9518**	0.9412	**0.9508**	**0.9480**	**0.9572**
		(1.2632)	(1.0950)	(**1.0773**)	(1.2201)	(1.2046)
	(1.00,1.00)	**0.9476**	0.9400	**0.9488**	0.9408	**0.9550**
		(1.6514)	(1.3822)	(**1.3539**)	(1.5796)	(1.5489)
(50, 100)	(0.25,0.25)	**0.9498**	0.9398	0.9410	**0.9484**	**0.9466**
		(0.1809)	(0.1736)	(0.1733)	(0.1787)	(**0.1770**)
	(0.25,0.50)	**0.9528**	**0.9490**	**0.9454**	**0.9488**	**0.9490**
		(0.2745)	(0.2636)	(**0.2625**)	(0.2712)	(0.2686)
	(0.25,0.75)	**0.9564**	**0.9484**	**0.9448**	**0.9526**	**0.9522**
		(0.4278)	(0.4067)	(**0.4043**)	(0.4211)	(0.4146)
(50,100)	(0.25,1.00)	**0.9500**	**0.9452**	**0.9448**	**0.9496**	**0.9496**
		(0.6457)	(0.6021)	(**0.5996**)	(0.6280)	(0.6150)
	(0.50,0.50)	**0.9452**	0.9406	0.9436	0.9434	**0.9462**

(n,n_2)	(α,α_2)	CP (AL)				
		GCI	**BCI**	**PBCI**	**BayCrI**	**HPD**
		(0.4081)	(0.3855)	(0.3833)	(0.4022)	(**0.3980**)
	(0.50,0.75)	**0.9450**	0.9400	0.9428	0.9444	**0.9466**
		(0.5264)	(0.4963)	(0.4924)	(0.5178)	(**0.5130**)
	(0.50,1.00)	**0.9492**	0.9428	0.9436	**0.9472**	**0.9492**
		(0.7145)	(0.6640)	(0.6596)	(0.6958)	(**0.6866**)
	(0.75,0.75)	**0.9480**	0.9386	0.9428	0.9438	**0.9458**
		(0.7182)	(0.6622)	(0.6555)	(0.7036)	(**0.6943**)
	(0.75,1.00)	**0.9488**	**0.9450**	**0.9492**	**0.9474**	**0.9538**
		(0.8728)	(0.8004)	(**0.7917**)	(0.8505)	(0.8421)
	(1.00,1.00)	**0.9464**	0.9432	**0.9462**	0.9424	**0.9502**
		(1.1483)	(1.0289)	(**1.0154**)	(1.1102)	(1.0927)

Notes: Bold represents values that satisfy criteria and the best-performing method.

Table 4. The daily wind speed data is from the Ayutthaya and Ratchaburi provinces in Thailand.

Provinces	Wind Speed (knots)								
Ayutthaya	6.6	4.7	4.2	9	9	4.4	2.8	2.8	4.4
	3.9	3.5	3.2	5.3	4.2	6.4	3.6	2.9	5.4
	8.7	10.2	14.4	9.6	16.3	16.5	14	9.7	7.1
	3.3	3.9	4.7	3.3	5.9	7.5	7.2	9.1	6.7
	3.5	4.8	4.4	3.9	5.7	7.8	6.8	7.3	5.9
	8.1	4.7	6.3	6.9	6.1	4.7	6	7.2	6.6
	6.5	6.2	6.3	4.1	5.1	14.6	22	18.8	13.2
	8.6	9.7	9.7	7.5	3.9	5.1	7.9	7.9	7.4
	6.4	7.5	7.4	6.2	7.4	4.5	3.8	6.4	7.2
	6.8	7.6	7.4	9	5	9.1	5.2	5.2	
Ratchaburi	5.2	5.2	2.6	5.2	6.6	3.9	5.0	4.9	5.5
	4.6	5.1	3.9	5.1	4.8	4.7	3.1	3.9	4.7
	5.6	5.6	5.8	6.4	7.7	10.4	9.9	7.7	2.6
	2.4	3.3	3.9	4.2	3.0	3.3	7.0	6.2	3.8
	2.1	3.7	4.0	4.1	4.2	6.8	6.5	6.5	7.1
	6.3	5.3	6.6	7.8	7.3	4.9	4.5	6.0	5.5
	5.1	6.7	5.1	4.5	6.3	8.0	16.8	12.9	7.8
	5.1	3.7	6.5	4.5	2.3	3.4	2.5	4.4	5.3
	5.6	4.7	6.0	5.1	7.5	4.4	5.1	6.1	5.5
	5.9	6.0	5.7	4.8	5.1	10.7	5.8	4.2	

Table 5. AIC and BIC values of Ayutthaya and Ratchaburi provinces are used to fit seven asymmetric distributions

Provinces	**Crite-ria**	**Distributions**					
		BirSau	**Cauchy**	**Logistic**	**Exponential**	**Weibull**	**Normal**
Ayutthaya	AIC	**439.34**	466.13	463.15	527.88	462.85	480.17
	BIC	**446.81**	471.11	468.12	530.37	467.83	485.15
Ratchaburi	AIC	**369.28**	379.92	375.47	484.03	391.33	396.34
	BIC	**376.76**	384.89	380.44	486.52	396.30	401.32

Table 6. Descriptive statistics for the wind speed data.

Province	**n**	**Min**	**Median**	**Mean**	**Max**	**Variance**
Ayutthaya	89	2.8	6.4	7.0579	22	12.4740
Ratchaburi	89	2.1	5.1	5.5179	16.8	4.8635

Table 7. The 95% CIs for the mean and difference of two means of wind speed for the Ayutthaya and Ratchaburi datasets

Methods	**Ayutthaya**		**Ratchaburi**		**Ayutthaya-Ratchaburi**	
	Interval	**Length**	**Interval**	**Length**	**Interval**	**Length**
GCI	6.4217-7.716	1.3499	5.1138-5.9871	0.8733	0.7408-2.3596	1.6188
BCI	6.3935-7.8827	1.4892	5.0861-6.0388	0.9526	0.7311-2.3936	1.6625
PBCI	6.4494-7.7233	**1.2739**	5.1258-5.9420	**0.8162**	0.7622-2.3227	**1.5605**
BayCrI	6.5072-7.8466	1.3394	5.1583-5.9922	0.8338	0.7847-2.4473	1.6626
HPD	6.4378-7.7552	1.3174	5.1585-5.9926	0.8342	0.7232-2.3612	1.6308