# New results on generalized quaternion algebra involving generalized Pell-Pell Lucas quaternions

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*Abstract:* This work presents a new sequence, generalized Pell-Pell-Lucas quaternions, we prove that the set of these elements forms an order of generalized quaternions with 3-parameters $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ as defined by ring theory. In addition, some properties of these elements are presented. The properties in this article refer to $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ algebras and sometimes to the 2-parameter algebra $\mathbb{H}(\alpha,\beta)$.

*Key-Words:* Quaternion algebra, Generalized quaternion algebra, Pell-Lucas quaternions, Generalized Pell-Pell-Lucas quaternions, Order, Centralizer.

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## 1 Introduction

In 1830, the Irish mathematician Sir William Rowan Hamilton began an exploration of complex numbers with the intention of generalization. After years of contemplation and extensive research, he finally introduced real quaternions in 1843 as a solution to this long-standing problem, [1], [2].

Then the researchers L. E. Dickson and L. W. Griffiths wrote two seminal articles on the subject of generalized quaternions, [3], [4]. Recently, the most general form of the quaternion algebra depending on 3-parameters (3PGQ) was introduced by [5], which prompted us to look for some properties associated with this algebra, which is called $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ in this article. For more information on the properties of this algebra (see, [6], [7]).

Nowadays, quaternions hold significant importance in various domains including computer science, quantum physics, and signal and color image processing, as evidenced by [8]. Furthermore, numerous researchers have explored various types of quaternion sequences such as Pseudo-Lucas Quaternions, Balancing Split Quaternions, and Pell-Lucas numbers, as documented in studies referenced by [9], [10]. The authors in [11], studied the quaternions whose coefficients were from the generalized Fibonacci and Lucas sequences. The authors in [12], studied the quaternions whose coefficients are Pell and PellLucas numbers. Liano and Wolch worked on Pell and Jacobsthal quaternions, [13]. Moreover, Catarino studied the Fibonacci quaternion polynomials and the modified Pell quaternions, [14], and obtained the norm values, generating functions, Binet formulas and identities (similar to Cassini) of these polynomials.

The Fibonacci, Lucas, Pell, and Pell-Lucas sequences are among the most popular and widely used sequences in the mathematical community because of their fascination and possible applications in other fields. Pell and Pell-Lucas numbers weave a common thread that spans analysis, geometry, trigonometry, and various areas of discrete mathematics, including disciplines such as number theory and linear algebra.

The study, [15], was able to introduce and generalize the Pell numbers and some interesting special results related to them. The study, [16], introduced split Pell and split Pell-Lucas quaternions and also considered some properties of these sequences, including the Catalan identity, the Cassini identity, and the Ducani identity. In this work, our main goal is to generate new quaternion sequences from two important families: Pell numbers and Pell-Lucas numbers, and to present some of their properties, and we have also enriched the repertoire of $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ quaternion algebras with new properties.

The following is the organization of this paper: Section 2 contains preliminary results for the algebra of quaternions, the Pell numbers, and the Pell-Lucas numbers. In Section 3, we detail the results obtained for the properties of the Pell-Pell-Lucas quaternion sequence generalized in the algebra $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$, and we strengthen these results by giving an order and a center of the algebra of quaternions Finally, Section 4 presents a conclusion and perspectives of research of this work in the practical area.

## 2 Definitions and Notations

In this section we state the definitions and the main results concerning the quaternions algebra has been introduce depending on 3-parameters (3PGQs), Generalized Quaternions (2-Parameter), the Pell and Pell-Lucas Numbers.

### 2.1 3-Parameter Generalized Quaternions

In the following, we'll go over some key concepts and notations that will help us understand and expand on this topic. From [5], we define the set of generalized quaternions with 3-parameters (3PGQ) as follows:

**Definition 2.1** *The set $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ defined by:*

$$\{a + be_1 + ce_2 + de_3 \mid a, b, c, d, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, \}$$

*where $e_1^2 = -\lambda_1\lambda_2$, $e_2^2 = -\lambda_1\lambda_3$, $e_3^2 = -\lambda_2\lambda_3$, $e_1e_2e_3 = -\lambda_1\lambda_2\lambda_3$ is called the set of generalized quaternions with 3-parameters (3PGQ).*

Each element $p = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ of the set $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ is called a 3-parameter generalized quaternion (3PGQ). The numbers $x_0, x_1, x_2, x_3$ are called components of $p$. The basis vectors $e_0, e_1, e_2, e_3$ of the $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ satisfy the following multiplication table:

**Table 1. Multiplication Table**

| $\cdot$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|---|
| $e_0$ | 1 | $e_1$ | $e_2$ | $e_3$ |
| $e_1$ | $e_1$ | $-\lambda_1\lambda_2$ | $\lambda_1e_3$ | $-\lambda_2e_2$ |
| $e_2$ | $e_2$ | $-\lambda_1e_3$ | $-\lambda_1\lambda_3$ | $\lambda_3e_1$ |
| $e_3$ | $e_3$ | $\lambda_2e_2$ | $-\lambda_3e_1$ | $-\lambda_2\lambda_3$ |

**Special cases:**

**(i)** If $\lambda_1 = 1$, $\lambda_2 = \alpha$, $\lambda_3 = \beta$, then we get the algebra of 2PGQs.

**(ii)** If $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$, then we get the algebra of Hamilton quaternions.

**(iii)** If $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -1$, then gives us the algebra of split quaternions.

**(iv)** If $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 0$, then we get the algebra of split semiquaternions.

Any $3PGQ$ $p = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ consists of two parts, the vector part and the scalar part: $p = S(p) + V(p)$ such as:

$$S(p) = x_0 \text{ and } V(p) = x_1e_1 + x_2e_2 + x_3e_3$$

The rules of addition, scalar multiplication and multiplication are defined on $\mathbb{k}$ as follows:
Let $p = x_0+x_1e_1+x_2e_2+x_3e_3$ and $q = y_0+y_1e_1+y_2e_2 + y_3e_3$ be 3PGQs and $\alpha$ be a real number.

- Addition:

$$\begin{aligned} p+q &= (S(p)+S(q)) + (V(p)+V(q)) \\ &= (x_0+y_0) + (x_1+y_1)e_1 + \\ &\quad (x_2+y_2)e_2 + (x_3+y_3)e_3. \end{aligned}$$

- Multiplication by scalar:

$$\alpha p = \alpha x_0 + \alpha x_1e_1 + \alpha x_2e_2 + \alpha x_3e_3 \text{ for all } \alpha \in \mathbb{R}.$$

- Multiplication: from the multiplication table, Table 1, we have

$$\begin{aligned} pq = {} & (x_0y_0 - \lambda_1\lambda_2x_1y_1 - \lambda_1\lambda_3x_2y_2 - \\ & \lambda_2\lambda_3x_3y_3) + e_1(x_0y_1 + y_0x_1 + \lambda_3x_2y_3 \\ & -\lambda_3x_3y_2) + e_2(x_0y_2 + y_0x_2 + \lambda_2x_3y_1 \\ & -\lambda_2x_1y_3) + e_3(x_0y_3 + y_0x_3 + \lambda_1x_1y_2 \\ & -\lambda_1x_2y_1) \end{aligned}$$

- The norm of a quaternion $p$ is:

$$N(p) = x_0^2 + \lambda_1\lambda_2x_1^2 + \lambda_1\lambda_3x_2^2 + \lambda_2\lambda_3x_3^2$$

**Definition 2.2** *A subring $\mathbb{O} \subseteq \mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ is an order in $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ if $\mathbb{O}$ is a $\mathbb{Z}$-lattice of $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$. That is, $\mathbb{O}$ is a finitely generated $\mathbb{Z}$-submodule of $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ (which is also a subring of $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ by* [17]).

### 2.2 2-Parameter Generalized Quaternions

Let $H(\alpha,\beta)$ be the generalized real quaternion algebra, the elements of $H(\alpha,\beta)$ are written in the form $p = x_0+x_1e_1+x_2e_2+x_3e_3$, where $x_i \in \mathbb{R}$, $e_1^2 = \alpha$, $e_2^2 = \beta$, $e_3 = e_1e_2 = -e_2e_1$. The following expressions represent the norm and the trace of a generalized quaternion $p$ :

$$N(p) = x_0^2 - \alpha x_1^2 - \beta x_2^2 + \alpha\beta x_3^2 \text{ and } t(a) = 2x_0.$$

As is well known, we have

$$p^2 - t(p)p + N(p) = 0 \text{ for all } p \in H(\alpha,\beta).$$

**Definition 2.3** *The quaternion algebra $\mathbb{A}$ is said to be a division algebra if for all $p \in \mathbb{A}^*$, $N(p) \neq 0$, otherwise $\mathbb{A}$ is called a split algebra.*

**Definition 2.4** *For $a \in \mathbb{H}_{\mathbb{Q}}(\alpha,\beta)$. The centralizer of the element $a$ is*

$$C(a) = \{x \in \mathbb{H}_{\mathbb{Q}}(\alpha,\beta) \mid ax = xa\}.$$

## 2.3 Properties of the Pell and Pell-Lucas Numbers

- Let $(P_n)_{n\geq 0}$ be a sequence of Pell numbers:

$$P_n = 2P_{n-1} + P_{n-2} \text{ where } n \geq 2 \qquad (1)$$

with $P_0 = 0, P_1 = 1$.

- Let $(Q_n)_{n\geq 0}$ be a sequence of Pell-Lucas numbers:

$$Q_n = 2Q_{n-1} + Q_{n-2} \text{ where } n \geq 2 \qquad (2)$$

with $Q_0 = 2, Q_1 = 2$.

- The Binets formulas of the $n^{th}$ Pell and Pell-Lucas numbers are:

$$\begin{aligned} P_n &= \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ for all } n \in \mathbb{N}, \\ Q_n &= \gamma^n + \delta^n \text{ for all } n \in \mathbb{N} \\ \text{with} \quad & \gamma = 1+\sqrt{2},\ \delta = 1-\sqrt{2}. \end{aligned} \qquad (3)$$

The following properties of Pell and Pell-Lucas numbers are known from [18], [19], [20].

**Proposition 2.5** *Let $(P_n)_{n\geq 0}$ be a sequence of Pell numbers and $(Q_n)_{n\geq 0}$ be the Pell-Lucas sequence. Then the following properties hold:*

**1)-** $P_n^2 + P_{n+1}^2 = P_{2n+1} \,\forall\, n \in \mathbb{N}$,

**2)-** $Q_n^2 + Q_{n+1}^2 = 8P_{2n+1} \,\forall\, n \in \mathbb{N}$,

**3)-** $P_{n+1}^2 - P_n^2 = \dfrac{Q_{2n+1} + 2(-1)^n}{4} \,\forall\, n \in \mathbb{N}$,

**4)-** $Q_{n+1}^2 - Q_n^2 = 8P_{2n+1} - 4(-1)^n \,\forall\, n \in \mathbb{N}$,

**5)-** $P_n^2 = \dfrac{Q_{2n} + 2(-1)^{n+1}}{8} \,\forall\, n \in \mathbb{N}$,

**6)-** $Q_n^2 = Q_{2n} + 2(-1)^n \,\forall\, n \in \mathbb{N}$,

**7)-** $P_{n+1}P_n = \dfrac{Q_{2n+1} - 2(-1)^n}{8} \,\forall\, n \in \mathbb{N}$,

**8)-** $Q_nQ_{n+1} - Q_{2n+1} = 2(-1)^n \,\forall\, n \in \mathbb{N}$,

**9)-** $P_{2n+1} = P_nQ_{n+1} + (-1)^n \,\forall\, n \in \mathbb{N}$,

**10)-** $P_nQ_m = P_{n+m} + (-1)^m P_{n-m} \,\forall\, n, m \in \mathbb{Z}$,

**11)-** $P_nP_{n+k} = \dfrac{1}{8}(Q_{2n+k} + (-1)^{n+1}Q_k) \,\forall\, n, k \in \mathbb{N}$,

**12)-** $Q_{n+2} + Q_{n-2} = 6Q_n \;\forall\; n \geq 2$,

**13)-** $P_{n+1} + P_{n-1} = Q_n \;\forall\; n \in \mathbb{N}^*$.

We will introduce some other properties of Pell and Pell-Lucas numbers. These properties will be useful later.

**Proposition 2.6** *Let $(P_n)_{n\geq 0}$ be the Pell sequence and $(Q_n)_{n\geq 0}$ be the Pell-Lucas sequence, then we have*

$$Q_nQ_{n+k} = Q_{2n+k} + (-1)^n Q_k \;\forall\; n, k \in \mathbb{N}. \qquad (4)$$

**Proof.** If we denote $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$, by Binets formula, we have

$$\begin{aligned} P_n =& \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ for all } n \in \mathbb{N} \text{ and} \\ Q_n =& \gamma^n + \delta^n \text{ for all } n \in \mathbb{N} \end{aligned}$$

$i)$ Let $m, p \in \mathbb{R}$, $p \leqslant m$, thus

$$\begin{aligned} Q_mQ_p - 8P_mP_p &= (\gamma^m + \delta^m)(\gamma^p + \delta^p) \\ &\quad -(\gamma^m - \delta^m)(\gamma^p - \delta^p) \\ &= 2\gamma^p\delta^p(\gamma^{m-p} + \delta^{m-p}) \\ &= 2(-1)^p Q_{m-p}. \end{aligned}$$

It results $Q_mQ_p - 8P_mP_p = 2(-1)^pQ_{m-p}$. So, $Q_mQ_p = 8P_mP_p + 2(-1)^pQ_{m-p}$. We use the Proposition 2.5 (11), we obtain

$$Q_mQ_p = Q_{m+p} + (-1)^p Q_{m-p} \,\forall\, n \in \mathbb{N}, p \leqslant m.$$

From this it follows that

$$Q_nQ_{n+k} = Q_{2n+k} + (-1)^n Q_k \text{ for all } n, k \in \mathbb{N}.$$

# 3 Generalized Pell-Pell Lucas numbers and generalized Pell-Pell Lucas Quaternions

Let $r$, $t$ be two arbitrary integers, and let $n$ be an arbitrary positive integer. The numbers in a sequence $(g_n)_{n\geqslant 1}$, where

$$g_{n+1} = rP_n + t\,Q_{n+1} \text{ where } n \geqslant 0$$

are called generalized Pell-Pell-Lucas numbers. To emphasize the presence of the integers $r$ and $t$, we will use $g_n^{r,t}$ instead of the notation $g_n$. Let $\mathbb{k}^{\mathbb{Q}}_{\lambda_1,\lambda_2,\lambda_3}$ be the generalized quaternion algebra over the rational field. We define the $n^{th}$ generalized Pell-Pell-Lucas quaternion to be an element of the form

$$G_n^{r,t} = g_n^{r,t}e_0 + g_{n+1}^{r,t}e_1 + g_{n+2}^{r,t}e_2 + g_{n+3}^{r,t}e_3$$

In the following proposition, we compute the norm for the $n^{th}$ generalized Pell-Pell-Lucas quaternions.

**Proposition 3.1** *Let $n$, $r$ be two positive integers and $t$ be an arbitrary integer. Let $G_n^{r,t}$ be the $n^{th}$ generalized Pell-Pell-Lucas quaternion. Then the norm of $G_n^{r,t}$ in the quaternion algebra $\mathbb{k}^Q_{\lambda_1,\lambda_2,\lambda_3}$ is given by:*

$$\begin{aligned}N(G_n^{r,t}) =& g_{2n}^{(2rt,\frac{6+\lambda_1\lambda_2}{8}r^2+t^2)}\\ &+g_{2n+2}^{(2r\lambda_1\lambda_2 t,\frac{-1}{8}r^2+\frac{\lambda_1\lambda_3}{8}+\lambda_1\lambda_2 t^2)}\\ &+g_{2n+4}^{(2r\lambda_1\lambda_3 t,\lambda_1\lambda_3 t^2+\frac{\lambda_2\lambda_3}{8}r^2)}+g_{2n+6}^{(2r\lambda_2\lambda_3 t,\lambda_2\lambda_3 t^2)}\\ &+g_2^{((1-\lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_2\lambda_3)(\frac{r^2}{4}+2t^2+2rt)(-1)^n,0)}\end{aligned}$$

**Proof.** We have

$$\begin{aligned}N(G_n^{r,t}) &= g_n^2+\lambda_1\lambda_2 g_{n+1}^2+\lambda_1\lambda_3 g_{n+2}^2+\lambda_2\lambda_3 g_{n+2}^2\\ &= (rP_{n-1}+tQ_n)^2+\lambda_1\lambda_2(rP_n+t\,Q_{n+1})^2\\ &+\lambda_1\lambda_3(rP_{n+1}+tQ_{n+2})^2+\lambda_2\lambda_3(rP_{n+2}\\ &+tQ_{n+3})^2\\ &= r^2P_{n-1}^2+t^2Q_n^2+2rtP_{n-1}Q_n+\\ &\lambda_1\lambda_2(r^2P_n^2+t^2Q_{n+1}^2+2rtP_nQ_{n+1})\\ &+\lambda_1\lambda_3(r^2P_{n+1}^2+t^2Q_{n+2}^2+2rtP_{n+1}Q_{n+2})\\ &+\lambda_2\lambda_3(r^2P_{n+2}^2+t^2Q_{n+3}^2+2rtP_{n+2}Q_{n+3})\\ &= r^2P_{n-1}^2+\lambda_1\lambda_2 r^2P_n^2+\lambda_2\lambda_3 r^2P_{n+2}^2\\ &+\lambda_1\lambda_3 r^2P_{n+1}^2+t^2Q_n^2+\lambda_1\lambda_2 t^2Q_{n+1}^2\\ &+\lambda_1\lambda_3 t^2Q_{n+2}^2+\lambda_2\lambda_3 t^2Q_{n+3}^2+2rtP_{n-1}Q_n\\ &+2rt\lambda_1\lambda_2P_nQ_{n+1}+2rt\lambda_1\lambda_3P_{n+1}Q_{n+2}\\ &+2rt\lambda_2\lambda_3P_{n+2}Q_{n+3}\end{aligned}$$

Using Proposition 2.5(5-6-10-12), we obtain

$$\begin{aligned}N(G_n^{r,t}) &= \tfrac{r^2}{8}(6Q_{2n}-Q_{2n+2}+2(-1)^n)\\ &+\lambda_1\lambda_2\tfrac{r^2}{8}(Q_{2n}+2(-1)^{n+1})\\ &+\lambda_1\lambda_3\tfrac{r^2}{8}(Q_{2n+2}+2(-1)^{n+2})\\ &+\lambda_2\lambda_3\tfrac{r^2}{8}(Q_{2n+4}+2(-1)^{n+3})\\ &+t^2(Q_{2n}+2(-1)^n)\\ &+\lambda_1\lambda_3 t^2(Q_{2n+4}+2(-1)^{n+2})\\ &+\lambda_2\lambda_3 t^2(Q_{2n+6}+2(-1)^{n+3})\\ &+2rt(P_{2n-1}+(-1)^n)+\\ &+\lambda_1\lambda_2 2rt(P_{2n+1}+(-1)^{n+1})\\ &+\lambda_1\lambda_3 2rt(P_{2n+3}+(-1)^{n+2})\\ &+\lambda_2\lambda_3 2rt(P_{2n+5}+(-1)^{n+3})\end{aligned}$$

$$\begin{aligned}&= 2rtP_{2n-1}+\lambda_1\lambda_2 2rtP_{2n+1}\\ &+\lambda_1\lambda_3 2rtP_{2n+3}+\lambda_2\lambda_3 2rtP_{2n+5}\\ &+(\tfrac{6r^2}{8}+\tfrac{\lambda_1\lambda_2 r^2}{8}+t^2)Q_{2n}\\ &+(\tfrac{-r^2}{8}+\tfrac{\lambda_1\lambda_3}{8}+\lambda_1\lambda_2 t^2)Q_{2n+2}\\ &+(\tfrac{\lambda_2\lambda_3 r^2}{8}+\lambda_1\lambda_3 t^2)Q_{2n+4}\\ &+\lambda_2\lambda_3 t^2Q_{2n+6}\\ &+\tfrac{r^2}{4}(-1)^n+\tfrac{\lambda_1\lambda_2 r^2}{4}(-1)^{n+1}\\ &+\tfrac{\lambda_1\lambda_3 r^2}{4}(-1)^{n+2}+\tfrac{\lambda_2\lambda_3 r^2}{4}(-1)^{n+3}\\ &+2t^2(-1)^n+2t^2\lambda_1\lambda_2(-1)^{n+1}\\ &+2t^2\lambda_1\lambda_3(-1)^{n+2}+2t^2\lambda_2\lambda_3(-1)^{n+3}\\ &+2rt(-1)^n+2rt\lambda_1\lambda_2(-1)^{n+1}\\ &+2rt\lambda_1\lambda_3(-1)^{n+2}+2rt\lambda_2\lambda_3(-1)^{n+3}\\ &= g_{2n+2}^{(2r\lambda_1\lambda_2 t,\frac{-1}{8}r^2+\frac{\lambda_1\lambda_3}{8}+\lambda_1\lambda_2 t^2)}\\ &+g_{2n+4}^{(2r\lambda_1\lambda_3 t,\lambda_1\lambda_3 t^2+\frac{\lambda_2\lambda_3}{8}r^2)}+g_{2n+6}^{(2r\lambda_2\lambda_3 t,\lambda_2\lambda_3 t^2)}\\ &+\tfrac{r^2}{4}(-1)^n+\tfrac{\lambda_1\lambda_2 r^2}{4}(-1)^{n+1}+2t^2(-1)^n\\ &+\tfrac{\lambda_1\lambda_3 r^2}{4}(-1)^n+\tfrac{\lambda_2\lambda_3 r^2}{4}(-1)^{n+1}\\ &+2t^2\lambda_1\lambda_2(-1)^{n+1}+2t^2\lambda_1\lambda_3(-1)^n\\ &+2t^2\lambda_2\lambda_3(-1)^{n+1}+2rt(-1)^n\\ &+2rt\lambda_1\lambda_2(-1)^{n+1}+2rt\lambda_1\lambda_3(-1)^n\\ &+2rt\lambda_2\lambda_3(-1)^{n+1}\\ &= g_{2n}^{(2rt,\frac{6}{8}r^2+\frac{\lambda_1\lambda_2}{8}r^2+t^2)}\\ &+g_{2n+2}^{(2r\lambda_1\lambda_2 t,\frac{-1}{8}r^2+\frac{\lambda_1\lambda_3}{8}+\lambda_1\lambda_2 t^2)}\\ &+g_{2n+4}^{(2r\lambda_1\lambda_3 t,\lambda_1\lambda_3 t^2+\frac{\lambda_2\lambda_3}{8}r^2)}\\ &+g_{2n+6}^{(2r\lambda_2\lambda_3 t,\lambda_2\lambda_3 t^2)}\\ &+\tfrac{r^2}{4}(-1)^n(1-\lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_2\lambda_3)\\ &+2t^2(-1)^n(1-\lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_2\lambda_3)\\ &+2rt(-1)^n(1-\lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_2\lambda_3)\\ &= g_{2n}^{(2rt,\frac{6+\lambda_1\lambda_2}{8}r^2+t^2)}+g_{2n+2}^{(2r\lambda_1\lambda_2 t,\frac{-1}{8}r^2+\frac{\lambda_1\lambda_3}{8}+\lambda_1\lambda_2 t^2)}\\ &+g_{2n+4}^{(2r\lambda_1\lambda_3 t,\lambda_1\lambda_3 t^2+\frac{\lambda_2\lambda_3}{8}r^2)}+g_{2n+6}^{(2r\lambda_2\lambda_3 t,\lambda_2\lambda_3 t^2)}\\ &+(1-\lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_2\lambda_3)(-1)^n(\tfrac{r^2}{4}+2t^2+2rt)\end{aligned}$$

so,

$$\begin{aligned}N(G_n^{r,t}) &= g_{2n}^{(2rt,\frac{6+\lambda_1\lambda_2}{8}r^2+t^2)}\\ &+g_{2n+2}^{(2r\lambda_1\lambda_2 t,\frac{-1}{8}r^2+\frac{\lambda_1\lambda_3}{8}+\lambda_1\lambda_2 t^2)}\\ &+g_{2n+4}^{(2r\lambda_1\lambda_3 t,\lambda_1\lambda_3 t^2+\frac{\lambda_2\lambda_3}{8}r^2)}\end{aligned}$$

$$+ g_{2n+6}^{(2r\lambda_2\lambda_3 t,\lambda_2\lambda_3 t^2)}$$
$$+ g_2^{((1-\lambda_1\lambda_2+\lambda_1\lambda_3-\lambda_2\lambda_3)(\frac{r^2}{4}+2t^2+2rt)(-1)^n,0)}$$

Using the generalized Pell-Pell-Lucas quaternions, we can construct an order of quaternion algebras, and we show that Pell-Pell-Lucas quaternions can also have an algebraic structure over $\mathbb{Q}$. The following remarks will help us.

**Remark 3.1** *Let $r$, $t$ be two arbitrary integers and $n$ be an arbitrary positive integer. Let $(g_n^{r,t})_{n\geq 1}$ be the generalized Pell-Pell-Lucas numbers. Then $rP_{n+1} + tQ_n = g_n^{r,t} + g_{n+1}^{2r,0}$ for all $n \in \mathbb{N}^*$.*

**Proof.**

$$\begin{aligned} rP_{n+1} + tQ_n &= r(2P_n + P_{n-1} + tQ_n) \\ &= rP_{n-1} + tQ_n + 2rP_n \\ &= g_n^{r,t} + g_{n+1}^{2r,0}. \end{aligned}$$

**Remark 3.2** *Let $r$, $t$ be two arbitrary integers and $n$ be an arbitrary positive integer. Let $(G_n^{r,t})_{n\geqslant 1}$ be the generalized Pell-Pell Lucas quaternion elements. Then $G_n^{r,t} = 0$ if and only $r = t = 0$.*

**Proof.** $\Leftarrow$). It is trivial.
$\Rightarrow$). If $G_n^{r,t} = 0$, since $\{e_0, e_1, e_2, e_3,\}$ is a basis in $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$, we obtain that

$$g_n^{r,t} = g_{n+1}^{r,t} = g_{n+2}^{r,t} = g_{n+3}^{r,t} = 0$$

It results

$$g_{n-1}^{r,t} = g_{n+1}^{r,t} - g_n^{r,t} = 0, \ldots, g_2^{r,t} = 0, g_1^{r,t} = 0,$$

therefore, $t = 0$. From $g_2^{r,t} = 0$, we obtain $r = 0$.

**Theorem 3.2** *Let $M$ be the set*

$$\left\{\sum_{i=1}^{n} 8G_{n_i}^{r_i,t_i} \mid n \in \mathbb{N}^*, r_i, t_i \in \mathbb{Z} \quad \forall i = 1 \ldots n\right\} \cup \{1\}.$$

*Then*

**(i)** *The set $M$ with addition and multiplication of quaternions has a ring structure.*

**(ii)** *The set $M$ is an order of the quaternion algebra $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$.*

**(iii)** $\left\{\sum_{i=1}^{n} 8G_{n_i}^{r_i',t_i'} \mid n \in \mathbb{N}^*, r_i', t_i' \in \mathbb{Q} \ \forall i = 1 \ldots n\right\} \cup \{1\}$ *is a $\mathbb{Q}$-algebra.*

**Proof.** $(i)$ Obviously it is.
$(ii)$ Using Remark 3.2, we first note that $0 \in M$. We now show that $M$ is a $\mathbb{Z}$-submodule of $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$.
Let $n, m \in \mathbb{N}^*, a, b, r, t, r', t' \in \mathbb{Z}$. It is easy to prove that

$$ag_n^{r,t} + bg_m^{r',t'} = g_n^{ar,at} + g_m^{br',bt'}$$

This implies that $aG_n^{r,t} + bG_m^{r',t'} = G_n^{ar,at} + G_m^{br',bt'}$. It is clear that $M$ is a $\mathbb{Z}$-submodule of the quaternion algebra $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$. Since this submodule basis is $\{e_0, e_1, e_2, e_3\}$, $M$ is a free $\mathbb{Z}$-module of rank 4. Now we prove that $M$ is a subring of $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$. It is sufficient to show that $8G_n^{r,t}.8\,G_m^{r',t'} \in M$. For this, if $m < n$, we compute

$$\begin{aligned} 8g_n^{r,t}.8g_m^{r',t'} &= 8(rP_{n-1} + tQ_n).8(r'P_{m-1} + t'Q_m) \\ &= 64rr'P_{n-1}P_{m-1} + 64rt'P_{n-1}Q_m \\ &+ 64tr'P_nQ_{m-1} + 64tt'Q_nQ_m \qquad (5) \end{aligned}$$

Using Proposition 2.5 (10, 11), Proposition 2.6, Remark 3.1 and the equality (5), we obtain:

$$\begin{aligned} 8g_n^{r,t}.8g_m^{r',t'} &= 8rr'(Q_{n+m-2} + (-1)^m Q_{n-m}) \\ &+ 64rt'(P_{m+n-1} + (-1)^m P_{n-m-1}) \\ &+ 64tr'(P_{m+n-1} + (-1)^m P_{n-m+1}) \\ &+ 64tt'(Q_{n+m} + (-1)^m Q_{n-m}) \\ &= 8(rr'Q_{n+m-2} + 8r't P_{n+m-1}) \\ &+ 8(8r't(-1)^m P_{n-m+1} + rr'(-1)^m Q_{n-m}) \\ &+ 64(rt')P_{n+m-1} + tt'Q_{n+m}) \\ &+ 64rt'(-1)^m P_{n-m-1} + tt'(-1)^m Q_{n-m} \\ &= 8g_{m+n-2}^{8r't,rr'} + 8g_{m+n-1}^{16r't,0} \\ &+ g_{n-m}^{8r't(-1)^m,rr'(-1)^m} + 8g_{n-m+1}^{8r't(-1)^m,0} \\ &+ 8g_{n+m}^{8rt',8tt'} + 8g_{n-m}^{8rt'(-1)^m,8tt'(-1)^m} \end{aligned}$$

Therefore, $8G_n^{r,t}.8G_m^{r',t'} \in M$. Consequently, $M$ is an order of the quaternion algebra $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$.
$(iii)$ is obvious.
It is known that if $r$ is an odd prime positive integer, the algebra $\mathbb{H}_{\mathbb{Q}}(-1, r)$ is a split algebra if and only if $r \equiv 1(mod4)$ (see, [21], [22]). In the following, we will show that this algebra contains an infinite number of invertible generalized Pell-Pell-Lucas quaternion elements. In this part we replace $(\lambda_1, \lambda_2, \lambda_3)$ by $(1, 1, -r)$, i.e. we focus on $\mathbb{k}_{1,1,-r}^{\mathbb{Q}} = \mathbb{H}_{\mathbb{Q}}(-1, r)$.

**Proposition 3.3** *Let $t$ be any integer and $n$, $r$ be two positive integers. Let $G_n^{r,t}$ be the $n^{th}$ generalized Pell-Pell Lucas quaternion. The norm of*

*$G_n^{r,t}$ in the quaternion algebra $\Bbbk^Q_{1,1,-r}$ has the form*

$$(i) N(G_n^{r,t}) = (r^2+2rt+r^3+14r^2t+48rt^2)P_{2n-1} + (8t^2-6r^3+2rt-72r^2t-240rt^2)P_{2n+1}. \quad (6)$$

*or*

$$(ii)\ N(G_n^{r,t}) = g_{2n}^{u,v}, where$$
$$u = r^2+7r^3+86r^2t+288rt^2$$
$$v = 8t^2-6r^3+2rt-72r^2t-240rt^2. \quad (7)$$

**Proof.** $(i)$

$$\begin{aligned} N(G_n^{r,t}) =& g_n^2+g_{n+1}^2-rg_{n+2}^2-rg_{n+2}^2 \\ =&(rP_{n-1}+tQ_n)^2+(rP_n+tQ_{n+1})^2- \\ & r(rP_{n+1}+tQ_{n+2})^2-r(rP_{n+2}+tQ_{n+3})^2 \\ =& r^2P_{n-1}^2+t^2Q_n^2+2rtP_{n-1}Q_n \\ &+(r^2P_n^2+t^2Q_{n+1}^2+2rtP_nQ_{n+1}) \\ &-r(r^2P_{n+1}^2+t^2Q_{n+2}^2+2rtP_{n+1}Q_{n+2}) \\ &-r(r^2P_{n+2}^2+t^2Q_{n+3}^2+2rtP_{n+2}Q_{n+3}) \\ =& r^2P_{n-1}^2+r^2P_n^2-r^3P_{n+1}^2-r^3P_{n+2}^2 \\ &+t^2Q_n^2+t^2Q_{n+1}^2-rt^2Q_{n+2}^2-rt^2Q_{n+3}^2 \\ &+2rtP_{n-1}Q_n+2rtP_nQ_{n+1} \\ &-2r^2tP_{n+1}Q_{n+2}-2r^2tP_{n+2}Q_{n+3} \\ =& r^2(P_{n-1}^2+P_n^2)-r^3(P_{n+1}^2+P_{n+2}^2) \\ &+t^2(Q_n^2+Q_{n+1}^2)+2rt(P_{n-1}Q_n+P_nQ_{n+1}) \\ &-2r^2t(P_{n+1}Q_{n+2}+P_{n+2}Q_{n+3}) \\ &-rt^2(Q_{n+2}^2+Q_{n+3}^2) \end{aligned}$$

Using Proposition 2.5, we obtain:

$$\begin{aligned} N(G_n^{r,t}) =& r^2P_{2n-1}+8t^2P_{2n+1} \\ &+2rt(P_{2n-1}+(-1)^n+P_{2n+1}+(-1)^{n+1}) \\ &-r^3P_{2n+3}-8rt^2P_{2n+5} \\ &-2r^2t(P_{2n+3}+(-1)^{n+2}+P_{2n+5}+(-1)^{n+3}) \\ =&(r^2+2rt)P_{2n-1}+(8t^2+2rt)P_{2n+1} \\ &+(-r^3-2r^2t)P_{2n+3}+(-8rt^2-2r^2t)P_{2n+5} \end{aligned}$$

Using Pell recurrence, we obtain:

$$P_{2n+3} = 6P_{2n+1}-P_{2n-1} \text{ and}$$
$$P_{2n+5} = 30P_{2n+1}-6P_{2n-1}$$

Thus, we conclude that

$$\begin{aligned} N(G_n^{r,t}) =&(r^2+2rt+r^3+14r^2t+48rt^2)P_{2n-1} \\ &+(8t^2-6r^3+2rt-72r^2t-240rt^2)P_{2n+1}. \end{aligned}$$

$(ii)$ according to i) we have:

$$\begin{aligned} N(G_n^{r,t}) =&(r^2+2rt+r^3+14r^2t+48rt^2)P_{2n-1} \\ &+(8t^2-6r^3+2rt-72r^2t-240rt^2)P_{2n+1}. \end{aligned}$$

Using Proposition 2.5 (13), we obtain :

$$\begin{aligned} N(G_n^{r,t}) =&(r^2+7r^3+86r^2t+288rt^2)P_{2n-1} \\ &+(8t^2-6r^3+2rt-72r^2t-240rt^2)Q_{2n} \\ =& uP_{2n-1}+vQ_{2n} \\ =& g_{2n}^{u,v}, \end{aligned}$$

where

$$u = r^2+7r^3+86r^2t+288rt^2 \text{ and}$$
$$v = 8t^2-6r^3+2rt-72r^2t-240rt^2.$$

**Proposition 3.4** *Let $n$ be an arbitrary positive integer. Let $(P_n)_{n\geq 0}$ be the Pell sequence and $(Q_n)_{n\geq 0}$ be the Pell Lucas sequence. Let $r$ be an odd prime positive integer, $r \equiv 1(mod4)$, $t$ be an arbitrary integer. Let $G_n^{r,t}$ be the $n^{th}$ generalized Pell-Pell Lucas quaternion and $\Bbbk_{1,1,-r}$ be the quaternion algebra. Then,*

$$N(G_n^{r,t}) \neq 0 \text{ for all } (n,t) \in \mathbb{N}^* \times \mathbb{N}.$$

**Proof.** From Proposition 3.3, we know that

$$\begin{aligned} N(G_n^{r,t}) =&(r^2+2rt+r^3+14r^2t+48rt^2)P_{2n-1} \\ &+(8t^2-6r^3+2rt-72r^2t-240rt^2)P_{2n+1}. \end{aligned}$$

Since $r \in \mathbb{N}^*$, it follows that $r^2+2rt+r^3+14r^2t+48rt^2 <$

$$-8t^2+6r^3-2rt+72r^2t+240rt^2.$$

Using the inequality $P_{2n-1} < P_{2n+1}$, we obtain that $N(G_n^{r,t}) < 0$ so $N(G_n^{r,t}) \neq 0$. From [23, Proposition 2.13], We are aware that the equation

$$ax = bx,\ a,b \in \mathbb{H}_{\mathbb{K}}(\alpha,\beta), \quad (8)$$

where $\mathbb{K}$ is an arbitrary field of $char(\mathbb{K}) \neq 0$, $a, b \notin \mathbb{K}$, $a \neq \bar{b}$, has the solutions of the form

$$x = \lambda[a-a_0+b-b_0]+\mu[N(a-a_0)-(a-a_0)(b-b_0)], \quad (9)$$

where, $\lambda, \mu \in \mathbb{K}$.
If $\mathbb{H}_{\mathbb{K}}(\alpha,\beta)$ is a division quaternion algebra or if $\mathbb{H}_{\mathbb{K}}(\alpha,\beta))$ is a split quaternion algebra and $N(a) \neq 0$, $N(b) \neq 0$.

**Proposition 3.5** *Let $n$ be a positive integer. Let $(P_n)_{n\geq 0}$ be the Pell sequence and $(Q_n)_{n\geq 0}$ be the Pell-Lucas sequence. Let $r$ be an odd prime positive integer, $r \equiv 1(mod4)$, $t$ be an arbitrary integer. Therefore, the centralizer of the element $G_n^{r,t} \in \mathbb{H}_{\mathbb{K}}(-1,r)$ is the set*

$$C(G_n^{r,t}) = \{G_n^{\varepsilon,\sigma}+\chi, \chi \in \mathbb{Q}\},$$

*where*

$$\varepsilon = 2\lambda r,$$
$$\sigma = 2\lambda t$$
$$\chi = g_n^{-2\lambda r,-2\lambda t} + g_{2n}^{2\mu(u-r^2-2rt),2\mu(v+\frac{r^2}{8}-t^2)} + 2\mu\varphi,$$

*with* $\lambda,\mu \in \mathbb{Q}$ *and* $\varphi = 2(-1)^{n+1}(\frac{r^2}{8}+t^2+rt)$.

**Proof.** Since

$$C(G_n^{r,t}) = \{x \in \mathbb{H}_{\mathbb{Q}}(-1,r) \mid xG_n^{r,t} = G_n^{r,t}x\},$$

using relations (8) and (9) for $a=b$, we obtain that the equation $xG_n^{r,t} = G_n^{r,t}x$ has the solutions of the form

$$x = 2\lambda[a-a_0] + 2\mu[N(a-a_0)], \ \ \lambda,\mu \in \mathbb{Q}$$

So

$$x = 2\lambda[G_n^{r,t} - g_n^{r,t}] + 2\mu[N(G_n^{r,t} - g_n^{r,t})]. \quad (10)$$

For $G_n^{r,t} = g_n^{r,t}e_0 + g_{n+1}^{r,t}e_1 + g_{n+2}^{r,t}e_2 + g_{n+3}^{r,t}e_3$, we have $N(G_n^{r,t}) = g_{2n}^{u,v}$ with $u$ and $v$ as in Proposition 3.3. From here, we have that

$$\begin{aligned} N(G_n^{r,t} - g_n^{r,t})) &= g_{2n}^{u,v} - (g_n^{r,t})^2 \\ &= g_{2n}^{u,v} - (rP_{n-1} + tQ_n)^2. \end{aligned}$$

Using Proposition 2.5, relations 6), 5) and 10), it results

$$\begin{aligned} N(G_n^{r,t} - g_n^{r,t}) &= g_{2n}^{u,v} - (rP_{n-1} + tQ_n)^2 \\ &= g_{2n}^{u,v} - r^2P_{n-1}^2 - t^2Q_n^2 \\ &\quad -2rtP_{n-1}Q_n. \end{aligned}$$

We have

$$\begin{aligned} Q_n^2 &= Q_{2n} + 2(-1)^n, \\ P_{n-1}^2 &= \tfrac{1}{8}Q_{2n-2} + 2(-1)^n), \\ P_{n-1}Q_n &= P_{2n-1} + (-1)^n, \end{aligned}$$

so

$$\begin{aligned} N(G_n^{r,t} - g_n^{r,t})) &= g_{2n}^{u,v} - r^2(\tfrac{1}{8}(Q_{2n-2} + 2(-1)^n)) \\ &\quad -t^2(Q_{2n} + 2(-1)^n) \\ &\quad -2rt(P_{2n-1} + (-1)^n) \\ &= g_{2n}^{u,v} - \tfrac{r^2}{8}Q_{2n-2} - t^2Q_{2n} \\ &\quad -2rtP_{2n-1} - \tfrac{2r^2}{8}(-1)^n \\ &\quad -2t^2(-1)^n - 2rt(-1)^n, \end{aligned}$$

we have $P_n = \frac{1}{8}(Q_{n+1} + Q_{n-1})$ so $Q_{2n-2} = 8P_{2n-1} - Q_{2n}$, we obtain:

$$\begin{aligned} N(G_n^{r,t} - g_n^{r,t}) &= g_{2n}^{u,v} - \tfrac{r^2}{8}(8P_{2n-1} - Q_{2n}) - t^2Q_{2n} \\ &\quad -2rtP_{2n-1} + 2(-1)^{n+1}(\tfrac{r^2}{8} + t^2 + rt) \\ &= g_{2n}^{u,v} - r^2P_{2n-1} + \tfrac{r^2}{8}Q_{2n} - t^2Q_{2n} \\ &\quad -2rtP_{2n-1} + 2(-1)^{n+1}(\tfrac{r^2}{8} + t^2 + rt) \\ &= g_{2n}^{u,v} + (-r^2 - 2rt)P_{2n-1} + (\tfrac{r^2}{8} - t^2)Q_{2n} \\ &\quad +2(-1)^{n+1}(\tfrac{r^2}{8} + t^2 + rt) \\ &= g_{2n}^{u,v} + g_{2n}^{-r^2-2rt,\frac{r^2}{8}-t^2} \\ &\quad +2(-1)^{n+1}(\tfrac{r^2}{8} + t^2 + rt) \\ &= g_{2n}^{u-r^2-2rt,v+\frac{r^2}{8}-t^2} + \varphi, \end{aligned}$$

where $\varphi = 2(-1)^{n+1}(\frac{r^2}{8} + t^2 + rt)$.
Using relation (10), we obtain

$$\begin{aligned} x &= 2\lambda[G_n^{r,t} - g_n^{r,t}] + 2\mu[g_{2n}^{u-r^2-2rt,v+\frac{r^2}{8}-t^2} + \varphi] \\ &= 2\lambda[G_n^{r,t} - g_n^{r,t}] + 2\mu[g_{2n}^{u-r^2-2rt,v+\frac{r^2}{8}-t^2} + \varphi] \\ &= G_n^{2\lambda r,2\lambda t} + g_n^{-2\lambda r,-2\lambda t} \\ &\quad + g_{2n}^{2\mu(u-r^2-2rt),2\mu(v+\frac{r^2}{8}-t^2)} + 2\mu\varphi. \end{aligned}$$

## 4 Conclusions

In this study we introduced a special set of elements, called Pell and Pell-Lucas quaternions, and showed that this set is an order of the quaternion algebra $\mathbb{k}_{\lambda_1,\lambda_2,\lambda_3}$ in the sense of ring theory. The determination of all properties of this algebra, as well as the circumstances under which it is a split algebra or a division algebra, will be highly intriguing, especially in the practical field. More precisely, the applications of Bell-Lucas quaternions are not as extensive as the applications of more familiar mathematical concepts such as complex numbers or quaternions themselves. Therefore, we will try to focus future work on studying the applied aspect of Bell-Lucas quaternions by addressing the following future works:

1)- Cryptographic algorithms,
2)- Image processing.

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