Dejdumrong Collocation Approach and Operational Matrix for a Class of Second-Order Delay IVPs: Error Analysis and Applications

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Abstract: In this paper, a collocation method based on the Dejdumrong polynomial matrix approach was used to estimate the solution of higher-order pantograph-type linear functional differential equations. The equations are considered with hybrid proportional and variable delays. The proposed method transforms the functional-type differential equations into matrix form. The matrices were converted into a system of algebraic equations containing the Dejdumrong polynomial. The coefficients of the Dejdumrong polynomial were obtained by solving the system of algebraic equations. Moreover, the error analysis is performed, and the residual improvement technique is presented. The presented methods are applied to three examples. Finally, the obtained results are compared with the results of other methods in the literature and were found to be better compared. All results in this study have been calculated using Matlab R2021a.

Key-Words: - Dejdumrong polynomial; Functional differential equations; Numerical solutions; Proportional and variable delays; Residual error analysis.

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1 Introduction

A multitude of physical phenomena cannot be sufficiently accounted for ordinary differential equations (ODEs) when the employed model constructed bases on a particular previous state in addition to its present state. As a consequence, DEs with time delays are utilized to model real-world scenarios for instance chemical engineering, electric circuits, fluid mechanics, human body control systems, multibody control systems, stage-structured populations, spread of bacteriophage infection, the dynamic diseases model in physiology, and the epidemic model in biology [1], [2], [3], [4], [5], [6], [7], [8].

Numerous numerical and analytical methodologies were developed in order to address nonlinear differential equations (NDEs) which involve proportional and unchanged delays. Some notable methods include the Aboodh transformation method [9], spectral method [10], residual power series method [11], Adomian decomposition method [12], [13], polynomial least squares approach [14], reproducing kernal Hilbert space [15], variable multistep techniques [16], differential transform method and its reduction [17], [18], the cuckoo optimization algorithm [19], Hermite wavelet-based approach [20], variational iteration approach [21], [22], [23], quasilinearization technique [24], expanded ansatz method [25], generalized Riccati equation mapping method [26], generic algorithm [27], and Kudryashov modified simplest equation method [28].

In contrast, NDEs involving variable delays have been the subject of relatively few studies. [29] have conducted research on the presence of positive solutions that repeat every ω-periodic. Asymptotic stability was evaluated using new criteria that were proposed in [30]. [31] and [32] have investigated the asymptotic behavior of solutions. The analysis of fixed points and stability is conducted in [33], [34], [35], [36] and included bibliography. There are only a few numerical strategies that have been used in order to solve equations of this kind. These techniques include a novel multi-step strategy [37], the Legendre-Gauss collocation approach [38], and the Runge-Kutta method employing Hermite interpolation [39].

There is significant interest in finding numerical solutions for ODEs, fractional differential equations (FDEs), and integro-differential equations. Recently, researchers have proposed methods for solving FDEs that are based on Pell-Lucas, Fibonacci, and Fermat polynomials [40], [41], [42], [43], [44], [45], [46], [47], [48]. Researchers were able to develop the operational matrix of fractional derivatives via these investigations. They also made the observation that the numerical solutions had less errors compared to the ones that were acquired through the use of orthogonal polynomials.

Investitigang the analytical solutions of nonlinear differential equations with delay variables is more challenging in comparison to linear differential equations. Another difference between linear and nonlinear differential equations is that the former often do not have analytical solutions. Consequently, numerical methods assume a critical role in the solution of these nonlinear differential equations.

Lately, there has been a significant emphasis among researchers on finding numerical solutions for nonlinear differential equations. We provide our precise numerical solutions, which exhibit increasing accuracy. It is worth mentioning that we may anticipate that a small number of Dejdamrong polynomial basis functions will be enough to get an approximate solution that closely matches the precise solution with a precision of up to 10 digits. Dejdamrong polynomial basis functions are utilized for the first time, to the best of our knowledge.

With regard to the current investigation, we take into consideration the NDE having variable delays of this kind

$$\begin{aligned}&\sum_{m=0}^{2}\sum_{n=0}^{1} P_{mn}(\nu)\chi^{(m)}(\nu-\hbar_{mn}(\nu))+\\&\sum_{r=0}^{2}\sum_{s=0}^{r} Q_{rs}(\nu)\chi^{(r)}(\nu)\chi^{(s)}(\nu)=h(\nu).\end{aligned} \tag{1}$$

subject to the given initial conditions (ICs)

$$\chi(a)=\eta_1 \quad \text{and} \quad \chi'(b)=\eta_2, \tag{2}$$

where $\chi(\nu)$ is the unknown function, $P_{mn}(\nu), Q_{rs}(\nu)$ and $h(\nu)$ are assumed to be continuous on the given domain $0 \leq a \leq \nu \leq b$. Also, the variable delays $\hbar_{mn}(\nu)$ are assumed to be so. Determining approximate solution to the considered problem Eq.(1)-Eq.(2) using the Dejdumrong collocation method is the objective of this analysis.

2 Dejdumrong Polynomial Representation

When dealing with polynomials of degree m, one may express them directly as [49], [50], [51], [52]:

$$\mathcal{D}_i^m(\nu)=\begin{cases}(3\nu)^i(1-\nu)^{i+3},\\ \text{for} \quad 0\leq i<\lceil 2^{-1}m\rceil-1,\\ (3\nu)^i(1-\nu)^{m-i},\\ \text{if} \quad i=\lceil 2^{-1}m\rceil-1,\\ 2\cdot 3^{i-1}(1-\nu)^i\nu^i,\\ \text{if} \quad i \quad \text{is even and} \quad i=2^{-1}m,\\ \mathcal{D}_{m-i}^m(1-\nu),\\ \text{for} \quad \lfloor 2^{-1}m\rfloor+1\leq i\leq m.\end{cases} \tag{3}$$

Definition 1. *The monomial matrix of Dejdumrong is denoted by [52]*

$$\mathcal{N}=\begin{bmatrix} n_{00} & n_{01} & \cdots & \cdots & n_{0m}\\ n_{10} & n_{11} & \cdots & \cdots & n_{1m}\\ \vdots & \vdots & \ddots & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ n_{m0} & n_{m1} & \cdots & \cdots & n_{mm}\end{bmatrix}_{(m+1)\times(m+1)} \tag{4}$$

where n_{rs} is given as:

$$n_{rs}=\begin{cases}(-1)^{(s-r)}3^r\begin{pmatrix} r+3\\ s-r\end{pmatrix},\\ \text{for } 0\leq r\leq\lceil 2^{-1}m\rceil-1,\\ (-1)^{(s-r)}3^r\begin{pmatrix} m-r\\ s-r\end{pmatrix},\\ \text{for r}=\lceil 2^{-1}m\rceil-1,\\ (-1)^{(s-r)}2(3^{r-1})\begin{pmatrix} r\\ s-r\end{pmatrix},\\ \text{for r}=2^{-1}\,m \text{ and } m \text{ is even},\\ (-1)^{(s-r)}3^{m-r}\begin{pmatrix} m-r\\ s-r\end{pmatrix},\\ \text{for r}=\lfloor 2^{-1}m\rfloor+1,\\ (-1)^{(s-m+r-1)}3^{m-k}\begin{pmatrix} m-r\\ s-m+r-3\end{pmatrix},\\ \text{for } \lfloor 2^{-1}m\rfloor+1\leq r\leq m.\end{cases} \tag{5}$$

with $\lfloor\frac{\nu}{2}\rfloor$ represents $GI \leq \nu$ and $\lceil\frac{\nu}{2}\rceil$ represents $LI \geq \nu$, where GI is the greatest integer and LI is smallest integer.

The following properties are satisfied by the Dejdumrong basis function, [53], [54], [55]:

1. The basis function of Dejdumrong is non-negative, which might be interpreted as,

$$\mathcal{D}_i^m(\nu)\geq 0, \forall i=0,1,\cdots,m.$$

2. The partitioning of unity, which is,

$$\sum_{i=0}^{m} \mathcal{D}_i^m(\nu) = 1.$$

3 Basic Matrix Relations

In this subsection, we use the Dejdumrong polynomial to depict the matrix representations of the problems Eq.(1)-Eq.(2).

Lemma 1. *One possible representation of the vector $\mathcal{D}_N(\nu)$ is as follows:*

$$\mathcal{D}_N(\nu) = T(\nu)\mathcal{N}_N^T, \tag{6}$$

where $T(\nu) = \begin{bmatrix} 1 & \nu & \nu^2 \cdots \nu^N \end{bmatrix}$ and $\mathcal{N}_N^T$ is given in Eq.(4).

Proof. By multiplying the vector $T(\nu)$ by the matrix $\mathcal{N}_N^T$ from the right side, the vector $\mathcal{D}_N(\nu) = T(\nu)\mathcal{N}_N^T$ is acquired. □

Lemma 2. *The form that may be used to represent the approximated solution based on the Dejdumrong polynomial in Eq.* (1) *is as follows:*

$$\chi(\nu) \cong \chi_N(\nu) = T(\nu)\mathcal{N}_N^T A_N, \tag{7}$$

where $A_N = [\ a_0 \quad a_1 \quad \cdots \quad a_N\ ]^T$.

Proof. By multiplying the vector $\mathcal{D}_N(\nu) = T(\nu)\mathcal{N}_N^T$ by the vector A_N from the right, the relation Eq.(7) is found. □

Lemma 3. *The matrix relations for the k^{th} derivatives of the solution form Eq.*(7) *are respectively as follows*

$$\chi^{(k)}(\nu) \cong \chi_N^{(k)}(\nu) = T(\nu)\Lambda^k \mathcal{N}_N^T A_N, \tag{8}$$

where,

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

.

Proof. When the k^{th} derivative of Eq.(7) are taken, we have

$$\chi^{(k)}(\nu) \cong \chi^{(k)}{}_N(\nu) = T^{(k)}(\nu)\mathcal{N}_N^T A_N, \tag{9}$$

Next, the k^{th} derivative of $T(\nu)$ are taken to obtain

$$T^{(k)}(\nu) = T(\nu)\Lambda^k. \tag{10}$$

Therefore, by substituting the corresponding values of Eq.(9) with the matrix relations Eq.(10) for the k^{th} derivative of the solution form Eq.(7)we get the desired results. □

Lemma 4. *The corresponding matrix relations for the proportional delay of the derivatives of the solution form Eq.*(7) *are as follows:*

$$\chi^{(k)}(\nu - \overline{h}_{mn}(\nu)) \cong \chi^{(k)}{}_N(\nu - \overline{h}_{mn}(\nu)) = T(\nu)\Omega_N(-\overline{h}_{mn}(\nu))\Lambda^k \mathcal{N}_N^T A_N,$$

where

$$\Omega_N(\eta) = \begin{bmatrix} \binom{0}{0}(-\overline{h}_{mn}(\nu))^0 & \binom{1}{0}(-\overline{h}_{mn}(\nu))^1 & \cdots & \binom{N}{0}(-\overline{h}_{mn}(\nu))^N \\ 0 & \binom{1}{1}(-\overline{h}_{mn}(\nu))^0 & \cdots & \binom{N}{1}(-\overline{h}_{mn}(\nu))^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(-\overline{h}_{mn}(\nu))^0 \end{bmatrix}$$

.

Proof. If $\nu - \overline{h}_{mn}(\nu)$ is written instead of ν in Eq.(8), then it is achieved

$$\chi^{(k)}(\nu - \overline{h}_{mn}(\nu)) \cong \chi_N^{(k)}(\nu - \overline{h}_{mn}(\nu)) = T(\nu)\Omega_N(-\overline{h}_{mn}(\nu))\Lambda^k \mathcal{N}_N^T A_N, \tag{11}$$

By multiplying the vector $T(\nu - \overline{h}_{mn}(\nu))$ by the vector $\Omega_N(-\overline{h}_{mn}(\nu))$ from the right side, on the other hand, gives us the result of

$$T_N(\nu - \overline{h}_{mn}(\nu)) = T_N(\nu)\Omega_N(-\overline{h}_{mn}(\nu)).$$

In addition, by using Eq. (8) we can obtain the matrix forms of $\chi^{(r)}(\nu)\chi^{(s)}(\nu)$ as:

$$\chi^{(r)}(\nu)\chi^{(s)}(\nu) = \overline{T}(\nu)\overline{\mathcal{N}_N^T}\overline{\Lambda^r}\,\overline{A}_N T(\nu)\Lambda^s \mathcal{N}_N^T. \tag{12}$$

where

$$\overline{T}(\nu)=\begin{bmatrix} T(\nu) & 0 & \cdots & 0 \\ 0 & T(\nu) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T(\nu) \end{bmatrix},$$

$$\overline{\mathcal{N}_N^T}=\begin{bmatrix} \mathcal{N}_N^T & 0 & \cdots & 0 \\ 0 & \mathcal{N}_N^T & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{N}_N^T \end{bmatrix},$$

$$\overline{V^r}=\begin{bmatrix} V^r & 0 & \cdots & 0 \\ 0 & V^r & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V^r \end{bmatrix},$$

$$\overline{A_N}=\begin{bmatrix} A_N & 0 & \cdots & 0 \\ 0 & A_N & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N \end{bmatrix}.$$

By substituting Eq.(11) and Eq.(12) in Eq.(1) yields

$$\begin{aligned}&\sum_{m=0}^{2}\sum_{n=0}^{1} P_{mn}(\nu)T(\nu)\Omega_N(-\hbar_{mn}(\nu))\Lambda^k\mathcal{N}_N^T A_N+\\&\sum_{r=0}^{2}\sum_{s=0}^{r} Q_{rs}(\nu)\overline{T}(\nu)\overline{\mathcal{N}_N^T}\overline{\Lambda^r A_N}T(\nu)\Lambda^s\mathcal{N}_N^T A_N=\\&H(\nu)\end{aligned} \tag{13}$$

As a result, the following set of adjustable collocation points is necessary to solve Eq.(13)

$$\nu_k=\frac{1}{2}-\frac{1}{2}\cos\left(\frac{k\pi}{N}\right), k=0,1,2,\ldots,N. \tag{14}$$

When these collocation points are substituted into the Eq.(13) one may derive as

$$\begin{aligned}&\sum_{m=0}^{2}\sum_{n=0}^{1} P_{mn}(\nu_k)T(\nu_k)\Omega_N(-\hbar_{mn}(\nu_k))\Lambda^k\mathcal{N}_N^T A_N+\\&\sum_{r=0}^{2}\sum_{s=0}^{r} Q_{rs}(\nu_k)\overline{T}(\nu_k)\overline{\mathcal{N}_N^T}\overline{\Lambda^r A_N}T(\nu_k)\Lambda^s\mathcal{N}_N^T A_N=\\&H(\nu_k), k=0,1,...,N.\end{aligned}$$

Or simply,

$$\begin{aligned}&\sum_{m=0}^{2}\sum_{n=0}^{1} P_{mn}T\Omega_N\Lambda^k\mathcal{N}_N^T A_N+\\&\sum_{r=0}^{2}\sum_{s=0}^{r} Q_{rs}\overline{TV^r}\overline{\mathcal{N}_N^T}\overline{A_N}TV^s\mathcal{N}_N^T A_N=\\&H, k=0,1,...,N.\end{aligned} \tag{15}$$

where

$$P_{mn}=\begin{bmatrix} P_{mn}(\nu_0) & 0 & \cdots & 0 \\ 0 & P_{mn}(\nu_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{mn}(\nu_N) \end{bmatrix},$$

$$Q_{rs}=\begin{bmatrix} Q_{rs}(\nu_0) & 0 & \cdots & 0 \\ 0 & Q_{rs}(\nu_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{rs}(\nu_N) \end{bmatrix},$$

$$H=\begin{bmatrix} h(\nu_0) \\ h(\nu_1) \\ \vdots \\ h(\nu_N) \end{bmatrix}, A_N=\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}, \overline{A_N}=\begin{bmatrix} A_N \\ A_N \\ \vdots \\ A_N \end{bmatrix},$$

$$\Omega_N=\begin{bmatrix} \Omega_N(-\hbar_{mn}(\nu_0)) & 0 & \cdots & 0 \\ 0 & \Omega_N(-\hbar_{mn}((\nu_1)) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Omega_N(-\hbar_{mn}(\nu_N)) \end{bmatrix}$$

$$T=\begin{bmatrix} T(\nu_0) \\ T(\nu_1) \\ \vdots \\ T(\nu_N) \end{bmatrix}, \overline{\mathcal{N}_N^T}=\begin{bmatrix} \mathcal{N}_N^T & 0 & \cdots & 0 \\ 0 & \mathcal{N}_N^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{N}_N^T \end{bmatrix}$$

$$\text{and}\quad \overline{T}=\begin{bmatrix} T(\nu_0) & 0 & \cdots & 0 \\ 0 & T(\nu_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T(\nu_N) \end{bmatrix}.$$

By using Eq. (33) in Eq. (32) we obtain

$$WA=H \quad \text{or} \quad [W;H], \tag{16}$$

where

$$\begin{aligned}W=&\sum_{m=0}^{2}\sum_{n=0}^{1} P_{mn}T\Omega_N\Lambda^k\mathcal{N}_N^T A_N+\\&\sum_{r=0}^{2}\sum_{s=0}^{r} Q_{rs}\overline{T\Lambda^r}\overline{\mathcal{N}_N^T}\overline{A_N}T\Lambda^s\mathcal{N}_N^T A_N.\end{aligned}$$

The initial form of system Eq.(2) is provided using the matrix for

$$\begin{aligned}&U_1=\chi(a)=\eta_1, \text{or} T(a)\mathcal{N}_N A=\eta_1,\\&U_2=\chi'(b)=\eta_2, \text{or} T(b)\Lambda\mathcal{N}_N A=\eta_2.\end{aligned} \tag{17}$$

Finally, the ICs mentioned in equation Eq.(17) have been substituted in the final three rows using the augmented matrix structure given in equation Eq.(16). This process will provide a new augmented form as:

$$\widetilde{W}A=\widetilde{H}\text{or}$$

$$\left[\widetilde{W};\widetilde{H}\right]=\left[\begin{array}{ccccccc} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & ; & h(\tau_0)\\ w_{1,0} & w_{1,2} & \cdots & w_{1,N} & ; & h(\tau_1)\\ \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N} & ; & h(\tau_{N-2})\\ U_{1,0} & U_{1,1} & \cdots & U_{1,N} & ; & \eta_1\\ U_{2,0} & U_{2,1} & \cdots & U_{2,N} & ; & \eta_2 \end{array}\right].$$

If rank $\widetilde{W}$ = rank $\left[\widetilde{W};\widetilde{H}\right]=N+1$, then it could be conclude that $A_N=\left(\widetilde{W}\right)^{-1}\widetilde{H}$ where the coefficient matrix of the Dejdumrong polynomial Eq.(7) is denoted by A_N. Consequently, the solution to Eq.(1) has been found. □

4 Errors Analysis

In this part, we will give the error analysis that was performed on the method that was employed. The issue will be given a residual correction process, which will attempt to provide an estimate of the absolute inaccuracy.

Let $\chi_N(\nu)$and$\chi(\nu)$ represent, respectively, the approximate and exact solutions to Eq.(1). For the purpose of estimating the error analysis, the subsequent process known as residual correction might be used. First, let's do some addition and subtraction on the term

$$\Re_N=\sum_{m=0}^{2}\sum_{n=0}^{1}P_{mn}(\nu)\chi_N^{(m)}(\nu-\hbar_{mn}(\nu))+\sum_{r=0}^{2}\sum_{s=0}^{r}Q_{rs}(\nu)\chi_N^{(r)}(\nu)\chi_N^{(s)}(\nu)-h(\nu)$$

to Eq.(1) produce the subsequent differential equation

$$\sum_{m=0}^{2}\sum_{n=0}^{1}P_{mn}(\nu)e_N^{(m)}(\nu-\hbar_{mn}(\nu))+\sum_{r=0}^{2}\sum_{s=0}^{r}Q_{rs}(\nu)e_N^{(r)}(\nu)e_N^{(s)}(\nu)=h(\nu)-\Re_N, \tag{18}$$

with the ICs

$$e_N(0)=0,\frac{de_N(0)}{d\nu}=0.$$

where $e_N=\chi(\nu)-\chi_N(\nu)$. For a given value M let $e^{\bullet}(\nu)$ be the approximate solution of Eq.(18), where is a polynomials degree of Eq.(18) and $M\geq N$.

Theorem 1. *Assume that $\chi_N(\nu)$ represents the approximate solution of equation* (1) *whereas $e^{\bullet}(\nu)$ is the approximate solution of equation* (18). *Furthermore, $\chi_N(\nu)+e^{\bullet}(\nu)$ may be considered an approximation solution to equation* (1) *with an error function denoted as $e_N(\nu)-e^{\bullet}(\nu)$.*

We term the approximate solution $\chi_N(\nu)+e^{\bullet}(\nu)$ as the corrected approximate solution. Note that if $\|e_N(\nu)-e_M^{\bullet}\|<\varepsilon$, thereafter, $e_M^{\bullet}$.might be used to estimate the AE. Moreover, if $\|e_N(\nu)-e_M^{\bullet}\|<\|\chi(\nu)-\chi_N(\nu)\|$, then $\chi_N(\nu)+e_M^{\bullet}$ is a more accurate solution than $\chi_N(\nu)$ in any given norm.

Theorem 2. *Consider $\chi(\nu)$ and $\chi_N(\nu)=T(\nu)\mathcal{N}_N^TA_N$ represent the exact solution and the Dejdumrong polynomial of the Eq.*(1) *with a given degree N. Furthermore, we make the assumption that $\chi_{NM}(\nu)=T(\nu)\widehat{A}$ is the expansion of the generalised Maclaurin series [56] of $\chi_N(\nu)$ with degree of N. Consequently, the AE of the polynomial solution $\chi_{NM}(\nu)$ for Dejdumrong polynomial solution is constrained as*

$$\|\chi(\nu)-\chi_N(\nu)\|_\infty\leq\frac{\chi^{(N+1)}(0)}{(N+1)!}\left\|\chi^{(N+1)}(\zeta)\right\|+\kappa_N\left(\left\|\widehat{A}\right\|_\infty+\|\mathcal{N}_N^T\|_\infty\|A\|_\infty\right),\quad\nu\in[0,b].$$

Proof. To begin, for proving the above theorem, we use the same procedure as in [56]. we may extract the following formula from the Maclaurin expansion $\chi_{NM}(\nu)$ with degrees N by adding and subtracting from the triangle inequality:

$$\begin{aligned}\|\chi(\nu)-\chi_N(\nu)\|_\infty &= \|\chi(\nu)-\chi_{NM}(\nu)+\chi_{NM}(\nu)-\chi_N(\nu)\|_\infty\\ &\leq \|\chi(\nu)-\chi_{NM}(\nu)\|_\infty+\|\chi_{NM}(\nu)-\chi_N(\nu)\|_\infty.\end{aligned}\tag{19}$$

From Eq.(6), the Dejdumrong polynomial solution $\chi_N(\nu)=\mathcal{D}_N(\nu)A$ is possible to be expressed using the matrix form $\chi_N(\nu)=T(\nu)\mathcal{N}_N^TA_N$ and $\chi_{NM}(\nu)=T(\nu)\widehat{A}$ is the truncated Maclaurin series of $\chi(\nu)$ having degree N, we can write

$$\begin{aligned}\|\chi_{NM}(\nu)-\chi_N(\nu)\|_\infty &= \left\|T(\nu)\left(\widehat{A}-\mathcal{N}_N^TA\right)\right\|_\infty\\ &\leq \|T(\nu)\|_\infty\left(\left\|\widehat{A}\right\|_\infty+\|\mathcal{N}_N^T\|_\infty\|A\|_\infty\right),\\ &\quad\nu\in[0,b].\end{aligned}\tag{20}$$

Since $\nu\in[0,b]$, , then the inequality will be given as $\|T(\nu)\|_\infty\leq\max\left\{b^N,1\right\}=\kappa_N$. Therefore, we are able to arrange Equation Eq.(20) as

$$\|\chi_{NM}(\nu)-\chi_N(\nu)\|_\infty \leq \kappa_N\left(\left\|\widehat{A}\right\|_\infty + \|\mathcal{N}_N^T\|_\infty \|A\|_\infty\right). \tag{21}$$

Conversely, it is understood that the residual term of the Maclaurin polynomial $\chi_{NM}(\nu)$, which has a degree of N is $\sum\limits_{n=N+1}^{\infty} \frac{\chi^{(n)}(0)}{n!}\nu^n$ So, we can write

$$\|\chi(\nu)-\chi_{NM}(\nu)\|_\infty \leq \left|\sum_{n=N+1}^{\infty} \frac{\chi^{(n)}(0)}{n!}\nu^n\right|, \quad \nu\in[0,b]. \tag{22}$$

Then, by using Eq.(19), Eq.(21) and Eq.(22), we obtain

$$\|\chi(\nu)-\chi_N(\nu)\|_\infty \leq \left|\sum_{n=N+1}^{\infty} \frac{\chi^{(n)}(0)}{n!}\tau^n\right| + \kappa_N\left(\left\|\widehat{A}\right\|_\infty + \|\mathcal{N}_N^T\|_\infty \|A\|_\infty\right), \nu\in[0,b]. \tag{23}$$

Given the existence of $\zeta \in (0,b)$ such that $\sum\limits_{n=N+1}^{\infty} \frac{\chi^{(n)}(0)}{n!}\nu^n = \frac{\nu^{N+1}}{(N+1)!}u^{N+1}(\zeta), \nu\in[0,b]$, in the In the residual term of Taylor's Theorem, the inequality (23) may be represented as

$$\|\chi(\nu)-\chi_N(\nu)\|_\infty \leq \frac{\chi^{(N+1)}(0)}{(N+1)!}\left\|\chi^{(N+1)}(\zeta)\right\| + \kappa_N\left(\left\|\widehat{A}\right\|_\infty + \|\mathcal{N}_N^T\|_\infty \|A\|_\infty\right), \nu\in[0,b]. \tag{24}$$

Hence, it may be concluded that the proof of the theorem has been completed. □

Theorem 3. *The Eq.*(25) *provides the convergence condition of the Dejdumrong polynomial solution* $\chi_N(\nu)=T(\nu)\mathcal{N}_N^T A_N$ *under the supposition that the maximal error in the interval* $0\leq\nu\leq b$ *is, in fact, equal to the upper bound Eq.*(25) *which is defined in Theorem 2*

$$\Delta\widetilde{A}_N + \|\mathcal{N}_N^T\|_\infty \Delta A_N < \frac{b^{N+1}}{k_N(N+1)!}\chi^{N+1}(0), \tag{25}$$

where $\chi_N(\tau)$ *is the Dejdumrong polynomial solution, and its coefficient matrix is represented by* A_N*. The coefficient matrix* $\widetilde{A}_N$ *represents the coefficients in the generalized Maclaurin polynomial of* $\chi(\tau)$ *with degrees* $N, \kappa_N=\max\{1,b^N\}$ *. The delta operator is defined as:* $\kappa_N=\max\{1,b^N\}$ *degree* $\widetilde{A}_N$*, while the definition of the delta operator is*

$$\Delta\widetilde{A}_N = \left\|\widetilde{A}_{N+1}\right\|_\infty - \left\|\widetilde{A}_N\right\|_\infty.$$

Proof. The same approach is used to establish the above theorem as in [56].The hypothesis of the theorem supposes that the maximum error is equivalent to its upper bound, which is stated in Theorem 2 Based on the results of Theorem 2, the maximum errors for $\chi_N(\nu)$ and $\chi_{N+1}(\nu)$ may be stated as

$$E_N^{\max} = \left|\sum_{n=N+1}^{\infty} \frac{\chi^{(n)}(0)}{n!}b^n\right| + \kappa_N\left(\left\|\widehat{A}_N\right\|_\infty + \|\mathcal{N}_N^T\|_\infty \|A_N\|_\infty\right)$$

and

$$E_{N+1}^{\max} = \left|\sum_{n=N+2}^{\infty} \frac{\chi^{(n)}(0)}{n!}b^n\right| + \kappa_{N+1}\left(\left\|\widehat{A}_{N+1}\right\|_\infty + \|\mathcal{N}_{N+1}^T\|_\infty \|A_{N+1}\|_\infty\right).$$

To ensure that the solution $\chi_N(\nu)$ converges, we want to identify the condition under which $E_{N+1}^{\max} < E_N^{\max}$ holds. Subsequently, we have the ability to write

$$E_{N+1}^{\max} - E_N^{\max} = -\left|\frac{\chi^{(N)}(0)}{N!}b^N\right| + \kappa_{N+1}\left(\left\|\widehat{A}_{N+1}\right\|_\infty + \|\mathcal{N}_{N+1}^T\|_\infty \|A_{N+1}\|_\infty\right)\|\mathcal{N}_{N+1}^T\|_\infty \|A_{N+1}\|_\infty\Big) - \kappa_N\left(\left\|\widehat{A}_N\right\|_\infty + \|\mathcal{N}_N^T\|_\infty \|A_N\|_\infty\right) < 0.$$

We are also aware that $\kappa_N < \kappa_{N+1}$. Then we get

$$\kappa_N\left(\left\|\widehat{A}_{N+1}\right\|_\infty + \|\mathcal{N}_{N+1}^T\|_\infty \|A_{N+1}\|_\infty - \left(\left\|\widehat{A}_N\right\|_\infty + \|\mathcal{N}_N^T\|_\infty \|A_N\|_\infty\right)\right) < \left|\frac{\chi^{(N+1)}(0)}{(N+1)!}b^{N+1}\right|.$$

Based on the fact that $\|\mathcal{N}_N^T\|_\infty < \|\mathcal{N}_{N+1}^T\|_\infty$, here is an example of an inequality that we may establish:

$$\kappa_N\left(\left\|\widehat{A}_{N+1}\right\|_\infty - \left\|\widehat{A}_N\right\|_\infty + \|\mathcal{N}_{N+1}\|_\infty(\|A_{N+1}\|_\infty - \|A_N\|_\infty)\right) < \frac{\chi^{(N+1)}(0)}{(N+1)!}b^{N+1}.$$

Here, by utilizing the operators $\Delta A_N = \|A_{N+1}\|_\infty - \|A_N\|_\infty$. and $\Delta \widetilde{A}_N = \left\|\widetilde{A}_{N+1}\right\|_\infty - \left\|\widetilde{A}_N\right\|_\infty$, we obtain

$$\Delta \widetilde{A}_N + \|\mathcal{N}_{N+1}^T\|_\infty \Delta A_N < \frac{\chi^{(N+1)}(0)}{\kappa_N (N+1)!} b^{N+1}.$$

Consequently, it could be noted that the proof has been completed. □

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5 Application

All the approaches discussed in Section3 and Section4 are now being evaluated using three different examples. Both tables and graphs are used to display the findings that were obtained. Additionally, comparisons are done with other methodologies that have been published in the literature. All of the results were computed with the help of MATLAB R2021a.

In this study, the symbol $\chi(\nu)$ denotes the exact solution, $\chi_N(\nu)$ corresponds to the Dejdumrong polynomial solution, $\chi_{N,M}(\nu)$ denotes the enhanced estimated solution, $|e_N(\nu)|$ symbolizes the function of the real error analysis, $|e_{N,M}(\nu)|$ stands for the function of the estimated AE and $|E_{N,M}(\nu)|$ symbolizes the function of the enhanced absolute error.

Example 1. We start by thinking about the nonlinear differential equation of second order with variable delays ν^3, ν^2 and $\frac{\nu}{2}$ [53],[54]

$$\chi''(\nu-\nu^3)+\chi(\nu-\nu^2)-\chi(\nu+\frac{\nu}{2})-\nu\big(\chi'(\nu)\big)^2 = h(\nu), \tag{26}$$

with the ICs

$$\chi(0) = 1, \chi'(0) = 1. \tag{27}$$

where $h(\nu) = \frac{(\nu-\nu^2)^2}{2} - \nu(1+\nu)^2 - \frac{\nu}{2} - \frac{17}{8}\nu^2 + 1$. The exact solution of Eq.(26) under the conditions Eq.(27) is $\chi(\nu) = \frac{\nu^2}{2} + \nu + 1$.
Now, let us investigate the solution of the equations Eq.(26)-Eq.(27) for the case when $N = 2$ in the form

$$\chi_2(\nu) = \sum_{r=0}^{2} a_i \mathcal{D}_N(\nu). \tag{28}$$

For $N = 2$, the collocation points are $\nu_0 = 0, \nu_1 = \frac{1}{2}$ and $\nu = 1$. According to the method in Section 3, from Eq.(15) the fundamental matrix equation becomes

$$T_1 \Lambda^2 \mathcal{N}_N^T + T_2 \mathcal{N}_N^T - T_3 \mathcal{N}_N^T - N_0 \overline{T}\,\overline{\mathcal{N}_N^T}\,\overline{A} T \mathcal{N}_N^T = H$$

where

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3/8 & 9/64 \\ 1 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/4 & 1/16 \\ 1 & 0 & 0 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3/4 & 9/16 \\ 1 & 3/2 & 9/4 \end{bmatrix}, \mathcal{N}^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix},$$

$$\Lambda^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Q_{00} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\overline{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\overline{\mathcal{N}_N^T} = \begin{bmatrix} \mathcal{N}_N^T & 0 & 0 \\ 0 & \mathcal{N}_N^T & 0 \\ 0 & 0 & \mathcal{N}_N^T \end{bmatrix}, A_N = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} A_N & 0 & 0 \\ 0 & A_N & 0 \\ 0 & 0 & A_N \end{bmatrix}, H = \begin{bmatrix} 1 \\ -7/8 \\ -45/8 \end{bmatrix}.$$

Therefore the augmented matrix is

$$[W;H] = \left[\begin{array}{ccc;c} 2 & -4 & 2 & 1 \\ -\frac{a_0}{16} - \frac{a_1}{8} - \frac{a_2}{16} - 4 & \frac{5}{2} - \frac{a_1}{16} - \frac{a_2}{32} - \frac{a_0}{32} & \frac{3}{2} - \frac{a_1}{16} - \frac{a_2}{32} - \frac{a_0}{32} & -\frac{7}{8} \\ -\frac{5}{2} & \frac{11}{4} & -a_2 - \frac{1}{4} & -\frac{45}{8} \end{array}\right] \tag{29}$$

Also, the matrix representations of the conditions Eq.(27)are as follows:

$$U_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, U_1 = \begin{bmatrix} -2 & 2 & 0 \end{bmatrix}.$$

Replace the first and the last rows of Eq.(29) we obtain

$$[\overline{W};\overline{H}] = \left[\begin{array}{ccc;c} 1 & 0 & 0 & 1 \\ \frac{a_2}{2} - \frac{a_0}{2} + \frac{5}{2} & -4 & \frac{a_0}{2} - \frac{a_2}{2} + \frac{3}{2} & -\frac{7}{8} \\ -2 & 2 & 0 & 1 \end{array}\right] \tag{30}$$

The Matlab program was used to solve the obtained system and so the Dejdumrong coefficients matrix have been calculated as $A = [1, 3/2, 5/2]^T$. By substituting this coefficients matrix A in Eq. (28), we get the approximate solution as $\chi(\nu) = 0.5\nu^2 + \nu + 1.0$, considered to be the exact solution.

Example 2 For the second example, let us take the second-order nonlinear differential equation with variable delay $\nu - \nu^3/8$, [53],[54]

$$\chi''(\nu) + \chi(\nu^3/8) + 2\chi(\nu) - \chi^2(\nu) = h(\nu), \nu \in [0, 1] \quad (31)$$

and the ICs

$$\chi(0) = 0, \chi'(0) = 1. \quad (32)$$

where $h(\nu) = \sin(\nu^3/8) + \sin(\nu) - \sin^2(\nu)$.
In Table 1 (Appendix), the actual absolute errors, the estimated absolute errors and the improved absolute errors are given. According to Table 1 (Appendix), we can infer three important conclusions. The first important result is that the errors decrease as the value of increases. The second important consequence is that the results of the estimated absolute errors are quite close to the results of the actual absolute errors. From this result, it can be said that the error estimation method described in Section 4 is effective. The final important result is that the improved absolute errors yield better results than the actual absolute errors at most points in the given range. From this result, it can be concluded that the technique of improving approximate solutions based on the residual function is effective. However, Table 2 (Appendix) displays Example 2's absolute error for several values of N. We observe that increasing the value of N yields an approximate solution that approaches the exact solution..
Table provides a presentation of the expected absolute errors, the actual absolute errors, and the improved absolute errors. Based on the data shown in Table 1 (Appendix), we may deduce three significant conclusions. The first significant finding is that the errors decrease as the value of N grows. It is also crucial to note that the results of the estimated absolute errors are pretty similar to the results of the real absolute errors. This is the second significant consequence. Based on this outcome, it can be concluded that the error estimate approach outlined in Section 4 is highly effective. The ultimate significant outcome is that the enhanced absolute errors provide better results compared to the current absolute errors at the majority of points in the given domine [0,1]. Based on this outcome, it can be inferred that the approach of enhancing approximation solutions using the residual function is efficacious. Yet the AE of example 2 for multiple values is presented in Table. It is clear from exploring Table that as the value of N is raised, an approximation solution is achieved that is a pretty close approximation to the exact solution.

Example 3 The third instance pertains to a second-order nonlinear differential equation that incorporates variable delays ν^2 and $-\nu/2$.

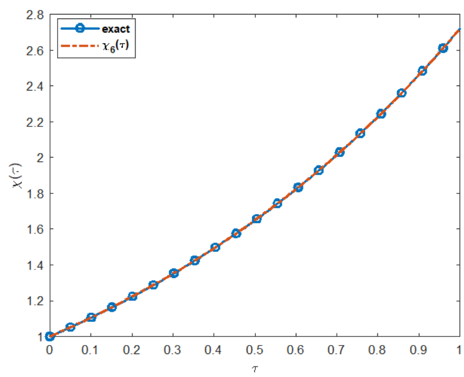

Fig. 1: Approximate and exact solutions for Example 3.

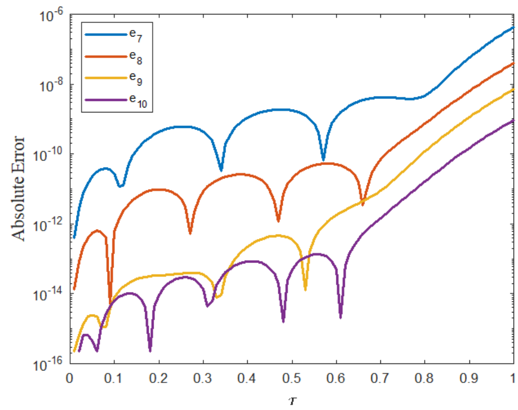

Fig. 2: The AE for Example 3 with several values of N.

$$\begin{aligned}&\chi''(\nu) + \chi'(\nu - \nu^2) - \nu^2\chi(\nu + \nu/2) - \\ &\chi'(\nu)\chi(\nu) + (\chi'(\nu))^2 = h(\nu),\ 0 \le \nu \le 1\end{aligned} \quad (33)$$

and the initial conditions

$$\chi(0) = 1, \chi'(0) = 1. \quad (34)$$

where $h(\nu) = e^\nu + e^{\nu-\nu^2} - \nu^2 e^{3\nu/2}$.

The analytical solution of this problem is $\chi(\nu) = e^\nu$. We solve the problem for several values of N For the values of $N = 4, 5, 7$ and 9, the absolute errors are shown in Table 3 (Appendix). It is clear that even with $N = 4$, one may get an accuracy of up to 4 decimal places. As N increases, the AE lowers for every single collecting point. Figure1 displays the analytical and approximate solution for the case where $N = 6$. The approximate solution is in close agreement with the exact solution.
Figure 2shows a comparison of the actual absolute error functions for $N = 5, N = 7$, and $N = 11$. Consequently, choosing a greater numerical value of N leads to a more precise outcome. The nonlinear dif-

ferential equations with variable delays were reported to be solved numerically using the Pell-Lucase and Lucase basis functions with the collection method. [53] and [54], in comparison with the present method, the Dejdamrong polynomial as a basis function, have given better results for absolution errors (Table 1 (Appendix) and Table 2 (Appendix)).

6 Conclusions

This study introduces a matrix approach that utilizes the Dejdumrong polynomial to solve functional differential equations of the pantograph type. These equations include hybrid delays that are both proportional and variable. In order to figure out the AE, the function of the residual errors are developed for these sorts of equations. Furthermore, the text specifies the execution of aforementioned technique and the processes for error analysis on specific problems. Upon examination of the difficulties, it becomes evident that the Dejdumrong polynomial coefficients may be readily obtained by the use of a computer program implemented in Matlab R2021a. If the truncation limit N is raised, it is possible to see that approximate solutions become more similar to the precise solutions. This is shown by the numerical results. Additionally, the method may be adapted to work with various kinds of equations and systems by making a few adjustments to it.

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APPENDIX

Table 1. Comparing the absolute errors, estimate, and absolute errors for the improved of the problem Eq.30 & Eq.32 for (N, M) = (4, 5), (4, 6), (7, 8), (7, 9), (10, 11), and (10, 12).

ν_i	Absolute errors	Estimated errors		Absolute errors for the improved	
	$\lvert e_5\rvert$	$\lvert e_{5,6}\rvert$	$\lvert e_{5,7}\rvert$	$\lvert E_{5,6}\rvert$	$\lvert E_{5,7}\rvert$
0.0	0.0000e+00	2.3798e-37	3.8968e-37	2.3798e-37	3.8968e-37
0.2	9.5303e-09	3.9484e-09	9.2624e-09	5.5820e-09	2.6789e-10
0.4	3.4432e-07	3.6505e-07	3.4282e-07	2.0725e-08	1.5033e-09
0.6	2.4805e-07	3.3208e-07	2.5695e-07	8.4037e-08	8.9049e-09
0.8	1.0733e-06	8.8959e-07	1.0756e-06	1.8370e-07	2.2670e-09
1.0	2.8254e-05	2.5584e-05	2.8333e-05	2.6699e-06	7.8980e-08
ν_i	$\lvert e_8\rvert$	$\lvert e_{8,9}\rvert$	$\lvert e_{8,10}\rvert$	$\lvert E_{8,9}\rvert$	$\lvert e_{8,10}\rvert$
0.0	0.0000e+00	1.0269e-40	0.0000e+00	1.0269e-40	0.0000e+00
0.2	1.0138e-11	1.0095e-11	1.0130e-11	4.3173e-14	8.9848e-15
0.4	2.3723e-11	2.3849e-11	2.3714e-11	1.2620e-13	8.3870e-15
0.6	6.0895e-11	6.1364e-11	6.0845e-11	4.6956e-13	5.0386e-14
0.8	7.8434e-11	8.0852e-11	7.9129e-11	2.4177e-12	6.9453e-13
1.0	3.1914e-09	3.2434e-09	3.1979e-09	5.1977e-11	6.4979e-12

Table 2. Comparison of the absolute errors for of Example 2 in [53]

ν_i	Ref. [53]			PM		
	$\lvert e_4\rvert$	$\lvert e_7\rvert$	$\lvert e_{10}\rvert$	$\lvert e_4\rvert$	$\lvert e_7\rvert$	$\lvert e_{10}\rvert$
0.0	0.0000e+00	6.3527e-22	1.2914e-22	0.0000e+00	0.0000e+00	0.0000e+00
0.2	1.3230e-05	7.2149e-10	1.4211e-13	4.4211e-06	1.1507e-10	1.5071e-15
0.4	3.8453e-05	1.4754e-09	2.8422e-13	1.3931e-05	2.6240e-10	1.7120e-14
0.6	4.2938e-05	2.1464e-09	2.2737e-13	7.2812e-05	4.6197e-10	2.3005e-14
0.8	2.1963e-04	2.7024e-09	4.5475e-13	8.9964e-05	9.8009e-10	4.1436e-14
1.0	1.2656e-03	1.07390e-07	1.5689e-11	1.2950e-03	4.6524e-08	2.2276e-12

Table 3. Comparison of the absolute errors for for $N = 5, 7$ and 9 with Ref. [54]with respect to Example 3

ν_i	Ref. [54]			PM		
	$\lvert e_5\rvert$	$\lvert e_7\rvert$	$\lvert e_9\rvert$	$\lvert e_5\rvert$	$\lvert e_7\rvert$	$\lvert e_9\rvert$
0.0	—	—	—	0.0000e+00	6.5457e-26	0.0000e+00
0.2	2.05e-06	2.79e-09	1.86e-12	3.2417e-08	4.2243e-10	9.2773e-15
0.4	6.22e-06	5.53e-09	3.56e-12	1.5154e-06	1.0791e-09	8.4035e-14
0.6	1.37e-05	8.75e-09	4.95e-12	1.3661e-06	1.1409e-09	4.8594e-14
0.8	1.85e-05	4.84e-09	8.55e-11	2.4782e-06	4.7415e-09	1.6358e-11
1.0	9.09e-05	5.35e-07	4.78e-09	1.2852e-04	4.2872e-07	9.3254e-10