Conjugate Graph and Conjugacy Class Graph related to Direct Product of Dihedral Groups

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Abstract: - Let G be a non-abelian group. The conjugate graph of G is a graph whose vertices are the non-central elements of G and two vertices are adjacent if they belong to the same conjugacy class. The conjugacy class graph of G is a graph whose vertices are the non-central conjugacy classes of G and two vertices a, b are adjacent if $gcd(|a|, |b|)$ is greater than one. In this paper, we explore the structures of these graphs for the groups $D_n \times D_m$ for odd and even values of n and m. The chromatic number and independence number of the conjugate graphs, their line graphs and complement graphs are found. We discuss various graph parameters like the existence of Eulerian and Hamiltonian cycles, planarity, connectedness, chromatic number, clique number, independence number, and diameter of the conjugacy class graphs of these groups.

Key-Words: - Conjugate graph, conjugacy class graph, line graph, complement graph, direct product, non-abelian group, dihedral group.

Received: September 22, 2023. Revised: April 21, 2024. Accepted: June 22, 2024. Published: July 19, 2024.

1 Introduction

Algebraic graph theory is an interesting subject concerned with the interplay between algebra and graph theory where algebraic tools are applied to yield useful insights of graph theoretic facts. This area of graph theory contributes to the study of properties of graphs by translating them into algebraic structures and the results and methods of algebra are applied to deduce theorems about the graphs. Also, many algebraic structures can be studied by translating them into graphs and using the properties of graphs. One such branch of algebraic graph theory involves the study of symmetry and regularity of graphs by using group theory. This paper contributes to the understanding of the intersection between graph theory and group theory, specifically in the context of non-abelian groups.

In 1990, a new graph called the *conjugacy class graph*, denoted by Γ^{cl}, related to conjugacy classes of non-abelian groups was introduced, [1]. The conjugacy class graph of a non-abelian group G, denoted by $\Gamma^{cl}(G)$, is a graph whose vertices are the non-central conjugacy classes of G and two distinct vertices are adjacent if the greatest common divisor of the sizes of the vertices (orders of the conjugacy classes) is greater than one. Motivated by the structure of conjugacy classes, another graph called *conjugate graph*, denoted by Γ^{C}, of a non-abelian group was defined, [2]. The vertices of this graph are the non-central elements of the group and two vertices are adjacent if they belong to the same conjugacy class. It was found that if S is a finite non-abelian simple group satisfying Thompson's conjecture and $\Gamma^{C}(G) \cong \Gamma^{C}(S)$, then $G \cong S$, where G is a finite non-abelian group. Many research articles have been published thereafter related to conjugate graphs of finite p-groups [3], regularity of graph related to conjugacy classes [4], graph properties of conjugate graphs and conjugacy class graphs of metacyclic 2-groups of order at most 32 [5], the conjugate graphs and generalized conjugacy class graphs of metacyclic 3-groups and metacyclic 5-group [6], and the non-conjugate graph associated with finite groups and the resolving polynomial for generalized quaternion groups, [7].

In this paper, we explore the conjugacy classes related to direct products of dihedral groups and discuss various graph parameters of the conjugate graphs and conjugacy class graphs of these groups. Graph parameters like chromatic number and regularity of graphs are crucial concept in graph theory that has numerous applications in various fields like computer science, operation research, network topology, cryptography, and game theory while the independence number of a graph serve as a useful tool in real world optimization problems.

This paper is divided into five sections. The first section is the introductory section where we give a brief definition of conjugacy classes, conjugate graph, and conjugacy class graph and an overview of the previously published works on this topic. In the second section, we state some preliminary definitions and results that have been referred to for our study. The third section involves detailed proof of our findings. In section 4, we summarize the key findings of our work, and discuss the methodology used and possible applications of the results presented here. Finally, in section 5, we have mentioned some of the potential areas for further research in this topic. Throughout the paper, the graphs referred to are undirected and simple graphs.

2 Basic Definitions and Results

In this section we state some basic definitions and results which have been referred to for establishing our main results.

Definition 2.1: **Conjugate elements and Conjugacy class**
Two elements $a, b \in G$ of a group G are said to be *conjugate* if there exists an element $c \in G$ such that $b = c^{-1}ac$. The set of all elements conjugate to a is called the *conjugacy class* of a and is denoted by $Cl(a)$. The conjugacy relation is an equivalence relation that partitions G into disjoint equivalence classes, called the conjugacy classes.

Definition 2.2: **Connected graph**
A graph G is said to be *connected* if there exists a path between any two distinct vertices in the graph; otherwise G is said to be *disconnected*.

Definition 2.3: **Planar Graph**
A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn in such a way that no edges cross each other.

Definition 2.4: **Diameter of a graph**
The *diameter* of a connected graph is the maximum distance between any two vertices in the graph.

Definition 2.5: **Chromatic number of a graph**
The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest number of colors required to color the vertices of G so that no two adjacent vertices share the same color.

Definition 2.6: **Clique and Clique number of a graph**
A *clique* of a graph G is a complete induced subgraph of G and the number of vertices in the largest clique of G is called the *clique number* of G.

Definition 2.7: **Independence number of a graph**
The *independence number* α of a graph is the maximum number of vertices from the vertex set of the graph such that no two vertices are adjacent.

Definition 2.8: **Eulerian and Hamiltonian graph**
A connected graph is said to be *Eulerian* if it contains an eulerian circuit, i.e., it contains a trail that visits every edge exactly once and starts and ends on the same vertex. A *Hamiltonian* circuit in a connected graph G is defined as a closed walk that traverses every vertex of G exactly once.

Definition 2.9: **Regular graph**
A graph is said to be *regular* if all its vertices have the same degree. A graph G is said to be k *−regular* if the degree of each vertex in G is k.

Definition 2.10: **Line graph**
Given a graph G, the *line graph* $L(G)$ of G is a graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent. Thus, the line graph $L(G)$ represents the adjacencies between the edges of G.

Definition 2.11: **Direct product of two groups**
For two groups G_1 and G_2, the direct product $G_1 \times G_2$ is defined as,

$$G_1 \times G_2 = \{(g_1, g_2): g_1 \in G_1, g_2 \in G_2\}$$

Two elements (g_1, h_1) and (g_2, h_2) are conjugate in $G \times H$ if and only if g_1 and g_2 are conjugate in G and h_1 and h_2 are conjugate in H. Thus, each conjugacy class in $G \times H$ is the cartesian product of a conjugacy class in G and a conjugacy class in H.

Proposition 2.1: A graph is said to be a planar graph if and only if it contains no subgraph isomorphic to $K_{3,3}$ or K_5, [8].

Proposition 2.2: A connected graph is eulerian if and only if every vertex has even degree, [8].

Proposition 2.3: A simple graph G is hamiltonian if the degree of every vertex in G is atleast $\frac{n}{2}$, where n is the number of vertices in G, [8].

Proposition 2.4: The line graph $L(K_n) = G$ of the complete graph K_n has the following properties:

- G has $\binom{n}{2}$ vertices.
- G is regular of degree $2(n-2)$.
- Every two non-adjacent points are mutually adjacent to exactly four points.
- Every two adjacent points are mutually adjacent to exactly $n-2$ points, [9].

3 Main Results

Before proceeding to our main findings, we first list the conjugacy classes of D_n. The dihedral group $D_n (n \geq 3)$ of order $2n$ is the group of symmetries of a regular n-sided polygon which consists of n reflections and n rotations. If a denotes any one of the n-reflections and b denote the rotation about the center through an angle $2\pi/_n$, then the group presentation of D_n is given by:

$$D_n = \langle a, b : a^2 = b^n = e, (ab)^2 = 1 \rangle$$

We have two cases here:

Case 1: When n is odd, then

$a = (1)(2 \;\; n)(3 \;\; n-1) \ldots (\frac{n+1}{2} \;\; \frac{n+3}{2})$

$b = (1 \;\; 2 \;\; 3 \;\; 4 \ldots . n-1 \;\; n)$

Case 2: When n is even, then

$a = (1)(2 \;\; n)(3 \;\; n-1) \ldots (\frac{n}{2} \;\; \frac{n+4}{2})(\frac{n+2}{2})$

$b = (1 \;\; 2 \;\; 3\, 4 \ldots n-1 \;\; n)$

The $2n$ distinct elements of D_n are:

$b, b^2, b^3, \ldots, b^n = e, ab, ab^2, ab^3, \ldots$
$ab^{n-1}, ab^n = a$

This paper is an extension of [10] where we studied the nature and properties of the elements of dihedral group and their conjugacy class size and it was found that if $k(D_n)$ denotes the number of conjugacy classes of D_n, then,

$$k(D_n) = \begin{cases} \frac{n+3}{2}, \text{if } n \text{ } is\ odd \\ \frac{n+6}{2}, \text{if } n \text{ } is\ even \end{cases}$$

When n is odd, the conjugacy classes of D_n are

$$\{e = b^n\},$$
$$\{a, ab, ab^2, ab^3, \ldots, ab^{n-2}, ab^{n-1}\},$$
$$\{b, b^{n-1}\},$$
$$\{b^2, b^{n-2}\}, \ldots,$$
$$\{b^{\frac{n-1}{2}}, b^{\frac{n+1}{2}}\}$$

When n is even, the conjugacy classes of D_n are

$$\{e = b^n\},$$
$$\{a, ab^2, ab^4, \ldots, ab^{n-2}\},$$
$$\{ab, ab^3, ab^5, \ldots, ab^{n-1}\},$$
$$\{b, b^{n-1}\},$$
$$\{b^2, b^{n-2}\}, \ldots,$$
$$\{b^{\frac{n}{2}}\}$$

The following results are thus obtained.

Theorem 1: The independence number α of the conjugate graph of $D_n \times D_m$ is given as:

$$\alpha(\Gamma^C(D_n \times D_m)) = \begin{cases} \left(\frac{n+3}{2}\right)\left(\frac{m+3}{2}\right) - 1, \text{if } m, n \equiv 1 (mod\ 2) \\ \left(\frac{n+6}{2}\right)\left(\frac{m+6}{2}\right) - 4, \text{if } m, n \equiv 0 (mod\ 2) \\ \left(\frac{n+3}{2}\right)\left(\frac{m+6}{2}\right) - 2, \text{if } m \equiv 0 (mod\ 2), \\ \qquad n \equiv 1 (mod\ 2) \end{cases}$$

Proof: Conjugate graphs are disconnected and union of complete graphs. So, the number of components gives the independence number of the conjugate graphs.

Case 1: Both m, n are odd.

Then there are $\left(\frac{n+3}{2}\right)\left(\frac{m+3}{2}\right)$ conjugacy classes in $D_n \times D_m$. Since the conjugate graphs do not contain the centre, there are $\left(\frac{n+3}{2}\right)\left(\frac{m+3}{2}\right) - 1$ components in the conjugate graph of $D_n \times D_m$ which is also the independence number of the graph.

Case 2: Both m, n are even.

Then there are $\left(\frac{n+6}{2}\right)\left(\frac{m+6}{2}\right)$ conjugacy classes in $D_n \times D_m$. The centre of D_n (for even value of n) has order 2 and so there are $\left(\frac{n+6}{2}\right)\left(\frac{m+6}{2}\right) - 4$ components in the conjugate graph of $D_n \times D_m$.

Case 3: m is even and n odd.

In this case there are $\left(\frac{n+3}{2}\right)\left(\frac{m+6}{2}\right)$conjugacy classes in $D_n \times D_m$. The centre of D_m has order 2 and the centre of D_n has order 1. So, there are $\left(\frac{n+3}{2}\right)\left(\frac{m+6}{2}\right) - 2$ components in the conjugate graph of $D_n \times D_m$.

Theorem 2: The chromatic number χ of $\Gamma^C(D_n \times D_m)$ is given by:

$$\chi\left(\Gamma^C(D_n \times D_m)\right) = \begin{cases} mn, \text{if } m,n \equiv 1(mod\ 2) \\ \dfrac{mn}{4}, \text{if } m,n \equiv 0(mod\ 2) \\ \dfrac{mn}{2}, \text{if } m \equiv 0(mod\ 2), \\ \qquad n \equiv 1(mod\ 2) \end{cases}$$

Proof: The conjugate graphs being disconnected and union of complete graphs, so the component of the conjugate graph with the highest number of vertices gives the chromatic number of the graph.
We have 3 cases:

Case 1: Both m, n are odd.
There are $\left(\frac{n+3}{2}\right)\left(\frac{m+3}{2}\right)$ conjugacy classes in $D_n \times D_m$ and orders of the conjugacy classes are $2,4, m, n, 2m, 2n$ and mn. Since $m, n \geq 3$, the conjugacy class with order mn is the largest.

Case 2: Both m, n are even.
There are $\left(\frac{n+6}{2}\right)\left(\frac{m+6}{2}\right)$ conjugacy classes in $D_n \times D_m$ and orders of the conjugacy classes are $2,4, m, n, \frac{m}{2}, \frac{n}{2}$ and $\frac{mn}{4}$. Since $m, n \geq 4$, the largest conjugacy class is of order $\frac{mn}{4}$.

Case 3: n odd, m even.
There are $\left(\frac{n+3}{2}\right)\left(\frac{m+6}{2}\right)$ conjugacy classes in $D_n \times D_m$ and orders of the conjugacy classes are $2,4, m, n, \frac{m}{2}, 2n$ and $\frac{mn}{2}$. Here, $n \geq 3$ and $m \geq 4$. So, the conjugacy class of highest order is of order $\frac{mn}{2}$.

Since each conjugacy class constitutes a component in the conjugate graph, the largest component in each case has order mn, $\frac{mn}{4}$ and $\frac{mn}{2}$ respectively.

Example 1: Consider $D_3 \times D_3$.
Conjugacy classes of D_3 are $\{e\}, \{a, ab, ab^2\}, \{b, b^2\}$
The 9 conjugacy classes of $D_3 \times D_3$ are $\{(e,e)\}$, $\{(e,a),(e,ab),(e,ab^2)\}$, $\{(e,b),(e,b^2)\}$, $\{(a,e),(ab,e),(ab^2,e)\}$, $\{(a,a),(a,ab),(a,ab^2),(ab,a),(ab,ab),(ab,ab^2),(ab^2,a)(ab^2,ab),(ab^2,ab^2)\}$, $\{(a,b),(a,b^2),(ab,b),(ab,b^2),(ab^2,b),(ab^2,b^2)\}$, $\{(b,e),(b^2,e)\}$, $\{(b,a),(b,ab),(b,ab^2),(b^2,a),(b^2,ab),(b^2,ab^2)\}$, $\{(b,b),(b,b^2),(b^2,b),(b^2,b^2)\}$

By definition, there are $9-1=8$ components in $\Gamma^C(D_3 \times D_3)$ with orders 2,2,3,3,4,6,6,9 and $\Gamma^C(D_3 \times D_3) = 2K_2 \cup 2K_3 \cup K_4 \cup 2K_6 \cup K_9$
Clearly, Independence number of $\Gamma^C(D_3 \times D_3) = 8$ and Chromatic number of $\Gamma^C(D_3 \times D_3) = 9$.

Theorem 3: The chromatic number of $\overline{\Gamma^C(D_n \times D_m)}$ is the number of components in $\Gamma^C(D_n \times D_m)$.

Proof: Let the number of components of $\Gamma^C(D_n \times D_m)$ be k. Each component forms a point (or vertex) as a set of non-adjacent vertices in $\overline{\Gamma^C(D_n \times D_m)}$. The k disconnected components in $\Gamma^C(D_n \times D_m)$ together form a complete graph of k vertices in $\overline{\Gamma^C(D_n \times D_m)}$. Therefore, k is the minimum number of colors required to color the vertices of $\overline{\Gamma^C(D_n \times D_m)}$ and so chromatic number = k = number of components of $\Gamma^C(D_n \times D_m)$.

Theorem 4: The independence number of $\overline{\Gamma^C(D_n \times D_m)}$ is given by:

$$\alpha\left(\overline{\Gamma^C(D_n \times D_m)}\right) = \begin{cases} mn, \text{if } m,n \equiv 1(mod\ 2) \\ \dfrac{mn}{4}, \text{if } m,n \equiv 0(mod\ 2) \\ \dfrac{mn}{2}, \text{if } m \equiv 0(mod\ 2), \\ \qquad n \equiv 1(mod\ 2) \end{cases}$$

Proof: Each component in $\Gamma^C(D_n \times D_m)$ forms a point (or vertex) as a set of non-adjacent vertices in $\overline{\Gamma^C(D_n \times D_m)}$. The vertices in $\overline{\Gamma^C(D_n \times D_m)}$ are such that each vertex is a set of non-adjacent vertices which are adjacent to every other vertex. Clearly, the point (vertex) with the highest number of non-adjacent vertices in $\overline{\Gamma^C(D_n \times D_m)}$ gives the independence number of $\overline{\Gamma^C(D_n \times D_m)}$, which is nothing but the component in $\Gamma^C(D_n \times D_m)$ with maximum number of vertices. Hence, the result follows from **Theorem 2**.

Theorem 5: The chromatic number of the line graphs of $\Gamma^C(D_n \times D_m) = G$ are given by:

$3 \leq \chi\big(L(G)\big) \leq 2mn - 3$, if $m, n \equiv 1(mod\ 2)$
$3 \leq \chi\big(L(G)\big) \leq \frac{mn}{2} - 3$, if $m, n \equiv 0(mod\ 2)$
$3 \leq \chi\big(L(G)\big) \leq mn - 3$, if $m \equiv 0(mod\ 2)$, $n \equiv 1(mod\ 2)$

Proof: The conjugate graphs are disconnected unions of complete graphs, so their line graphs are again disconnected but unions of regular graphs.

The largest component in the line graph correspond to the largest component in the conjugate graph. Since the largest component in the conjugate graphs have sizes mn, $\frac{mn}{4}$ and $\frac{mn}{2}$, so the largest component in the line graph comprises of a regular graph of $\binom{mn}{2}$ vertices of degree $2(mn-2)$, $\binom{mn/4}{2}$ vertices of degree $2(\frac{mn}{4}-2)$, $\binom{mn/2}{2}$ vertices of degree $2(\frac{mn}{2}-2)$ respectively for the three cases. Since the chromatic number is at most one greater than maximum degree, so,

$$\chi \leq \begin{cases} 2mn-3, \text{for } m,n \equiv 1(mod\ 2) \\ \frac{mn}{2}-3, \text{for } m,n \equiv 0(mod\ 2) \\ mn-3, \text{for } m \equiv 0(mod\ 2), \\ \qquad n \equiv 1(mod\ 2) \end{cases}$$

Also, the line graph corresponding to a complete graph of order n is such that every two adjacent points are mutually adjacent to exactly $n-2$ points. So, the minimum chromatic number for a line graph corresponding to a complete graph is at least 3. Hence the result.

Theorem 6: The conjugacy class graph of $D_n \times D_m$ is a connected graph.

Proof: The conjugacy class graphs are such that each vertex represents a set of vertices with same order and adjacent to each other. The adjacency between any two pair of vertices indicates that every point in a vertex is adjacent to each point in the other vertex and so on. Conjugacy classes of same order, say n, are represented by a vertex indexed $[n]$.
We have 3 cases here:

Case 1: If both m,n are odd, the vertices with cardinality $2m$ and $2n$ are adjacent to all the vertices. The graph structures are either of the following two (Figure 1) depending on whether $gcd(m,n)=1$ or $gcd(m,n)>1$:

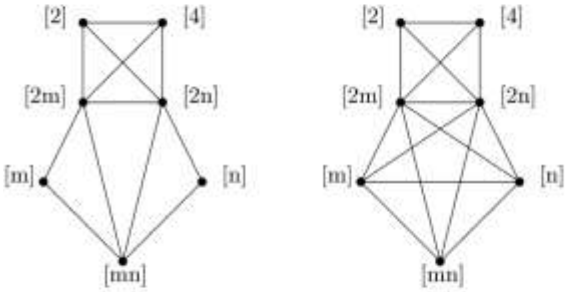
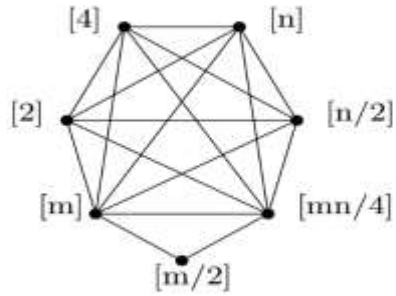
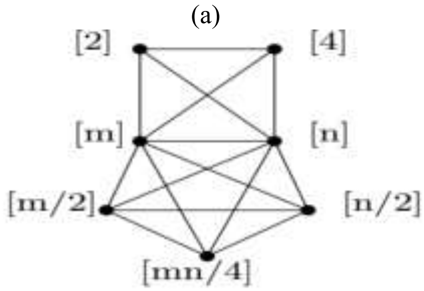


Fig. 1: $\Gamma^{cl}(D_n \times D_m)$ when both m,n are odd

Case 2: If both m,n are even, the graph is complete (Figure 4) or the vertices are adjacent to either of the set of vertices $[m], [\frac{mn}{4}]$ or $[m],[n]$ for different values of m and n (Figure 2).



(a)



(b)

Fig. 2: $\Gamma^{cl}(D_n \times D_m)$ when both m,n are even

Case 3: When n is odd and m is even, all the vertices are either adjacent to the vertices $[m],[2n]$ or $[2n],[\frac{mn}{2}]$. The graph structures are similar to the above structures discussed above.

Hence for every value of m and n, $\Gamma^{cl}(D_n \times D_m)$ is connected.

Example 2: Consider $D_3 \times D_3$
Since, there are 9 conjugacy classes in $D_3 \times D_3$ and conjugacy class graph do not contain the centre, so there are $9-1=8$ vertices in $\Gamma^{cl}(D_3 \times D_3)$. The orders of the conjugacy classes are 2,2,3,3,4,6,6,9 and hence the vertices will be represented by $[2],[3],[4],[6],[9]$. The graph is shown in Figure 3.

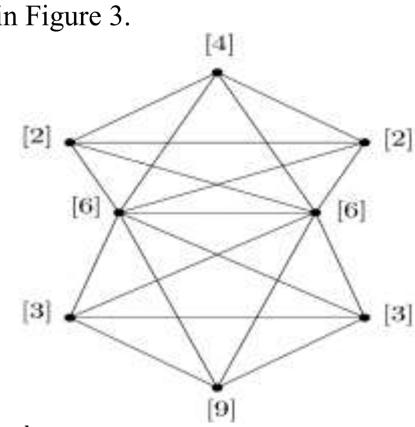


Fig. 3: $\Gamma^{cl}(D_3 \times D_3)$

Theorem 7: The diameter of $\Gamma^{cl}(D_n \times D_m)$ is 2.

Proof: By definition, diameter is the maximum distance between any two vertices in a graph, where distance is the shortest path between any two vertices. As seen in the previous theorem, for each case, there exists at least two vertices that are either adjacent to every other vertex in $\Gamma^{cl}(D_n \times D_m)$ or connected by a path of length 2 and hence diameter of $\Gamma^{cl}(D_n \times D_m)$ is 2.

Theorem 8: $\Gamma^{cl}(D_n \times D_m)$ is both Eulerian and Hamiltonian for even values of $m, n, \frac{m}{2}, \frac{n}{2}$.

Proof: For even values of m, there are 2 conjugacy classes of order 1, $\frac{m}{2} - 1$ conjugacy classes of order 2, 2 conjugacy classes of order $\frac{m}{2}$ in D_m.

The number of conjugacy classes of D_m being $\frac{m+6}{2}$, there will be $(\frac{n+6}{2})(\frac{m+6}{2})$ conjugacy classes in $D_n \times D_m$. So, the number of vertices in $\Gamma^{cl}(D_n \times D_m)$ is $\left(\frac{n+6}{2}\right)\left(\frac{m+6}{2}\right) - 4$.

There are $m + n - 4$ vertices of order 2, $(\frac{m}{2} - 1)(\frac{n}{2} - 1)$ vertices of order 4, $n - 2$ vertices of order m, $m - 2$ vertices of order n, 4 vertices of order $\frac{m}{2}$, 4 vertices of order $\frac{n}{2}$, 4 vertices of order $\frac{mn}{4}$.

Since $\frac{m}{2}, \frac{n}{2}$ are both even, all the vertices are adjacent to each other, i.e., the graph is a complete graph. Also, $\left(\frac{n+6}{2}\right)\left(\frac{m+6}{2}\right) - 4$ being an odd number, order of each vertex is even and equal. Hence the graph is both Hamiltonian and Eulerian.

Example 3:

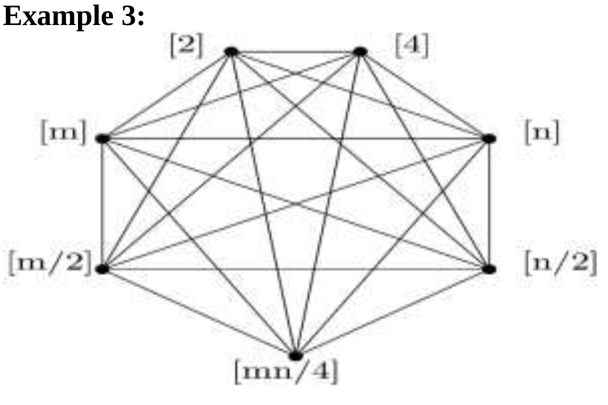


Fig. 4: $\Gamma^{cl}(D_n \times D_m)$ (for even values of $m, n, \frac{m}{2}, \frac{n}{2}$)

Theorem 9: The clique number ω of $\Gamma^{cl}(D_n \times D_m)$ is given by:

(i) Both n, m are odd.

$$\omega(\Gamma^{cl}(D_n \times D_m)) = n + m - 2 + \frac{(n-1)(m-1)}{4}$$

(ii) Both n, m are even.

$$\omega(\Gamma^{cl}(D_n \times D_m) = \begin{cases} \left(\frac{m+6}{2}\right)\left(\frac{n+6}{2}\right) - 4, & \text{if } m, n \equiv 0 (mod\ 4) \\ 2m + 2n + \left(\frac{m}{2} - 1\right)\left(\frac{n}{2} - 1\right), & if\ m \equiv 0 (mod\ 4),\ n \equiv 2 (mod\ 4) \\ 2m + 2n - 8 + \left(\frac{m}{2} - 1\right)\left(\frac{n}{2} - 1\right), & if\ m, n \equiv 2 (mod\ 4) \end{cases}$$

(iii) n odd, m even.

$$\omega(\Gamma^{cl}(D_n \times D_m)) = \begin{cases} 2n + m - 4 + \left(\frac{m}{2} - 1\right)\left(\frac{n-1}{2}\right), & \text{if } m \equiv 2 (mod\ 4) \\ 2n + m + \left(\frac{m}{2} - 1\right)\left(\frac{n-1}{2}\right), & \text{if } m \equiv 0 (mod\ 4) \end{cases}$$

Proof: **Case 1**: Both m, n are odd.
The vertices with orders $2, 4, 2m, 2n$ form a complete subgraph for the subcases $gcd(m, n) = 1$ and $gcd(m, n) > 1$, which is also the maximal subgraph for both the cases. Hence the clique number for $\Gamma^{cl}(D_n \times D_m)$ is the total number of vertices of order $2, 4, 2m, 2n$ which is $m + n - 2 + \frac{(n-1)(m-1)}{4}$.

Case 2: Both m, n are even.
We have 3 subcases here:
Subcase (i): Both $\frac{m}{2}, \frac{n}{2}$ are even.
In the previous theorem, we have already proved that the conjugacy class graph is a complete graph $K_{\left(\frac{m+6}{2}\right)\left(\frac{n+6}{2}\right)-4}$ which gives the clique number as $\left(\frac{m+6}{2}\right)\left(\frac{n+6}{2}\right) - 4$.

Subcase (ii): One of $\frac{m}{2}, \frac{n}{2}$ is even and the other one odd.

The vertices $[2],[4],[m],[n],\left[\frac{n}{2}\right],[\frac{mn}{4}]$ form a complete maximal subgraph of $\Gamma^{cl}(D_n \times D_m)$ which gives the clique number as $2m+2n+(\frac{m}{2}-1)(\frac{n}{2}-1)$.

Subcase (iii): Both $\frac{m}{2},\frac{n}{2}$ are odd.
If $\gcd\left(\frac{m}{2},\frac{n}{2}\right)=1$, then we have one complete subgraph comprising of the vertices $[2],[4],[m],[n]$ which is also the maximal subgraph of $\Gamma^{cl}(D_n \times D_m)$. If $\gcd\left(\frac{m}{2},\frac{n}{2}\right)>1$, then $\Gamma^{cl}(D_n \times D_m)$ comprises of two complete subgraphs with vertices $[2],[4],[m],[n]$ and $[m],[n],[\frac{m}{2}],[\frac{n}{2}],[\frac{mn}{4}]$. For both the cases, the complete subgraph with vertices $[2],[4],[m],[n]$ forms the maximal complete subgraph. Hence the result.

Case 3: m even, n odd.
We have 2 subcases here:
Subcase (i): $\frac{m}{2}$ is odd.
The vertices $[2],[4],[m],[2n]$ form a maximal complete subgraph and so the clique number is the total number of vertices of order $2,4,m,2n$ which is $2n+m-4+(\frac{m}{2}-1)(\frac{n-1}{2})$.

Subcase (ii): $\frac{m}{2}$ is even.
The vertices with orders $2,4,m,2n,\frac{m}{2},\frac{mn}{2}$ form a maximal complete subgraph and hence the clique number is $2n+m+(\frac{m}{2}-1)(\frac{n-1}{2})$.

Theorem 10: $\Gamma^{cl}(D_n \times D_m)$ is non-planar for all values of m,n.

Proof: As seen in the previous theorem, for all the cases, the clique number is at least 5 $(m,n\geq 3)$, i.e, all the graphs contain a K_5 graph. Hence, $\Gamma^{cl}(D_n \times D_m)$ is non-planar for all values of m,n.

Theorem 11: The chromatic number χ of $\Gamma^{cl}(D_n \times D_m)$ is given by:

(i) Both n,m are odd.
$$\chi = n+m-2+\frac{(n-1)(m-1)}{4}$$

(ii) Both n,m are even.
$$\chi = \left(\frac{m+6}{2}\right)\left(\frac{n+6}{2}\right)-4, \quad \text{if } m,n \equiv 0(mod\ 4)$$

$$\chi = 2m+2n+\left(\frac{m}{2}-1\right)\left(\frac{n}{2}-1\right), \quad \text{if } m\equiv 0(mod\ 4), n\equiv 2(mod\ 4)$$

$$\chi \leq 2m+2n-1+\left(\frac{m}{2}-1\right)\left(\frac{n}{2}-1\right), \quad \text{if } m,n \equiv 2(mod\ 4)$$

(iii) n odd and m even.
$$\chi = 2n+m+\left(\frac{m}{2}-1\right)\left(\frac{n-1}{2}\right), \quad \text{if } m\equiv 0(mod\ 4)$$

$$\chi \leq 2n+m-2+\left(\frac{m}{2}-1\right)\left(\frac{n-1}{2}\right), \quad \text{if } m\equiv 2(mod\ 4)$$

Proof: We have three cases:
Case 1: Both m,n are odd.
Orders of the conjugacy classes will be as follows:

$$\underbrace{2,2,\dots,2}_{(\frac{n-1}{2}+\frac{m-1}{2})times}, \underbrace{4,4,\dots,4}_{(\frac{n-1}{2})(\frac{m-1}{2})times}, m, n, \underbrace{2m,2m,\dots,2m}_{(\frac{n-1}{2})times}, \underbrace{2n,2n,\dots,2n}_{(\frac{m-1}{2})times}, mn$$

Clearly, the chromatic number is at least $n+m-2+\frac{(n-1)(m-1)}{4}$. Since there are at least two vertices of order 2 and one vertex of order 4 and the vertices $[2],[4]$ being non-adjacent to all $[m],[n],[mn]$ vertices, the chromatic number is $n+m-2+\frac{(n-1)(m-1)}{4}$.

Case 2: Both m,n are even.
There are $m+n-4$ vertices of order 2, $(\frac{m}{2}-1)(\frac{n}{2}-1)$ vertices of order 4, $n-2$ vertices of order m, $m-2$ vertices of order n, 4 vertices of order $\frac{m}{2}$, 4 vertices of order $\frac{n}{2}$ and 4 vertices of order $\frac{mn}{4}$.

If $\frac{m}{2},\frac{n}{2}$ are both even, the graph is a complete graph and so there is nothing to prove.

If one of $\frac{m}{2},\frac{n}{2}$ is even and the other odd, the chromatic number is at least $2m+2n+(\frac{m}{2}-1)(\frac{n}{2}-1)$. There are at least four vertices of order 2 and these vertices being non-adjacent to the four vertices of order $\frac{m}{2}$, the chromatic number of $\Gamma^{cl}(D_n \times D_m)$ is same as the chromatic number of the maximal complete subgraph of $\Gamma^{cl}(D_n \times D_m)$ which is $2m + 2n + (\frac{m}{2} - 1)(\frac{n}{2} - 1)$.

For odd values of both $\frac{m}{2}, \frac{n}{2}$, the maximal complete subgraph of $\Gamma^{cl}(D_n \times D_m)$ will be $K_{2m+2n-8+(\frac{m}{2}-1)(\frac{n}{2}-1)}$ formed by the vertices $[2], [4], [m], [n]$. Since there are at least four vertices of order 2, one vertex of order 4, two vertices of order m, n each, and considering the non-adjacency of [2] and [4] with $[\frac{mn}{4}], \left[\frac{n}{2}\right], [\frac{m}{2}]$, the chromatic number is at most $2m + 2n - 1 + (\frac{m}{2} - 1)(\frac{n}{2} - 1)$.

Case 3: m even, n odd.
Orders of the conjugacy classes will be as follows:

$$\underbrace{2,2,\ldots,2}_{(\frac{m}{2}+n-2)times}, \underbrace{4,4,\ldots,4}_{\left(\frac{n-1}{2}\right)\left(\frac{m-1}{2}\right)times}, \underbrace{m,m,\ldots,m}_{(n-1)times}, n, n, \frac{m}{2}, \frac{m}{2}, \underbrace{2n,2n,\ldots,2n}_{\left(\frac{m}{2}-1\right)times}, \frac{mn}{2}, \frac{mn}{2}.$$

Clearly, for even values of $\frac{m}{2}$, the chromatic number is same as the clique number which is $2n + m + (\frac{m}{2} - 1)(\frac{n-1}{2})$.

For odd values of $\frac{m}{2}$, the maximal complete subgraph of $\Gamma^{cl}(D_n \times D_m)$ will be $K_{m+2n-4+(\frac{m}{2}-1)(\frac{n-1}{2})}$ formed by the vertices $[2], [4], [m], [2n]$. Since there are at least three vertices of order 2, one vertex of order 4, two vertices of order m, one vertex of order $2n$, and considering the non-adjacency of [2] and [4] with $\left[\frac{mn}{2}\right], \left[\frac{m}{2}\right], [n]$, the chromatic number is at most $m + 2n - 2 + (\frac{m}{2} - 1)(\frac{n-1}{2})$.

Theorem 12: The independence number α of $\Gamma^{cl}(D_n \times D_m)$ is such that $1 \leq \alpha \leq 3$.

Proof: We have three cases here:

Case 1: Both m, n are odd.
Since both m, n are odd, the graph structure will be either of the two types depending on whether $gcd(m, n) = 1$ or $gcd(m, n) > 1$. In the first case, the vertices $[m]$, $[n]$ and [2] (or [4]) form an independent set. While in the second case, the vertices [2] (or [4]) and $[mn]$ (or any one of $[m], [n]$) form an independent set.

Case 2: Both m, n are even.
We have 3 subcases here:
Subcase (i): Both $\frac{m}{2}$ and $\frac{n}{2}$ are even.
In this case, all the vertices will have even order and so gcd of the orders of any two vertices will be atleast 2. Hence, we get a complete graph $K_{\left(\frac{m+6}{2}\right)\left(\frac{n+6}{2}\right)-4}$ which gives the independence number as 1.

Subcase (ii): One of $\frac{m}{2}$ and $\frac{n}{2}$ is odd and the other even.
Suppose $\frac{n}{2}$ is odd. In that case, the vertices $[\frac{n}{2}]$ and [2] (or any one of $[4], [m], [\frac{m}{2}]$) form an independent set.

Subcase (iii): Both $\frac{m}{2}$ and $\frac{n}{2}$ are odd.
In this case we will have two different graph structures depending on whether $gcd(\frac{m}{2}, \frac{n}{2}) = 1$ or $\gcd(\frac{m}{2}, \frac{n}{2}) > 1$. So, the independence number is at most 3 formed by the vertices [2] (or [4]), $[\frac{m}{2}]$ and $[\frac{n}{2}]$.

Case 3: n is odd and m is even.
We have 2 subcases here:
Subcase (i): $\frac{m}{2}$ is even.
In this case, all the vertices have even order except for the vertex $[n]$. So, any one of the vertices $\left[\frac{m}{2}\right]$, $[m], [2], [4]$ together with the vertex $[n]$ form an independent set if $gcd(n, \frac{m}{2}) = 1$. If $gcd(n, \frac{m}{2}) > 1$, then the vertex $[n]$ together with the vertex [2] (or [4]) form an independent set.

Subcase (ii): $\frac{m}{2}$ is odd.
The graph structures for this case are similar to the graph structures discussed in **Case 2 Subcase (iii)** and hence the independence number is at most 3.

4 Conclusion

In this paper, various graph parameters of conjugate graphs and conjugacy class graphs related to direct products of dihedral groups are obtained. The chromatic number and independence number of the conjugate graphs of $D_n \times D_m$ and the complement of the conjugate graphs and their line graphs are computed with the help of conjugacy classes of D_n. The conjugacy class graphs are found to be nonplanar connected graphs with diameter 2. The chromatic number and clique number of these graphs are found and a bound for the independence number is obtained. The conjugacy class graphs of $D_n \times D_m$ are found to be both Eulerian and Hamiltonian for certain values of n and m.

To summarize, the conjugacy class graphs related to the direct product of dihedral groups are constructed using the size of conjugacy classes of dihedral groups, which allowed us to obtain different parameters of the concerned graphs.

The method followed herein to obtain these results can be extended to any non-abelian group for which the conjugacy classes can be found, manually or with the help of a computer.

Work is in progress to compute the conjugacy classes related to the direct product of symmetric and alternating groups and thereby obtain the graph structures for these groups.

5 Future Study

The generalized conjugacy class graphs of dihedral groups and generalized quaternion groups and their related direct product graphs can be found. Also, the partition dimension of conjugacy class graphs related to the above-mentioned groups will be an interesting topic of further research, and bounds for partition dimension can be established, if it exists.

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On behalf of all the authors, the corresponding author states that the manuscript is an original work of the authors and all authors made a significant contribution to this paper.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

The authors declare that there is no conflict of interest.

Conflict of Interest

No funding was received for conducting this study. This manuscript has not been submitted for publication elsewhere and has not been published in any other journal.