Estimates and Radii of Convexity in Some Classes of Regular Functions

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Abstract: - A class $C_n(\lambda,\delta,a,\gamma)$ is being introduced regular in the circle $E=\{z:|z|<1\}$ functions $f(z)$, satisfying the condition $|((1-\lambda z^n)(1-\delta z^n)f'(z))^{1/\gamma}-a|\le a,\ z\in E,$ where $\lambda,\delta\ge 0, 0<\gamma\le 1, a>1/2, n\ge 1$. Class $C_n(\lambda,\delta,a,\gamma)$ generalizes various subclasses of close-to-convex functions, including functions which are convex in a certain direction and functions with limited rotation. Estimates of the derivative and logarithmic derivative of the function $f(z)\in C_n(\lambda,\delta,a,\gamma)$ are found, and also the radii of the convexity of the class $C_n(\lambda,\delta,a,\gamma)$. The case is also considered when the function $f(z)$ has gaps in the expansion in a row. Similar results are formulated for the class $T_n(\lambda,\delta,a,\gamma)$ of functions $F(z)$, satisfying the condition $|((1-\lambda z^n)(1-\delta z^n)F(z)/z)^{1/\gamma}-a|\le a,\ z\in E,$ which generalizes classes of typically real and close-to-starlike functions. All results are accurate. With the appropriate selection of parameter values of $\lambda,\delta,a,\gamma,n$ both new and previously published results are obtained.

Key-Words: - geometric theory of functions, estimates of regular functions, radii of convexity, close-to-convex functions, typically real functions, close-to-starlike functions.

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1 Introduction

Let $\Re$ denotes a class of functions $\varphi(z)=c_0+c_1z+a_2z^2+\ldots,$ regular in the circle $E=\{z:|z|<1\}$, $\Re_n(c_0)$ – function class $\varphi(z)\in\Re$ with decomposition $\varphi(z)=c_0+c_nz^n+a_{n+1}z^{n+1}+\ldots,n\ge 1.$ Function class $f(z)\in\Re$, normalized by the condition $f(0)=f'(0)-1=0,$ with decomposition of the form

$$f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\ldots,n\ge 1,z\in E, \quad (1)$$

we denote by $\mathcal{N}_n$, and $\mathcal{N}_1\equiv\mathcal{N}$. Let S,S^0,S^* and K denote, respectively, the classes of univalent, convex, starlike and close-to-convex functions $f(z)\in\mathcal{N}$.

Close-to-convex functions are introduced in the works [1], [2], where it is established that the function $f(z)\in\mathcal{N}$ is a univalent in E if and only if there exists a function $g(z)\in S^0$ such that

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)\ge 0,\ z\in E. \quad (2)$$

Each function $g(z)$ corresponds to its class K_g close-to-convex functions $f(z)$, satisfying condition (2), and $K=\cup K_g\ \forall g(z)\in S^0$. In this case, the functions $f(z)$ class K_g to a certain extent, they inherit the properties of the function $g(z)$. Of these, classes are often explored $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3$ close-to-convex functions $f(z)\in\mathcal{N}$, satisfying the conditions

$$\mathcal{F}_1:\operatorname{Re}((1-z^2)f'(z))\ge 0, \quad (3)$$

$$\mathcal{F}_2:\ \operatorname{Re}((1-z)^2f'(z))\ge 0, \quad (4)$$

$$\mathcal{F}_3:\ \operatorname{Re}((1-z)f'(z))\ge 0 \quad (5)$$

or their generalizations, while the classes $\mathcal{F}_1,\mathcal{F}_2$ characterized by an original geometric property of the area $f(E)$– convexity in the direction of the imaginary axis, [3], [4] (for the class $\mathcal{F}_1$) and convexity in the positive direction of the real axis [5] (for the class $\mathcal{F}_2$), what is related to the simplicity of the geometric properties of convex functions

$$g_1(z) = \frac{1}{2}\ln\frac{1+z}{1-z},\ g_2(z) = \frac{z}{1-z},\ g_3(z) = -\ln(1-z),$$

on the basis of which these classes are built.

Works [3], [4] marked the beginning of a whole series of studies of the class $\mathcal{F}_1$ of functions convex in the direction of the imaginary, as well as various generalizations of this class. This includes, for example, articles [6], [7]. In [6], the class $\mathcal{F}_1$ is investigated, in [7] – class $C_\beta(\lambda)$ functions $f(z) \in \mathcal{N}$, satisfying the condition

$$\mathrm{Re}\,[e^{i\beta}(1-\lambda z^2) f'(z)] \ge 0, 0 \le \lambda \le 1,\ \beta \in R,$$

where both the distortion, growth and coverage theorems and coefficient estimates are investigated. At $\lambda = 1, \beta = 0$ this condition sets the class $\mathcal{F}_1$, and at $\lambda \to 0$ a simple parametric transition from the class $\mathcal{F}_1$ to the class of functions with limited rotation, is performed which is set using the condition $\mathrm{Re}[e^{i\beta} f'(z)] \ge 0$ and it is well known as a classic sign of univalentness from [8], [9].

In [10], [11] is introduced a subclass $\mathcal{K}(\gamma) \subset K$ functions of close-to-convex order γ, satisfying the condition

$$\left|\arg\frac{f'(z)}{g'(z)}\right| \le \gamma\frac{\pi}{2},\ g(z) \in S^{\circ}, 0 < \gamma \le 1, z \in E \quad (6)$$

which aroused great interest due to such a visual geometric property as reachability from outside the area $f(\mathcal{E})$ solution angles $(1-\gamma)\pi$. By analogy, subclasses were also introduced $\mathcal{F}_1(\gamma)$, $\mathcal{F}_2(\gamma)$, $\mathcal{F}_3(\gamma)$ functions satisfying, respectively, the conditions

$$\left|arg((1-z^2) f'(z))\right| \le \gamma\frac{\pi}{2},\ \left|arg((1-z)^2 f'(z))\right| \le \gamma\frac{\pi}{2}$$

and $\left|arg((1-z) f'(z))\right| \le \gamma\frac{\pi}{2}$.

Almost simultaneously, in articles [12], [13], the angular reachability of two classes of functions generalizing classes was investigated $\mathcal{F}_1, \mathcal{F}_2$. At the same time, in [12] emphasis is placed on obtaining distortion and growth theorems, as well as finding the radii of convexity, and in the article [13] the geometric properties of the domain of values of functions from the introduced classes are investigated.

By analogy with condition (2) for the class K of close-to-convex functions, in [14] introduced the class CS^* close-to-starlike functions $F(z) \in \mathcal{N}$ satisfying the condition $\mathrm{Re}\frac{F(z)}{g(z)} \ge 0$, where $g(z) \in S^*, z \in E$. The subclasses of the CS^* class are the classes CS_1^*, CS_2^*, CS_3^* of close-to-starlike functions $f(z)$, respectively satisfying the conditions:

$$Re\,((1-z^2)F(z)/z) \ge 0,$$
$$Re\,((1-z)^2 F(z)/z) \ge 0,$$
$$Re\,((1-z)F(z)/z) \ge 0.$$

There is a simple relationship between classes $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3$ and CS_1^*, CS_2^*, CS_3^*, which is expressed by the ratio:

$$f(z) \in C_k \Leftrightarrow F(z) = zf'(z) \in CS_k^*, k = 1,2,3$$

Classes CS_1^*, CS_2^*, CS_3^* were studied in articles [15], [16], [17], [18], [19] from the point of view of estimating the coefficients and finding the radii of star formation of these classes relative to various subclasses of the class S^*.

Thus, based on the above-described literature review, the issue of studying the extreme properties of various N_n subclasses remains relevant at present.

In this paper, a fairly wide class of regular functions is introduced and in this class, irreducible distortion theorems and exact convexity radii are found and also, as a consequence, similar results are obtained in other subclasses of close-to-convex functions. In addition, in the class of functions introduced in the article, exact theorems on distortions and coverings, as well as starlikeness radii, generalizing classes of typically real and close-to-starlike functions, are obtained.

2 Problem Formulation

The purpose of this article is to introduce a sufficiently wide class of regular functions with decomposition of the form (1), including the subclasses of close-to-convex functions listed in the introduction with obvious geometric properties, in order to conduct their study from a unified point of view. As an application of the main results for close-to-convex functions, using the ratio $F(z)=zf'(z)$, these results are transferred to subclasses of close-to-starlike functions, complementing or generalizing some of the known results.

Definition. *We will assume that the function* $f(z)$ *belongs to the class* $C_n(\lambda,\delta,a,\gamma)$, $\lambda,\delta\ge 0$, $0<\gamma\le 1, a>1/2, n\ge 1$, *if and if* $f(z)\in\mathcal{N}$ *satisfies the condition*

$$\left|\left((1-\lambda z^n)(1-\delta z^n)f'(z)\right)^{1/\gamma}-a\right|\le a,\ z\in E \qquad (7)$$

in doing so

$$\frac{\lambda}{1+\lambda}+\frac{\delta}{1+\delta}\le\frac{1}{n}. \qquad (8)$$

It follows from (7) that $\operatorname{Re}((1-\lambda z^n)(1-\delta z^n)f'(z))^{1/\gamma}\ge 0$ for all $a>1/2$, which in terms of dependence of functions can be written as $(1-\lambda z^n)(1-\delta z^n)f'(z)\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$. That is $C_n(\lambda,\delta,a,\gamma)\subset C_n(\lambda,\delta,\infty,\gamma)$ and the condition is fulfilled

$$|\arg((1-\lambda z^n)(1-\delta z^n)f'(z))|\le\gamma\frac{\pi}{2},\ z\in E.$$

Therefore, the functions $f(z)$ from $C_n(\lambda,\delta,a,\gamma)$ is univalent and close-to-convex of the order γ, since satisfy condition (6) with convex in E function $g(z)=\int_0^z\frac{dt}{(1-\lambda t^n)(1-\delta t^n)}$. Indeed, $g'(z)=\frac{1}{(1-\lambda z^n)(1-\delta z^n)}$ and therefore

$$\operatorname{Re} z\frac{g''(z)}{g'(z)}=\operatorname{Re}\frac{\lambda nz^n}{1-\lambda z^n}+\operatorname{Re}\frac{\delta nz^n}{1-\delta n^n}\ge-\frac{\lambda n}{1+\lambda}-\frac{\delta n}{1+\delta}\ge-1,$$
$z\in E.$

Due to condition (8), so $g(z)\in S^{\circ}$.

We note also that the classes $\mathcal{F}_1(\gamma), \mathcal{F}_2(\gamma), \mathcal{F}_3(\gamma)$, as well as $C_\beta(\lambda)$ at $\beta=0$, are subclasses of the class $C_n(\lambda,\delta,a,\gamma)$.

Consider the function $f_0:E\to C$, which is defined by the formula

$$f_0(z)=\int_0^z\left(\frac{a(1+t)}{a-(a-1)t}\right)^{\gamma}\frac{dt}{(1-\lambda t^n)(1-\delta t^n)}$$

where $\lambda,\delta,a,\gamma,n$ are fixed real parameters $n\in N$, $\lambda,\delta\ge 0,\ 0<\gamma\le 1,\ a>\frac{1}{2}, n\ge 1$ and the inequality (8).

Then $((1-\lambda z^n)(1-\delta z^n)f'(z))^{1/\gamma}=w(z)$, where $((1-\lambda z^n)(1-\delta z^n)f'(z))^{1/\gamma}=w(z)$ – mapping the disk E to a disk $\{w:|w-a|<a\}$.

Therefore, $f_0(z)$ satisfies condition (7). Hence, $f_0(z)\in C_n(\lambda,\delta,a,\gamma)$ and $C_n(\lambda,\delta,a,\gamma)\ne\varnothing$. This function will be one of the extremal functions in Theorems 1 and 2. Other examples of functions $f_1(z)$ and $f_2(z)$ from $C_n(\lambda,\delta,a,\gamma)$ can be found in the proofs of Theorems 1 and 2.

The main research method of the article is a fairly easy-to-use and at the same time effective method of subordination of regular functions. They say that the regular function $\varphi(z)$ in E is subordinate to the function $\varphi_0(z)$ and write $\varphi(z)\prec\varphi_0(z)$, if there exists a function $\omega(z),\omega(0)=0,|\omega(z)|\le 1$ in E such that $\varphi(z)=\varphi_0(\omega(z))$. In the case of a univalent function $\varphi_0(z)$, the subordination ratio $\varphi(z)\prec\varphi_0(z)$ means that $\varphi(E)\subset\varphi_0(E)$ and $\varphi(0)=\varphi_0(0)$, which implies the embedding of closed regions $\varphi(|z|\le r)\subseteq\varphi_0(|z|\le r^n)$ at $0\le r<1$ for the function $\varphi(z)\in\mathrm{R}_n(\varphi_0(0))$.

In the works of many authors, the proof of various estimates related to the function $\varphi(z)$ is based directly on the use of the relation $\varphi(z)=\varphi_0(\omega(z))$ using estimates for $\omega(z)$. Another approach to obtaining estimates of the function $\varphi(z)$ is to use the embedding ratio of closed regions $\varphi(|z|\le r)\subseteq\varphi_0(|z|\le r^n)$. Hence, based on the geometric properties of the set $\varphi_0(|z|\le r^n)$, estimates of the real, imaginary part, argument and modulus of the function $\varphi(z)$ are easily derived. Adjacent to this approach is the finding of estimates $|\varphi'(z)|$ based on the properties of the inner radius of the region.

3 Problem Solution

3.1 Distortion Theorem and Radius of Convexity of Class $C_n(\lambda,\delta,a,\gamma)$

Theorem 1. *Let* $f(z)\in C_n(\lambda,\delta,a,\gamma)$. *Then when* $|z|=r<1$ *there are accurate estimates*

$$\left(\frac{a(1-r)}{a+(a-1)r}\right)^{\gamma}\frac{1}{(1+\lambda r^n)(1+\delta r^n)}\le |f'(z)|\le$$

$$\le\left(\frac{a(1+r)}{a-(a-1)r}\right)^{\gamma}\frac{1}{(1-\lambda r^n)(1-\delta r^n)},\qquad(9)$$

$$\left|z\frac{f''(z)}{f'(z)}-\frac{\lambda nz^n}{1-\lambda z^n}-\frac{\delta nz^n}{1-\delta z^n}\right|\le\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}.\qquad(10)$$

Proof. In terms of dependence, condition (7) means that

$$\varphi(z)=(1-\lambda z^n)(1-\delta z^n)f'(z)\prec\varphi_0(z)=\left(\frac{a(1+z)}{a-(a-1)z}\right)^{\gamma},\qquad(11)$$

where $w(z)=\dfrac{a(1+z)}{a-(a-1)}$ at $a>1/2$ displays a circle E on the circle $\{w:|w\text{-}a|\le a\}$. due to dependence, the inclusion of areas is performed $\varphi(|z|\le r)\subset\varphi_0(|z|\le r)$ at any $r, 0\le r<1$, that is, taking into account the type of area $\varphi_0(E)$ in the circle $|z|\le r$ inequalities are fulfilled

$$\left(\frac{a(1-r)}{a+(a-1)r}\right)^{\gamma}\le|\varphi(z)|=$$

$$=\left|(1-\lambda z^n)(1-\delta z^n)f'(z)\right|\le\left(\frac{a(1+r)}{a-(a-1)r}\right)^{\gamma}\qquad(12)$$

Due to $\lambda>0,\delta>0$, then when $|z|=r<1$ we have

$$(1-\lambda r^n)(1-\delta r^n)\le\left|(1-\lambda z^n)(1-\delta z^n)\right|\le(1+\lambda r^n)(1+\delta r^n).$$

Combining this estimate with the estimate (12), we come to (9).

To prove the estimate (10), we use lemma 1, which follows by replacing $\varphi(z)=p^{\gamma}(z)$ from the results from [20] for the class of functions $p(z)=1+c_nz^n+c_{n+1}z^{n+1}+\dots, n\ge1, z\in E,$ satisfying the condition $|p(z)-a|\le a, a>1/2$.

Lemma 1. *If* $\varphi(z)$ *from* $\mathcal{R}_n(1)$ *satisfies the condition* $\left|(\varphi(z))^{1/\gamma}-a\right|\le a,\ a>1/2,\ 0<\gamma\le1,\ z\in E,$ *then when* $|z|=r<1$ *there is an accurate assessment*

$$\left|z\frac{\varphi'(z)}{\varphi(z)}\right|\le\frac{\gamma(2a-1)nr^n}{(1-r^n)(a+(a-1)r^n)},\qquad(13)$$

which is achieved for the function $\varphi(z)=\varphi_0(z^n)$ *at the point* $z=\sqrt[n]{-1}r$, *where* $\varphi_0(z)=\left(\dfrac{a(1+z)}{a-(a-1)z}\right)^{\gamma}$.

Due to $\varphi(z)\prec\varphi_0(z)$, then considering that $f(z)\in\mathcal{N}$, that of (13) when $n=1$ we get the estimate (10):

$$\left|z\frac{\varphi'(z)}{\varphi(z)}\right|=\left|z\frac{f''(z)}{f'(z)}-\frac{\lambda nz^n}{1-\lambda z^n}-\frac{\delta nz^n}{1-\delta z^n}\right|\le\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}.$$

Let us now prove that estimates (9) and (10) cannot be improved.

For the function

$$f_0(z)=\int_0^z\left(\frac{a(1+t)}{a-(a-1)t}\right)^{\gamma}\frac{dt}{(z-\lambda t^n)(1-\delta t^n)}\in C_n(\lambda,\delta,a,\gamma)$$

we have

$$f_0'(z)=\left(\frac{a(1+z)}{a-(a-1)z}\right)^{\gamma}\frac{1}{(1-\lambda z^n)(1-\delta z^n)},$$

$$z\frac{f_0''(z)}{f_0'(z)}-\frac{\lambda nz^n}{1-\lambda z^n}-\frac{\delta nz^n}{1-\delta z^n}=\frac{\gamma(2a-1)z}{(1+z)(a-(a-1)z)}.$$

Therefore, the right estimate in (9) is achieved for the function $f_0(z)$ at the point $z=r$, and the estimate (10) is achieved for the function $f_0(z)$ at the point $z=-r$, because

$$\left.\left(z\frac{f_0''(z)}{f_0'(z)}-\frac{\lambda nz^n}{1-\lambda z^n}-\frac{\delta nz^n}{1-\delta z^n}\right)\right|_{z=-r}=-\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}.$$

If n – odd and $z=-r$, that $z^n=-r^n$. Therefore, in the left estimate (9) for the function $f_0(z)$ at the point $z=-r$ the equal sign is reached again, since

$$f_0'(z)\big|_{z=-r}=\left(\frac{a(1-r)}{a+(a-1)r}\right)^{\gamma}\frac{1}{\left(1+\lambda r^n\right)\left(1+\delta r^n\right)}$$

Let now $n-$ even.

If $n=2,6,10,\ldots,2(2k-1),\ldots,k\in\mathrm{N}$. Then $i^n=-1$ and for $z=ir$ we have $z^n=i^n r^n=-r^n, iz=-r$. Therefore, for the function

$$f_1(z)=\int_0^z\left(\frac{a(1+it)}{a-(a-1)it}\right)^{\gamma}\frac{dt}{(1-\lambda t^n)(1-\delta t^n)}\in C_n(\lambda,\delta,a,\gamma)$$

at the point $z=ir$ we get

$$f_1'(z)\big|_{z=ir}=\left(\frac{a(1+iz)}{a-(a-1)iz}\right)^{\gamma}\frac{1}{(1-\lambda z^n)(1-\delta z^n)}\bigg|_{z=ir}=$$

$$=\left(\frac{a(1-r)}{a+(a-1)r}\right)^{\gamma}\frac{1}{(1+\lambda r^n)(1+\delta r^n)},$$

which proves that the left estimate (9) is not improved when $n=2(2k-1),k\in\mathrm{N}$.

If $n=4,8,12,\ldots,4k,\ldots,k\in\mathrm{N}$, that $i^n=1$ and for $z=i^{\frac{n+2}{n}}r$ we have $z^n=i^{n+2}r^n=-r^n$. Besides, $i^{\frac{n-2}{n}}z=i^{\frac{n-2}{n}}\cdot i^{\frac{n+2}{n}}r=i^2r=-r$. Therefore , at the point $z=i^{\frac{n+2}{n}}r$ or the function

$$f_2(z)=\int_0^z\left(\frac{a(1+i^{\frac{n-2}{n}}t)}{a-(a-1)i^{\frac{n-2}{n}}t}\right)^{\gamma}\frac{dt}{(1-\lambda t^n)(1-\delta t^n)}\in C_n(\lambda,\delta,a,\gamma)$$

we get

$$f_2'(z)\Big|_{z=i^{\frac{n+2}{n}}}=\left(\frac{a(1+i^{\frac{n-2}{n}}z)}{a-(a-1)i^{\frac{n-2}{n}}z}\right)^{\gamma}\frac{1}{(1-\lambda z^n)(1-\delta z^n)}\bigg|_{z=i^{\frac{n+2}{n}}r}=$$

$$=\left(\frac{a(1-r)}{a+(a-1)r}\right)^{\gamma}\frac{1}{(1+\lambda r^n)(1+\delta r^n)},$$

which proves the unimprovability of the left estimate (15) in this case as well.
The theorem is proved.

At $\delta=0$ from theorem 1 follows

Corollary 1. *If the function* $f(z)$ *satisfies the condition*

$$\left|((1-\lambda z^n)f'(z))^{1/\gamma}-a\right|\le a\,,$$

then at $|z|=r<1$ *there are accurate estimates*

$$\left(\frac{a(1-r)}{a+(a-1)r}\right)^{\gamma}\frac{1}{1+\lambda r^n}\le|f'(z)|\le\left(\frac{a(1+r)}{a-(a-1)r}\right)^{\gamma}\frac{1}{1-\lambda r^n}$$

,

$$\left|z\frac{f''(z)}{f'(z)}-\frac{\lambda nz^n}{1-\lambda z^n}\right|\le\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}.$$

We consider the limiting case of corollary 1 when $a\to\infty$.

Corollary 2. *Where* $f(z)\in C_n(\lambda,0,\infty,\gamma)$, *so* $f(z)$ *satisfies the condition*

$$\left|\arg((1-\lambda z^n)f'(z))\right|\le\gamma\frac{\pi}{2},z\in E\,.$$

Then at $|z|=r<1$ *accurate estimates are made*

$$\left(\frac{1-r}{1+r}\right)^{\gamma}\frac{1}{1+\lambda r^n}\le|f'(z)|\le\left(\frac{1+r}{1-r}\right)^{\gamma}\frac{1}{1-\lambda r^n},$$

$$\left|z\frac{f''(z)}{f'(z)}-\frac{\lambda nz^n}{1-\lambda z^n}\right|\le\frac{2\gamma r}{1-r^2}.$$

Corollary 2 at $n=2,\lambda=\gamma=1$ obtained in [6], when $n=2,\lambda=1,\,0<\gamma\le 1-$ in [12], and for the case $n=2,0\le\lambda\le1,\gamma=1,$ coincides with a special case of grades for a class $C_\beta(\lambda)$ at $\beta=0$ from [7].

Note also that when $n=\lambda=1,a\to\infty$ corollary 2 gives estimate for the class $\mathcal{F}_3(\gamma)$.

At $\lambda=\delta\,,a\to\infty$ the following result follows from Theorem 1.

Corollary 3. *If* $f(z)\in C_n(\lambda,\delta,a,\gamma)$, *so* $f(z)$ *satisfies the condition* $\left|\arg((1-\lambda z^n)^2f'(z))\right|\le\gamma\frac{\pi}{2}$, $z\in E$, *then when* $|z|=r<1$ *accurate estimates are made*

$$\left(\frac{1-r}{1+r}\right)^{\gamma}\frac{1}{(1+\lambda r^n)^2}\le|f'(z)|\le\left(\frac{1+r}{1-r}\right)^{\gamma}\frac{1}{(1-\lambda r^n)^2},$$

$$\left|z\frac{f''(z)}{f'(z)}-\frac{2\lambda nz^n}{1-\lambda z^n}\right|\le\frac{2\gamma r}{1-r^2}.$$

We note that when $n=\lambda=1$ from corollary 3, the estimate for the class are obtained $\mathcal{F}_2(\gamma)$.

Theorem 2. *The exact radius of convexity* r_0 *of class* $C_n(\lambda,\delta,a,\gamma)$ *is determined as the only root of the equation on the interval* $(0;1)$

$$1-\frac{y(2a-1)r}{(1-r)(a+(a-1)r)}-\frac{\lambda nr^n}{1+\lambda r^n}-\frac{\delta nr^n}{1+\delta r^n}=0 \quad (14)$$

Proof. Let's check that in the circle $|z|\le r_0$ the convexity condition is satisfied $1+\operatorname{Re} z\frac{f''(z)}{f'(z)}\ge 0$. Due to the estimate (10) we get

$$\left|\operatorname{Re}\left(z\frac{f''(z)}{f'(z)}-\frac{\lambda nz^n}{1-\lambda z^n}-\frac{\delta nz^n}{1-\delta z^n}\right)\right|\le\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)},$$

from where for all $z,|z|\le r$ we find

$$1+\operatorname{Re} z\frac{f''(z)}{f'(z)}\ge 1-\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}+ \\ +\min_{|z|\le r}\operatorname{Re}\left(\frac{\lambda nz^n}{1-\lambda z^n}\right)+\min_{|z|\le r}\operatorname{Re}\left(\frac{\delta nz^n}{1-\delta z^n}\right)$$

Because for any $t,0<t\le 1$, equality is fulfilled

$$\min_{|\zeta|\le r}\left(\frac{tn\zeta}{1-t\zeta}\right)=\left.\frac{tn\zeta}{1-t\zeta}\right|_{z=-r}=-\frac{tnr}{1+tr},$$

that

$$1+\operatorname{Re} z\frac{f''(z)}{f'(z)}\ge 1-\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}- \\ -\frac{\lambda nr^n}{1+\lambda r^n}-\frac{\delta nr^n}{1+\delta r^n}.$$

Therefore, if $r=r_0\in(0;1)$ – the root of equation (14), then in the circle $|z|\le r_0$ the convexity condition is met.

We now show that equation (14) has a single root r_0 on the interval $(0;1)$. We denote

$$\psi_0(r)=1-\frac{\lambda nr^n}{1+\lambda r^n}-\frac{\delta nr^n}{1+\delta r^n},\ M(r)=\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}.$$

Because $\psi_0(r)$ decreases by $(0;1)$ from $\psi_0(0)=1$ before $\psi_0(1)$, in doing so $\psi_0(1)\in[0;1]$ in virtue of (8), $M(r)$ increases by $(0;1)$ from 0 before $+\infty$, then on $(0;1)$ there is a single point r_0, in which $\psi_0(r)=M(r)$, that was what needed to be proved.

We show that the radius of the convexity cannot be improved.

For odd n we consider the extremal function

$$f_0(z)=\int_0^z\left(\frac{a(1+t)}{a-(a-1)t}\right)^{\gamma}\frac{dt}{(1-\lambda t^n)(1-\delta t^n)}\in C_n(\lambda,\delta,a,\gamma)$$

At the point $z=-r$, where $r=r_0$, considering that $z^n=-r^n$, we get

$$1+z\left.\frac{f_0''(z)}{f_0'(z)}\right|_{z=-r}=\left.\left(\begin{gathered}1+\frac{\gamma(2a-1)z}{(1+z)(a-(a-1)z)}+\\+\frac{\lambda nz^n}{1-\lambda z^n}+\frac{\delta nz^n}{1-\delta z^n}\end{gathered}\right)\right|_{z=-r}=$$

$$=1-\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}-\frac{\lambda nr^n}{1+\lambda r^n}-\frac{\delta nr^n}{1+\delta r^n}=0.$$

That is, for odd n the radius of convexity is accurate.

If $n-$ even, then, as in proving the accuracy of the left estimate (9), we consider two cases.

1) Let $n=2,6,10,..2(2k-1)...k\in\mathrm{N}$. Then $i^n=-1$ and $z^n=-r^n$ for $z=ir$. For the function

$$f_1(z)=\int_0^z\left(\frac{a(1+it)}{a-(a-1)it}\right)^{\gamma}\frac{dt}{(1-\lambda t^n)(1-\delta t^n)}\in C_n(\lambda,\delta,a,\gamma)$$

we have

$$1+z\frac{f_1''(z)}{f_1'(z)}=1+\frac{\gamma(2a-1)iz}{(1+iz)(a-(a-1)iz)}+\frac{\lambda nz^n}{1-\lambda z^n}+\frac{\delta nz^n}{1-\delta z^n},$$

and at the point $z=ir$, where $r=r_0$, we find

$$1+z\left.\frac{f_1''(z)}{f_{01}'(z)}\right|_{z=ir}=1-\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}-\frac{\lambda nz^n}{1+\lambda z^n}-\frac{\delta nz^n}{1+\delta z^n}=0$$

Therefore, in this case, the radius of convexity cannot be improved.

2) Let $n=4,8,12,\ldots,4k,\ldots,k\in\mathrm{N}$. we denote $\varepsilon=i^{\frac{n+2}{n}}$. Then $i^n=1$, $\varepsilon^n=-1$ and for $z=-\varepsilon r$ we get $z^n=-r^n,\varepsilon^{-1}z=-r$. Because for the function

$$f_2(z)=\int_0^z\left(\frac{a(1+\varepsilon^{-1}t)}{a-(a-1)\varepsilon^{-1}t}\right)^{\gamma}\frac{dt}{(1-\lambda t^n)(1-\delta t^n)}\in C_n(\lambda,\delta,a,\gamma)$$

we have:

$$1+z\frac{f_2''(z)}{f_2'(z)}=1+\frac{\gamma(2a-1)\varepsilon^{-1}z}{(1+iz)(a-(a-1)\varepsilon^{-1}z)}+\frac{\lambda nz^n}{1-\lambda z^n}$$

Because of this, at the point $z=-\varepsilon r$, where $r=r_0$, find

$$1+z\left.\frac{f_2''(z)}{f_2'(z)}\right|_{z=-\varepsilon r}=1-\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}-\frac{\lambda nr^n}{1+\lambda r^n}-\frac{\delta nr^n}{1+\delta r^n}=0,$$

which proves that the radius of convexity is unimprovable in this case as well.

Theorem 2 is proved.

At $\delta=0, n=2$ theorem 2 implies

Corollary 4. *Accurate radius of convexity* r_0 *function class* $f(z)\in\mathcal{N}$*, satisfying the condition*

$$\left|((1-\lambda z^2)f'(z))^{1/\gamma}-a\right|\le a,\ 0\le\lambda\le 1, 0<\gamma\le 1, a>1/2,\ z\in E,$$

is determined as the only root of the equation on the interval $(0;1)$

$$(1-\lambda z^2)(1-r)(a+(a-1)r)-\gamma(2a-1)r(1+\lambda r^2)=0.$$

At $\lambda=1, a\to\infty$ from Corollary 4 we obtain the radius of convexity of the class $\mathcal{F}_1(\gamma)$ from [12], and for $\gamma=1, a\to\infty-$ class $\mathcal{F}_1$ radius of convexity from [6].

3.2 Distortion Theorems and Radii of Convexity in the Case When $f(z)\in\mathbf{N}_n$

Theorem 3. *Let the function* $f(z)$ *with a decomposition of the form (1) belongs to the class* $C_n(\lambda,\delta,a,\gamma)$. *Then at* $|z|=r, 0<r<1$ *there are exact estimates*

$$\left(\frac{a(1-r^n)}{a+(a-1)r^n}\right)^{\gamma}\frac{1}{(1+\lambda r^n)(1+\delta r^n)}\le|f'(z)|\le$$

$$\le\left(\frac{a(1+r^n)}{a-(a-1)r^n}\right)^{\gamma}\frac{1}{(1-\lambda r^n)(1-\delta r^n)} \tag{15}$$

$$\left|z\frac{f''(z)}{f'(z)}-\frac{\lambda nz^n}{1-\lambda z^n}-\frac{\delta nz^n}{1-\delta z^n}\right|\le$$

$$\le\frac{\gamma(2a-1)nr^n}{(1-r^n)(a+(a-1)r^n)} \tag{16}$$

and the exact radius of convexity r_0 *class* $C_n(\lambda,\delta,a,\gamma)$ *is determined as the only root of the equation on the interval* $(0;1)$

$$1-\frac{\gamma(2a-1)nr^n}{(1-r^n)(a+(a-1)r^n)}-$$

$$-\frac{\lambda nr^n}{1+\lambda r^n}-\frac{\delta nr^n}{1+\delta r^n}=0 \tag{17}$$

The proof of Theorem 3 is carried out similarly to the proofs of Theorems 1 and 2, only Lemma 1 is applied for the case $n\ge 1$ taking into account the fact that $f(z)\in\mathcal{N}_n$ and therefore $\varphi(z)$ from (11) has a decomposition of the form $\varphi(z)=1+c_nz^n+c_{n+1}z^{n+1}+\ldots,n\ge 1, z\in E$.

Corollary 5. *Let a function* $f(z)$ *belong to the class*

$$C_n(0,0,\infty,\gamma)=\left\{f(z)\in\mathcal{N}_n; \left|\arg\ f'(z)\right|\le\gamma\frac{\pi}{2}, 0<\gamma\le 1\right\}.$$

Then at $|z|=r, 0<r<1$ *there are exact estimates*

$$((1-r^n)/(1+r^n))^{\gamma}\le|f'(z)|\le((1+r^n)/(1-r^n))^{\gamma},$$

$$\left|z\frac{f''(z)}{f'(z)}\right|\le\frac{2\gamma nr^n}{1-r^{2n}}$$

and the exact radius r_0 *of the convexity class* $C_n(0,0,\infty,\gamma)$ *is determined by the formula* $r_0=(\sqrt{\gamma^2n^2+1}-\gamma n)^{1/n}$.

Corollary 5 follows from Theorem 3 for $\lambda=\delta=0, a\to\infty$.

This result for $\gamma=1$ coincides with the estimates and the radius of convexity $r_0=\sqrt{2}-1$ lass of functions with limited rotation $\mathrm{Re}\, f'(z)\ge 0, z\in E,$ obtained in [21].

At $\delta=0,\lambda=\gamma=1,n=2$ and at $\delta=\lambda=0$ the following corollaries follow from Theorem 3.

Corollary 6, [22]. *Accurate radius of the convexity* r_0 *of function class* $f(z)\in\mathcal{N}_2$*, satisfying the condition* $\left|(1-z^2)f'(z)-a\right|\le a, a>1/2,\ z\in E$, is determined as the only root of the equation on the interval $(0;1)$

$$(a-1)r^6-(5a-4)r^4-(5a-1)r^2+a=0.$$

Corollary 7. *Accurate radius of the convexity* r_0 *of function class* $f(z)\in\mathcal{N}_2$, *satisfying the* *condition* $\left| f'(z))^{1/\gamma} - a \right| \le a, a > 1/2,\ z \in E$ *is determined by the formula*

$$r_0 = \begin{cases} \dfrac{1+\gamma(2a-1)n - \sqrt{(1+\gamma(2a-1)n)^2 + 4a(a-1)}}{2(1-a)}, & a \ne 1; \\ (1+\gamma n)^{-1/n}, & a = 1. \end{cases} \quad (18)$$

At $\gamma = 1$ this result was obtained in [22].

3.3 Generalization of Classes of Typically Real and Close-To-Starlike Functions

In [23] introduced typically real functions, that is, the functions $F(z), F(0)=0, F'(0)=1,$ which takes real values at points $z \in (-1;1)$ and at other points of the circle E satisfy the condition $\operatorname{Im} F(z) \cdot \operatorname{Im} z > 0.$ In [23] he also obtained a membership criterion $F(z)$ to class T of typically real functions:

$$F(z) \in T \Leftrightarrow \operatorname{Re}\left(\frac{1-z^2}{z} F(z) \right) \ge 0, z \in E. (19)$$

By analogy with the class $C_n(\lambda,\delta,a,\gamma),$ $\lambda,\delta \ge 0, 0<\gamma\le 1, a>1/2, n\ge 1,$ we introduce a class $T_n(\lambda,\delta,a,\gamma)$ functions $F(z) \in \mathcal{N},$ satisfying the condition

$$\left| \left(\frac{(1-\lambda z^n)(1-\delta z^n)}{z} F(z) \right)^{1/\gamma} - a \right| \le a, z \in E, (20)$$

In doing so

$$\frac{\lambda}{1+\lambda} + \frac{\delta}{1+\delta} \le \frac{1}{n}. \quad (21)$$

Comparing (7) and (20), we get that:

$$f(z) \in C_n(\lambda,\delta,a,\gamma) \Leftrightarrow F(z) = zf'(z) \in T_n(\lambda,\delta,a,\gamma)\, (22)$$

and at $\delta = 0, n = 2, \lambda = \gamma = 1, a \to \infty$ class $T_n(\lambda,\delta,a,\gamma)$ is transformed into a class T of typically real functions.

Wherein, $T_2(1,0,a,\gamma) \subset T = T_2(1,0,\infty,1)$ for all $a > 1/2, 0 < \gamma \le 1.$

On the other hand, when $\lambda = \delta = 0, \gamma = 1, a \to \infty$ we obtain the class of close-to-starlike functions introduced in [14] using the condition $\operatorname{Re}\frac{F(z)}{z} \ge 0, z \in E.$ Wherein, $T_n(0,0,a,\gamma)$ is a subclass of the class of close-to-starlike functions for all $a > 1/2, 0<\gamma\le 1.$ We note also that if the class functions $C_n(\lambda,\delta,a,\gamma)$ are univalent, then the class functions $T_n(\lambda,\delta,a,\gamma)$ generally speaking, they are not.

Relation (22), taking into account the equality $\dfrac{zf''(z)}{f'z} = \dfrac{zF'(z)}{F(z)} - 1$ makes it easy to transfer all results for a class $C_n(\lambda,\delta,a,\gamma)$ per class $T_n(\lambda,\delta,a,\gamma)$.

Theorem 4. *Let* $F(z) \in \mathcal{N} \cap T_n(\lambda,\delta,a,\gamma).$ *Then at* $|z| = r < 1$ *there are exact estimates*

$$\left(\frac{a(1-r)}{a+(a-1)r} \right)^{\gamma} \frac{r}{(1+\lambda r^n)(1+\delta r^n)} \le |F(z)| \le$$

$$\le \left(\frac{a(1+r)}{a-(a-1)r} \right)^{\gamma} \frac{r}{(1-\lambda r^n)(1-\delta r^n)}, \quad (23)$$

$$\left| z\frac{F'(z)}{F(z)} - 1 - \frac{\lambda n z^n}{1-\lambda z^n} - \frac{\delta n z^n}{1-\delta z^n} \right| \le$$

$$\le \frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}. \quad (24)$$

and the exact radius of starlikeness r^* *of the class* $T_n(\lambda,\delta,a,\gamma)$ *is determined as the only root of equation (14) on the interval* $(0;1)$.

If, in addition to the conditions of the theorem, the function $F(z)$ *expands into a series of the form (1), then we have exact estimates*

$$\left(\frac{a(1-r^n)}{a+(a-1)r^n} \right)^{\gamma} \frac{r}{(1+\lambda r^n)(1+\delta r^n)} \le |F(z)| \le$$

$$\le \left(\frac{a(1+r^n)}{a-(a-1)r^n} \right)^{\gamma} \frac{r}{(1-\lambda r^n)(1-\delta r^n)}, \quad (25)$$

$$\left| z\frac{F'(z)}{F(z)} - 1 - \frac{\lambda n z^n}{1-\lambda z^n} - \frac{\delta n z^n}{1-\delta z^n} \right| \le$$

$$\le \frac{\gamma(2a-1)nr^n}{(1-r^n)(a+(a-1)r^n)}. \quad (26)$$

and the exact radius of starlikeness r^* *of the class* $T_n(\lambda,\delta,a,\gamma)$ *is defined as the only root of equation (17) on the interval* $(0;1)$.

Corollary 8. *Let* $F(z) \in T_n(\lambda,0,a,\gamma),$ *that is* $F(z)$ *satisfies the condition* $\left| \left((1-\lambda z^n)\dfrac{F(z)}{z} \right)^{1/\gamma} - a \right| \le a,\quad z \in E.$ *Then at* $|z| = r < 1$ *there are exact estimates*

$$\left(\frac{a(1-r)}{a+(a-1)r}\right)^{\gamma}\frac{r}{1+\lambda r^n}\le|F(z)|\le\left(\frac{a(1+r)}{a-(a-1)r}\right)^{\gamma}\frac{r}{1-\lambda r^n},$$

$$\left|z\frac{F'(z)}{F(z)}-1-\frac{1+\lambda(n-1)z^n}{1-\lambda z^n}\right|\le\frac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}$$

and the exact starlikeness radius r^* of the class $T_n(\lambda,0,a,\gamma)$ is determined as the only root of the equation on the interval $(0;1)$

$$\begin{cases} 1-\dfrac{\gamma(2a-1)nr^n}{(1-r^n)(a+(a-1)r^n)}-\dfrac{\lambda nr^n}{1+\lambda r^n}=0 \\ \text{for the case when } F(z)\in\mathcal{N}_n; \\ 1-\dfrac{\gamma(2a-1)r}{(1-r)(a+(a-1)r)}-\dfrac{\lambda nr^n}{1+\lambda r^n}=0 \\ \text{for the case when } F(z)\in\mathcal{N}. \end{cases}$$

At $\delta=0, n=2, \lambda=\gamma=1, a\to\infty$ class $T_n(\lambda,\delta,a,\gamma)$ coincides the class T and from Corollary 8 we obtain the well-known estimates [23] in the class T

$$\frac{(1-r)r}{(1-r)(1+r^2)}\le|F(z)|\le\frac{r}{(1-r)^n},\left|z\frac{F'(z)}{F(z)}-\frac{1+z^2}{1-z^2}\right|\le\frac{2r}{1-r^2}$$

and exact radius of starlikeness $r^*=\frac{1}{2}\left(\sqrt{5}+1-\sqrt{2(\sqrt{5}+1)}\right)$ of class T, previously found in [24].

We note also that when $n=2, \delta=0, \lambda=\gamma=1$ Corollary 8 the exact radius of starlikeness of the subclass of typically real functions $F(z)\in\mathcal{N}_2$, satisfying the condition $\left|(1-z^2)\frac{F(z)}{z}-a\right|\le a, z\in E,$ as the only root of the equation on the interval $(0;1)$ $(a-1)r^6-(5a-4)r^4-(5a-1)r^2+a=0,$ found in [22].

At $\lambda=\delta=0$ it turns out the class $\left\{F(z):\left|\left(\frac{F(z)}{z}\right)^{1/\gamma}-a\right|\le a, z\in E\right\},$ the exact radius starlikeness of which, in the case $F(z)\in\mathcal{N}_n$, is determined by formula (18), which coincides with the result from [22].

From here at $a\to\infty$ we get class $\left\{F(z):\left|\arg\frac{F(z)}{z}\right|\le\gamma\frac{\pi}{2}, z\in E\right\},$ whose exact radius of starlikeness is $r^*=(\sqrt{\gamma^2n^2+1}-\gamma n)^{1/n}$. At $n=\gamma=1$ is get the radius of starlikeness $r^*=\sqrt{2}-1$. from [21] of the class of close-to-starlike functions $F(z)\in\mathcal{N}$, satisfying the condition $\operatorname{Re}\frac{F(z)}{z}\ge 0, z\in E.$

Note. Recently, a whole series of articles has been published, [16], [17], [18], [19], devoted to finding the radii of starlikeness of various classes of close-to-starlike functions with respect to some subclasses of the class S^* (Recently, a whole series of articles has been published, [23], [24], devoted to finding the radii of starlikeness of various classes of close-to-starlike functions with respect to some subclasses of the class:

$$S_\beta^*=\left\{F(z):Re\, z\frac{F'(z)}{F(z)}\ge\beta, 0\le\beta<1, z\varsigma E\right\}$$

starlike functions $f(z)$ of order β, $S_L^*=\left\{F(z):\left|\left(z\frac{F'(z)}{F(z)}\right)^2-1\right|\le 1, z\varsigma E\right\}$ lemniscate starlike functions and others).

If in Theorems 2-3 instead of the convexity condition we use the condition $1+Re\, z\frac{f''(z)}{f'(z)}\ge\beta$ convexity of order $\beta, 0\le\beta<1,$ and accordingly in Theorem 4 – condition $Re\, z\frac{F'(z)}{F(z)}\ge\beta$ starlike of order β, then we obtain the exact radius of starlikeness of order β of class $T_n(\lambda,0,\alpha,\gamma)$ as the only one on the interval (0;1) root of the equation:

$$\begin{cases} 1-\beta-\dfrac{\gamma(2a-1)nr^n}{(1-r^n)(a+(a-1)r^n}-\dfrac{\lambda nr^n}{1+\lambda r^n}-\dfrac{\delta nr^n}{1+\delta r^n}=0 \\ \text{for the case when } F(z)\in\mathcal{N}_n; \\ 1-\beta-\dfrac{\gamma(2a-1)r}{(1-r)(a+(a-1)r}-\dfrac{\lambda nr^n}{1+\lambda r^n}-\dfrac{\delta nr^n}{1+\delta r^n}=0 \\ \text{for the case when } F(z)\in\mathcal{N}. \end{cases} \tag{27}$$

In special cases, when $F(z)\in N$, we obtain a number of well-known results.

1) For $n=2, \gamma=\lambda=1, \delta=0, a\to\infty$ we obtain the star radius [16] of order β class

$$K_1=\left\{F(z): \mathrm{Re}\frac{1-z^2}{z}F(z)\ge 0, z\in E\right\}$$

as the only root of the equation (0;1) on the interval

$$(1+\beta)r^4-2r(r^2+r+1)+(1-\beta)=0.$$

2) For $n=\gamma=\lambda=1, \delta=0, a\to\infty$ we obtain a star radius of order β class

$$F_3=\left\{F(z): \mathrm{Re}\frac{1-z}{z}F(z)\ge 0, z\in E\right\}$$

from [18] as the only root of the equation $\beta r^2-3r+1+\beta=0$, , on the interval (0;1), that is

$$r^*=\frac{3-\sqrt{9-4\beta(1-\beta)}}{2\beta}$$

3) For $n=\gamma=\lambda=\delta=1, a\to\infty$ we obtain a star radius of order β class

$$F_4=\left\{F(z): \mathrm{Re}\frac{(1-z)^2}{z}F(z)\ge 0, z\in E\right\}$$

from [18] as the only root of the equation $(1+\beta)r^2-4r+1-\beta=0$, on the interval (0;1), that is

$$r^*=\frac{2-\sqrt{3+\beta^2}}{1+\beta}.$$

Other applications of the $T_n(\lambda,\delta,a,\gamma)$ class of almost star-like functions, as well as promising tasks for further research, are to find estimates of the coefficients of functions of the $T_n(\lambda,\delta,a,\gamma)$ class, as well as the radii of the starlikeness of this class relative to such subclasses of the S^* class as $S^*(\alpha)=S^*\left((1+z)/(1-z)^{\alpha}\right)$ is a highly starlike, $S_P^*=S^*\left(1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1+\sqrt{z}}\right)\right)\right)$ parabolic, $S_L^*=S^*\left(\sqrt{1+z}\right)$ lemniscatic, $S_\pi^*=S^*\left(z+\sqrt{1+z^2}\right)$ moon-shaped, $S_{car}^*=S^*\left(1+ze^z\right)$ cardioid and other starlike functions.

4 Conclusion

The article introduces a class $C_n(\lambda,\delta,a,\gamma)$ of functions $f(z)$, satisfying the condition $|((1-\lambda z^n)(1-\delta z^n)f'(z))^{1/\gamma}-a|\le a, z\in E.$ It is shown that all functions of this class are close-to-convex and its subclasses are the classes of functions convex in a certain direction and functions with bounded rotation. In class $C_n(\lambda,\delta,a,\gamma)$ exact distortion theorems and exact convexity radii are found, which generalize previously known results. The case when the function $f(z)$ in the expansion in a power series has omissions of terms.

It is also considered the class $T_n(\lambda,\delta,a,\gamma)$ of functions $F(z)$, associated with class $C_n(\lambda,\delta,a,\gamma)$ functions $f(z)$ ratio $F(z)=zf'(z).$ Class $T_n(\lambda,\delta,a,\gamma)$ is a generalization of classes of typically real and close-to-starlike functions. Exact covering theorems and exact radii of starlikeness are obtained in the class, which generalize previously published results for typically real and close-to-starlike functions.

Thus, the authors of the article, based on a single approach, solved a number of extreme problems for fairly wide subclasses of close-to-convex and close-to-starlike functions. At the same time, along with the generalization of known results, new results are obtained in particular cases.

In conclusion, we note that the results of the article admit of a simple generalization if, instead of conditions (7)-(8), we use conditions of the form

$$\left|\left(\prod_{i=1}^{k}(1-\lambda_i z^n)\right)f'(z))^{\frac{1}{\gamma}}-a\right|\le a, k\ge 1, z\in E, (7')$$

and

$$\sum_{i=1}^{k}\frac{\lambda_i}{1+\lambda_i}\le\frac{1}{n}. \qquad (8')$$

It is known that the solution of many extreme problems for subclasses of functions $f(z)$ from $\mathcal{N}_n$ is reduced to finding, for $|z|=r, 0<r<1$, accurate estimates of the functionals

$$\max_{|z|=r}\left|z\frac{\varphi'(z)}{\varphi(z)}\right|, \min_{|z|=r}\left\{\mu Re\varphi(z)+\eta Re\, z\frac{\varphi'(z)}{\varphi(z)}\right\} (28)$$

on the class $\mathrm{P}_n(\varphi_0)$ defining the class of functions $f(z)$. In this regard, the application of evaluation (13) for the class of regular functions $\varphi(z)$ from R_n satisfying the condition $\left|(\varphi(z))^{1/\gamma}-a\right|\le a$,

$a > 1/2, 0 < \gamma \le 1$, is promising, as it will further solve a number of extreme problems related to various subclasses of univalent functions, which in this article is demonstrated by the example of the class $C_n(\lambda, \delta, a, \gamma)$.

As directions for further research, we note the following.

1) Introduction of a unified class $\mathrm{P}_n(\varphi_0)$ of regular functions $\varphi(z)$ from $\mathrm{R}_n(c_0)$ satisfying the subordination condition $\varphi(z) \prec \varphi_0(z)$, $\varphi_0(z) \in \mathrm{P}^*$, and finding accurate estimates of functionals (28) in this class. Here, the class of functions $\varphi_0(z)$ is denoted by P^*, which conformally map the circle E to a region starlike relative to the point $w = 1$, belonging to the half-plane $\mathrm{Re}\, w > 0$ and symmetrical with respect to the real axis.

2) Application of the obtained estimates of the functionals (28) for the study of unified subclasses of close-to-convex functions, including convex ones in a certain direction. This includes classes of functions for which $\frac{f'(z)}{g'(z)} \prec \varphi_0(z)$, $\varphi_0(z) \in \mathrm{P}_n(\varphi_0)$, and $g(z)$ is a convex function.

3) Introduction of new subclasses of double close-to-convex, close-to-starlike and double close-to-starlike functions based on the class $\mathrm{P}_n(\varphi_0)$ and investigation of extreme properties of functions of these classes using functional estimates (28).

4) Finding estimates of coefficients for functions of the classes under consideration.

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