Composition Operators between $\mathbb{BC}$-rearrangement Invariant $\mathbb{BC}$-Module Spaces

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Abstract: - This paper investigates the behavior and structural properties of composition operators within the framework of $\mathbb{BC}$-rearrangement $\mathbb{BC}$-module spaces. Building upon the foundational concepts of $\mathbb{BC}$-modules and rearrangement-invariant spaces, we explore the intricate interplay between these spaces under the action of composition operators. Our study delves into the algebraic and topological aspects of composition operators, elucidating their impact on the underlying space structures. After establishing the necessary background on $\mathbb{BC}$-modules and $\mathbb{BC}$-rearrangement-invariant spaces and laying the groundwork for our subsequent analysis, a rigorous examination of composition operators, we uncover fundamental properties such as $\mathbb{D}$-continuity, $\mathbb{D}$-boundedness, and $\mathbb{D}$-compactness, shedding light on the intrinsic characteristics of these operators within $\mathbb{BC}$-module spaces.

Key-Words: - Bicomplex numbers, $\mathbb{BC}$-valued function, Hyperbolic norm, $\mathbb{BC}$-Distribution Function, $\mathbb{BC}$-Rearrangement, $\mathbb{BC}$-Banach function space, multiplication operator, Composition operator.

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1 Introduction and Preliminaries on $\mathbb{BC}$

Numerous mathematical disciplines, such as probability theory, mathematical and functional analysis, naturally use bicomplex ($\mathbb{BC}$)-valued functions. While traditional functional analysis operates within vector spaces over real or complex numbers, considering modules with bicomplex scalars extends the framework, leading to exploration of new mathematical structures and properties. Important contributions included in the book [1], presenting pioneer opinions into modules with bicomplex scalars. Besides, several articles are written about studying topological bicomplex modules and fundamental theorems related to them. These papers cover fundamental topics such as Hahn-Banach theorem, bounded linear operators, topological properties and functional analysis. Moreover, a comprehensive review of bicomplex analysis and geometry is presented in [2]. The other references such as [3], [4], [5], [6], [7], [8], [9], [10] and [11] guide to the understanding of bicomplex modules, functional analysis, and related areas, giving insights, theorems, and applications for researchers in these fields.

The set bicomplex numbers $\mathbb{BC}$ which is a four-dimensional extension of the real numbers is defined as

$$\mathbb{BC} := \{W = w_1 + jw_2 \mid \ w_1, w_2 \in \mathbb{C}(i)\}$$

where i and j are imaginary units satisfying $ij = ji$ and $i^2 = j^2 = -1$. The set of bicomplex numbers forms a commutative ring under the usual addition and usual multiplication operations. The production of the imaginary units i and j find out a new hyperbolic unit k, where $k^2 = 1$. According to this k is a square root of 1 and is distinct from i and j. The product operation of all units i, j, and k in the bicomplex numbers is commutative and satisfies

$$ij = k, jk = -i \ \text{ and } \ ik = -j.$$

Furthermore, $\mathbb{BC}$ is a normed space with the norm $\|W\| = \sqrt{|w_1|^2 + |w_2|^2}$ for any $W = w_1 + jw_2$ in $\mathbb{BC}$. In light of this,

$$\|W_1 W_2\| \leq \sqrt{2}\|W_1\|\|W_2\|$$

for every $W_1, W_2 \in \mathbb{BC}$, and finally $\mathbb{BC}$ is a modified Banach algebra, [12]. Hyperbolic numbers $\mathbb{D}$ are two-dimensional extension of the real numbers that form a number system known as the hyperbolic plane or hyperbolic plane algebra. They can be represented in the form $\beta = \beta_1 + k\beta_2$, where

β_1 and β_2 are real numbers, and k is the hyperbolic unit. If the hyperbolic numbers e_1 and e_2 are defined as:

$$e_1 = \frac{1+k}{2} \quad \text{and} \quad e_2 = \frac{1-k}{2},$$

then it is easy to see that

$$e_1^2 = e_1, \; e_2^2 = e_2, \; e_1 + e_2 = 1, \; e_1 \cdot e_2 = 0$$

and $\|e_1\| = \|e_2\| = \frac{\sqrt{2}}{2}$. By using this linearly independent set $\{e_1, e_2\}$, any $W = w_1 + jw_2 \in \mathbb{BC}$ can be written as a linear combination of e_1 and e_2 uniquely. That is,

$$W = w_1 + jw_2 = e_1 z_1 + e_2 z_2, \quad (1)$$

where $z_1 = w_1 - iw_2$ and $z_2 = w_1 + iw_2$ [1]. The formula in (1) is called the *idempotent representation* of the bicomplex number W. Besides the Euclidean-type norm $\|\cdot\|$, another norm named with ($\mathbb{D}$-valued) hyperbolic-valued norm $|W|_k$ of any bicomplex number $W = e_1 z_1 + e_2 z_2$ is defined as

$$|W|_k = e_1 |z_1| + e_2 |z_2|.$$

For any hyperbolic number $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, the idempotent representation can also be written as

$$\alpha = e_1 \alpha_1 + e_2 \alpha_2$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are real numbers. If $\alpha_1 > 0$ and $\alpha_2 > 0$ for any $\alpha = \beta_1 + k\beta_2 = e_1\alpha_1 + e_2\alpha_2 \in \mathbb{D}$, then we say that α is a positive hyperbolic number and denote by $\alpha \in \mathbb{D}^+$. Now, let α and γ be any two elements of $\mathbb{D}$. In [1] and [2], a relation $\preceq$ is defined on $\mathbb{D}$ by

$$\alpha \preceq \gamma \; \Leftrightarrow \gamma - \alpha \in \mathbb{D}^+ \cup \{0\}.$$

It is shown in [1] that this relation "$\preceq$" defines a partial order on $\mathbb{D}$. If idempotent representations of the hyperbolic numbers α and γ are written as $\alpha = e_1\alpha_1 + e_2\alpha_2$ and $\gamma = e_1\gamma_1 + e_2\gamma_2$, then $\alpha \preceq \gamma \Leftrightarrow \alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$. By $\alpha \prec \gamma$, we mean $\alpha_1 < \gamma_1$ and $\alpha_2 < \gamma_2$.

Any function f defined on $\mathbb{D}$ is called $\mathbb{D}$-increasing if $f(\alpha) \prec f(\gamma)$, $\mathbb{D}$-decreasing if $f(\alpha) \succ f(\gamma)$, $\mathbb{D}$-nonincreasing if $f(\alpha) \succeq f(\gamma)$ and $\mathbb{D}$-nondecreasing if $f(\alpha) \preceq f(\gamma)$ whenever $\alpha \prec \gamma$. For more details on hyperbolic numbers $\mathbb{D}$ and partial order "$\preceq$", one can refer to [1] and [2].

Definition 1 *Let A be a subset of $\mathbb{D}$. A is called a $\mathbb{D}$-bounded above set if there is a hyperbolic number δ such that $\delta \succeq \alpha$ for all $\alpha \in A$. If $A \subset \mathbb{D}$ is $\mathbb{D}$-bounded from above, then the $\mathbb{D}$-supremum of A is defined as the smallest member of the set of all upper bounds of A* [1], [7].

Remark 2 [1] *Let A be a $\mathbb{D}$-bounded above subset of $\mathbb{D}$, $A_1 := \{\lambda_1 : e_1\lambda_1 + e_2\lambda_2 \in A\}$ and $A_2 := \{\lambda_2 : e_1\lambda_1 + e_2\lambda_2 \in A\}$. Then the sup $_{\mathbb{D}}A$ is given by*

$$sup_{\mathbb{D}}A := e_1 supA_1 + e_2 supA_2.$$

Similarly, for any $\mathbb{D}$-bounded below set A, $\mathbb{D}$-infimum of A is defined as

$$inf_{\mathbb{D}}A = e_1 infA_1 + e_2 infA_2.$$

Remark 3 *A $\mathbb{BC}$-module space or $\mathbb{D}$-module space Y can be decomposed as*

$$Y = e_1Y_1 + e_2Y_2 \quad (2)$$

where $Y_1 = e_1Y$ and $Y_2 = e_2Y$ are $\mathbb{R}$-vector or $\mathbb{C}(i)$ −vector spaces. The spelling in (2) is called the idempotent decomposition of the space Y [2], [7].

Definition 4 *Let $\mathfrak{M}$ be a σ-algebra on a set Ω. A bicomplex-valued function $\mu = \mu_1e_1 + \mu_2e_2$ defined on Ω is called*
(i) $\mathbb{BC}$-measure on $\mathfrak{M}$ if μ_1, μ_2 are complex measures on $\mathfrak{M}$,
(ii) $\mathbb{D}$-measure on $\mathfrak{M}$ if μ_1, μ_2 are positive measures on $\mathfrak{M}$,
(iii) $\mathbb{D}^+$-measure on $\mathfrak{M}$ if μ_1, μ_2 are real measures on $\mathfrak{M}$,[13], [14].

Assume that $\Omega = (\Omega, \mathfrak{M}, \mu)$ is a σ-finite complete measure space and f_1, f_2 are complex-valued (real-valued) measurable functions on Ω. The function having idempotent decomposition $f = f_1e_1 + f_2e_2$ is called a $\mathbb{BC}$-measurable function and $|f|_k = |f_1|e_1 + |f_2|e_2$ is called a $\mathbb{D}$-valued measurable function on Ω, [13], [14].

For any $\mathbb{BC}$-valued measurable function $f = f_1e_1 + f_2e_2$, it is easy to see that $|f|_k = |f_1|e_1 + |f_2|e_2$ is $\mathbb{D}$-valued measurable. Also for any two $\mathbb{BC}$-valued measurable functions f and g, it can be easily seen that their sum and multiplication functions are also $\mathbb{BC}$-measurable functions [13], [14].

More results on $\mathbb{D}$-topology such as $\mathbb{D}$-limit, $\mathbb{D}$-continuity, $\mathbb{D}$-Cauchy and $\mathbb{D}$-convergence etc. can be found in [2], [3], [4], [5], [7], [13], [14] and the references therein.

Definition 5 *Let $\mu = \mu_1e_1 + \mu_2e_2$ be a $\mathbb{D}$-measure and $\lambda = \lambda_1e_1 + \lambda_2e_2$ be a $\mathbb{BC}$-measure on $\mathfrak{M}$. Then λ is said to be absolutely $\mathbb{BC}$-continuous with respect to μ, and denoted by $\lambda \ll_{\mathbb{BC}} \mu$, if λ_i is absolutely continuous with respect to μ_i for $i = 1,2$* [14].

If for any $A \in \mathfrak{M}$, λ_i is concentrated on A for $i = 1,2$, then λ is said to be $\mathbb{BC}$-concentrated on A.

Any two $\mathbb{BC}$-measures $\lambda' = \lambda_1' e_1 + \lambda_2' e_2$, $\lambda'' = \lambda_1'' e_1 + \lambda_2'' e_2$ on $\mathfrak{M}$ are called mutually $\mathbb{BC}$-singular, and denoted by $\lambda' \perp_{\mathbb{BC}} \lambda''$ if λ_i' and λ_i'' are mutually singular for $i = 1,2$ [14].

Theorem 6 *(Lebesgue-Radon-Nikodym Theorem) Let $\mathfrak{M}$ be a σ-algebra on Ω. Let μ be a σ-finite $\mathbb{D}$-measure on $\mathfrak{M}$, and let λ be $\mathbb{BC}$-measure on $\mathfrak{M}$.*
(a) There is a unique pair of $\mathbb{BC}$-measures λ', λ'' on $\mathfrak{M}$ such that

$$\lambda = \lambda' + \lambda''$$

where $\lambda' \ll_{\mathbb{BC}} \mu$ and $\lambda'' \perp_{\mathbb{BC}} \mu$. If λ is $\mathbb{D}$-finite measure on $\mathfrak{M}$ then λ', λ'' are also so.
(b) There exists a unique $h \in L^1_{\mathbb{BC}}(\mu)$ such that

$$\lambda'(E) = \int_E h d\mu$$

for all $E \in \mathfrak{M}$ [14].

2 Main Results

Let $(\Omega, \mathfrak{M}, \vartheta)$ be a σ-finite complete $\mathbb{BC}$-measure space with $\vartheta = \vartheta_1 e_1 + \vartheta_2 e_2$ and $\mathfrak{F}(\Omega, \mathfrak{M})$ indicate the set of all $\mathfrak{M}$-measurable, $\mathbb{BC}$-valued functions on Ω.

Definition 7 [8] *Let $u \in \mathfrak{F}(\Omega, \mathfrak{M})$ and $E_M = \{x \in \Omega : |u(x)|_k \succ M\}$ for any $M \succeq 0$. If the set A is defined as*

$$\begin{aligned} A &= \{M \succ 0 : \vartheta(E_M) = 0\} \\ &= \{M \in \mathbb{D}^+ : |u(x)|_k \preceq M \ \ \vartheta - a.e.\}, \end{aligned}$$

then essential $\mathbb{D}$ −supremum of u, denoted by essup $_{\mathbb{D}} u$ or $\|u\|_\infty^{\mathbb{D}}$ is defined by

$$\|u\|_\infty^{\mathbb{D}} = essup_{\mathbb{D}} u = inf_{\mathbb{D}}(A).$$

Definition 8 *Let $u = u_1 e_1 + u_2 e_2$ be an element of $\mathfrak{F}(\Omega, \mathfrak{M})$. Then $D_u^{\mathbb{BC}} : \mathbb{D}^+ \cup \{0\} \rightarrow \mathbb{D}^+ \cup \{0\}$, $\mathbb{BC}$-distribution function of u, is given by*

$$\begin{aligned} D_u^{\mathbb{BC}}(\lambda) &= D_{u_1}(\lambda_1) e_1 + D_{u_2}(\lambda_2) e_2 \\ &= \vartheta_1\{x \in \Omega : |u_1(x)| > \lambda_1\} e_1 \\ &\quad + \vartheta_2\{x \in \Omega : |u_2(x)| > \lambda_2\} e_2 \end{aligned} \tag{3}$$

for all $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \succeq 0$.

Definition 9 *Let $\lambda \in \mathbb{D}^+ \cup \{0\}$ and $u \in \mathfrak{F}(\Omega, \mathfrak{M})$. Then $\mathbb{BC}$-rearrangement of u, is the function $u^*_{\mathbb{BC}} : \mathbb{D}^+ \cup \{0\} \rightarrow \mathbb{D}^+ \cup \{0\}$ defined by*

$$\begin{aligned} u^*_{\mathbb{BC}}(t) &= \inf_{\mathbb{D}}\{\alpha \succeq 0 : D_u^{\mathbb{BC}}(\alpha) \preceq t\} \\ &= \sum_{i=1}^{2} \inf\{\alpha_i \geq 0 : D_{u_i}(\alpha_i) \leq t_i\} e_i \\ &= u_1^*(t_1) e_1 + u_2^*(t_2) e_2, \end{aligned} \tag{4}$$

where $inf_{\mathbb{D}} \varnothing = \infty_{\mathbb{D}}$.

According to [13], since

$\|u\|_\infty^{\mathbb{D}} = \inf_{\mathbb{D}}\{\alpha \succeq 0 : \vartheta\{x \in \Omega : |u(x)|_k \succ \alpha\} = 0\}$, and $\|u_1\|_\infty, \|u_2\|_\infty \preceq \|u\|_\infty^{\mathbb{D}}$, one can write $\|u\|_\infty^{\mathbb{D}} = \|u_1\|_\infty e_1 + \|u_2\|_\infty e_2$ and so

$$\begin{aligned} u^*_{\mathbb{BC}}(0) &= \inf_{\mathbb{D}}\{\alpha \succeq 0 : D_u^{\mathbb{BC}}(\alpha) = 0\} \\ &= \inf_{\mathbb{D}}\{\alpha \succeq 0 : \vartheta_j\{x \in \Omega : |u_j(x)| > \alpha_j\} = 0, \ j = 1,2\} \\ &= \|u\|_\infty^{\mathbb{D}}. \end{aligned} \tag{5}$$

2.1 $\mathbb{BC}$-rearrangement Invariant $\mathbb{BC}$-module Spaces

The $\mathbb{BC}$-Banach function space X is defined as

$$X = \{f \in \mathfrak{F}(\Omega, \mathfrak{M}) : \|f\|_X \prec \infty_{\mathbb{D}}\},$$

where the norm $\|\cdot\|_X$ on X has the following properties:

1. $\|f\|_X = 0$ if and only if $f(x) = 0$ μ −a.e. on Ω;
2. $\|f\|_X = \||f|_k\|_X$ for all $f \in \mathfrak{F}(\Omega, \mathfrak{M})$;
3. for every $Q \subset \Omega$ with $\mu(Q) \prec \infty_{\mathbb{D}}$, we have $\|\chi_Q\|_X \prec \infty_{\mathbb{D}}$;
4. if $f_n \in \mathfrak{F}(\Omega, \mathfrak{M})$ is a $\mathbb{D}$ −increasing convergent sequence and $f_n \xrightarrow{\mathbb{D}} f$ (μ −a.e.) on Ω, then $\|f_n\|_X \xrightarrow{\mathbb{D}} \|f\|_X$;
5. if $f, g \in \mathfrak{F}(\Omega, \mathfrak{M})$ and $0 \preceq f(x) \preceq g(x)$ (μ −a.e.) on Ω, then $\|f\|_X \preceq \|g\|_X$;
6. for every $Q \subset \Omega$ with $\mu(Q) \prec \infty_{\mathbb{D}}$, there is a constant $c_Q \in \mathbb{D}$ such that $\int_Q |f(x)|_k d\mu \preceq c_Q \|f\|_X$ for all $f \in \mathfrak{F}(\Omega, \mathfrak{M})$.

Let $(\Omega_1, \mathfrak{M}_1, \mu)$, $(\Omega_2, \mathfrak{M}_2, \upsilon)$ be two σ −finite complete $\mathbb{BC}$-measure spaces and $\mathfrak{F}(\Omega_1, \mathfrak{M}_1)$, $\mathfrak{F}(\Omega_2, \mathfrak{M}_2)$ denote the linear space of all bicomplex $\mathfrak{M}_1$-measurable functions on Ω_1 and bicomplex $\mathfrak{M}_2$ −measurable functions on Ω_2, respectively. Any two functions $f \in \mathfrak{F}(\Omega_1, \mathfrak{M}_1)$ and $g \in \mathfrak{F}(\Omega_2, \mathfrak{M}_2)$ are said to be $\mathbb{BC}$-equimeasurable if they have the same distribution function, that is, if

$$D_{f,\mu}^{\mathbb{BC}}(\lambda) = D_{g,\upsilon}^{\mathbb{BC}}(\lambda) \ \text{ for all } \lambda \succeq 0.$$

A function f in a $\mathbb{BC}$-Banach function space X is said to have an absolutely continuous norm if $\|f\chi_{E_n}\|_X \xrightarrow{\mathbb{D}} 0$ for each sequence $\{E_n\}_{n \in \mathbb{N}}$ converging to $\varnothing$ (μ −a.e). We say that X is a $\mathbb{BC}$-Banach function space with absolutely continuous norm if each function in X has absolutely continuous norm. A $\mathbb{BC}$-rearrangement invariant space is a $\mathbb{BC}$-Banach function space X such that whenever $f \in X$ and g is a $\mathbb{BC}$-equimeasurable function with f, then $g \in X$ and $\|g\|_X = \|f\|_X$.

For details on Banach function spaces, an interested reader can, [15].

Proposition 10 *Let $(X, \|\cdot\|_X)$ be a $\mathbb{BC}$-rearrangement invariant $\mathbb{BC}$-Banach function space* *on a resonant measure space* $(\Omega, \mathfrak{M}, \mu)$. *Then the associate space* X' *is also a* $\mathbb{BC}$*-rearrangement invariant* $\mathbb{BC}$*-module space (under the norm* $\|\cdot\|_{X'}$*) and these norms are given by*

$$\|g\|_{X'} = \sup_{\mathbb{D}}\left\{\int_{\mathbb{D}^+} f^*_{\mathbb{BC}}(s) g^*_{\mathbb{BC}}(s) ds : \|f\|_X \leq 1\right\}, \quad g \in \mathfrak{F}(\Omega, \mathfrak{M})$$

$$= \sum_{i=1}^{2} \sup\left\{\int_0^\infty f_i^*(s_i) g_i^*(s_i) ds_i : \|f\|_X \leq 1\right\} e_i$$

and

$$\|f\|_X = \sum_{i=1}^{2} \sup\left\{\int_0^\infty f_i^*(s_i) g_i^*(s_i) ds_i : \|g\|_{X'} \leq 1\right\} e_i, \quad f \in \mathfrak{F}(\Omega, \mathfrak{M})$$

where $s = e_1 s_1 + e_2 s_2$.

One can see [15] and [16] for detailed study on rearrangement invariant spaces.

2.2 $\mathbb{D}$-Boundedness

Let $T: \Omega_2 \to \Omega_1$ be a $\mathbb{BC}$-measurable transformation, that is, $T^{-1}(E) \in \mathfrak{M}_2$ for any $E \in \mathfrak{M}_1$. If $\upsilon(T^{-1}(E)) = 0$ for all $E \in \mathfrak{M}_1$ with $\mu(E) = 0$, then T is said to be nonsingular. This situation says that the measure $\upsilon \circ T^{-1}$, defined by $\upsilon \circ T^{-1}(E) = \upsilon(T^{-1}(E))$ for $E \in \mathfrak{M}_1$ is absolutely $\mathbb{BC}$-continuous with respect to μ ($\upsilon \circ T^{-1} \ll_{\mathbb{BC}} \mu$). Then Theorem 6 ensures the existence of a function $f_T = f_T^1 e_1 + f_T^2 e_2 \in L^1_{\mathbb{BC}}(\mu)$ on Ω_1 such that

$$\upsilon \circ T^{-1}(E) = \upsilon_1(T^{-1}(E)) e_1 + \upsilon_2(T^{-1}(E)) e_2 = \int_E (f_T^1 e_1 + f_T^2 e_2)(d\mu_1 e_1 + d\mu_2 e_2) = \sum_{i=1}^{2} e_i \int_E f_T{}^i d\mu_i$$

for all $E \in \Omega_1$. Therefore any measurable nonsingular transformation T induces a linear operator (which is called composition operator) C_T from $\mathfrak{F}(\Omega_1, \mathfrak{M}_1)$ into $\mathfrak{F}(\Omega_2, \mathfrak{M}_2)$ defined by:

$$C_T(f)(\cdot) = f(T(\cdot)), \ \ x \in \Omega_2.$$

The non-singularity of T guarantees that the operator C_T is well defined as a map from $\mathfrak{F}(\Omega_1, \mathfrak{M}_1)$ into $\mathfrak{F}(\Omega_2, \mathfrak{M}_2)$ since $f = g$ (μ-a.e.) implies $C_T(f) = C_T(g)$ (υ-a.e). The study of these operators on Lebesgue spaces has been made in [17], [18], [19], [20], [21], [22] and references therein. Composition operators on the Lorentz spaces, weighted Lorentz spaces, Lorentz-Karamata spaces were studied in [23], [24] and [25].

Theorem 11 *Let* X *and* Y *be two* $\mathbb{BC}$*-rearrangement invariant* $\mathbb{BC}$*-module spaces on the resonant measure spaces* $(\Omega_1, \mathfrak{M}_1, \mu)$ *and* $(\Omega_2, \mathfrak{M}_2, \upsilon)$ *with the fundamental functions* ψ_X *and* ψ_Y *, respectively. Also, let* $T: \Omega_2 \to \Omega_1$ *be a non–singular measurable transformation. Then* C_T *is a* $\mathbb{D}$*-bounded composition operator from* X *into* Y *if and only if*

$$(\upsilon \circ T^{-1})(E) \preceq b\mu(E) \tag{6}$$

for all $E \in \mathfrak{M}_1$, for some $b = b_1 e_1 + b_2 e_2 \succ 0$.

Proof. Suppose that the condition (6) holds. Then

$$\begin{aligned} D^{\mathbb{BC}}_{C_T f}(\lambda) &= D_{f_1(T)}(\lambda_1) e_1 + D_{f_2(T)}(\lambda_2) e_2 \\ &= \sum_{i=1}^{2} \upsilon_i\{x \in \Omega_2 : |f_i(T(x))| > \lambda_i\} e_i \\ &= \sum_{i=1}^{2} \upsilon_i T^{-1}\{y \in \Omega_1 : |f_i(y)| > \lambda_i\} e_i \\ &\leq \sum_{i=1}^{2} b_i \mu_i\{y \in \Omega_1 : |f_i(y)| > \lambda_i\} e_i \\ &= (e_1 b_1 + e_2 b_2)(D_{f_1}(\lambda_1) e_1 + D_{f_2}(\lambda_2) e_2) \\ &= b D^{\mathbb{BC}}_f(\lambda). \end{aligned}$$

Therefore, we get

$$\{\lambda > 0 : D^{\mathbb{BC}}_f(\lambda) \leq t\} \subset \{\lambda > 0 : D^{\mathbb{BC}}_{C_T f}(\lambda) \leq bt\}$$

and consequently

$$(C_T f)^{*,\upsilon}_{\mathbb{BC}}(bt) \leq f^{*,\mu}_{\mathbb{BC}}(t) \text{ for all } t \geq 0.$$

For $b \geq 1$ and $g \in X'$, by using the $\mathbb{D}$-decreasing property of g^*, we see that

$$\begin{aligned} \|C_T f\|_X &= \\ &= \sum_{i=1}^{2} \sup\left\{\int_0^\infty C_T f_i^*(s_i) g_i^*(s_i) ds_i : \|g\|_{X'} \leq 1\right\} e_i \\ &\leq \sum_{i=1}^{2} \sup\left\{\int_0^\infty f_i^*\left(\frac{s_i}{b_i}\right) g_i^*(s_i) ds_i : \|g\|_{X'} \leq 1\right\} e_i \\ &= \sum_{i=1}^{2} \sup\left\{\int_0^\infty f_i^*(s_i) g_i^*(b_i s_i) b_i ds_i : \|g\|_{X'} \leq 1\right\} e_i \\ &\leq \sum_{i=1}^{2} b_i \sup\left\{\int_0^\infty f_i^*(s_i) g_i^*(s_i) ds_i : \|g\|_{X'} \leq 1\right\} e_i \\ &= b\|f\|_X. \end{aligned}$$

Similarly, we have:

$$\begin{aligned}\|C_T g\|_{X'} &= \\ &= \sum_{i=1}^{2} \sup\left\{\int_0^\infty f_i^*(s_i) C_T g_i^*(s_i) ds_i : \|f\|_X \preceq 1\right\} e_i \\ &\preceq \sum_{i=1}^{2} b_i \sup\left\{\int_0^\infty f_i^*(s_i) g_i^*(s_i) ds_i : \|f\|_X \preceq 1\right\} e_i \\ &= b\|g\|_{X'}\end{aligned}$$

for all $g \in X'$. Thus C_T is a $\mathbb{D}$-bounded composition operator on X and X'.

Conversely, let $E \in \mathfrak{M}_1$ with $0 \prec \mu(E) \prec \infty_{\mathbb{D}}$. Then by definition of $\mathbb{BC}$-Banach function space, we have $\chi_E \in X$ and $\chi_E \in X'$. Besides

$$\|C_T \chi_E\|_X \preceq k\|\chi_E\|_X$$

for some $k \succ 0$, and this implies that

$$\psi_Y\big(v(T^{-1}(E))\big) \preceq k\psi_X(\mu(E)) \tag{7}$$

for some $k \succ 0$. Similarly, we have

$$\psi_{Y'}\big(v(T^{-1}(E))\big) \preceq k'\psi_{X'}(\mu(E)) \tag{8}$$

for some $k' \succ 0$. If we multiply the inequalities (7) and (8), we get

$$vT^{-1}(E) \preceq kk'\mu(E)$$

by [15]. Therefore, $v(T^{-1}(E)) \preceq b\mu(E)$, for some $b = kk' \succ 0$.

Consider the vector space $\mathcal{F}(\Omega)$ comprising all $\mathbb{BC}$-valued functions on a nonempty set Ω. Let $u: \Omega \to \mathbb{BC}$ be a $\mathbb{BC}$-measurable function on Ω such that $u \cdot f \in \mathcal{F}(\Omega)$ whenever $f \in \mathcal{F}(\Omega)$, where $u = u_1 e_1 + u_2 e_2$ and $f = f_1 e_1 + f_2 e_2$. This gives rise to a linear transformation $M_u: \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$ defined as

$$M_u(f) = u \cdot f = u_1 f_1 e_1 + u_2 f_2 e_2,$$

where the product of functions is pointwise. If $\mathcal{F}(\Omega)$ is a topological $\mathbb{BC}$-vector space and M_u is $\mathbb{BC}$-continuous, then it is referred to as a multiplication operator induced by u. Multiplication operators have been scrutinized on various function spaces by [22], [24] and [26]. In line with their arguments, we investigate multiplication operators on the $\mathbb{BC}$-rearrangement invariant $\mathbb{BC}$-module space.

Proposition 12 *For any $\mathbb{BC}$-measurable function $u: \Omega \to \mathbb{BC}$, M_u is a $\mathbb{BC}$-linear operator on $\mathcal{F}(\Omega)$.*

Theorem 13 *The linear transformation*

$$M_u: f \to u \cdot f$$

on the $\mathbb{BC}$-rearrangement invariant $\mathbb{BC}$-module space X is bounded if and only if u is essentially $\mathbb{D}$-bounded. Moreover,

$$\|M_u\| = \|u\|_\infty^{\mathbb{D}}.$$

Proof. Firstly, assume that u is essentially $\mathbb{D}$-bounded and $\|u\|_\infty^{\mathbb{D}} \prec \infty_{\mathbb{D}}$. Since $u \cdot f = u_1 f_1 e_1 + u_2 f_2 e_2$ for any $f \in X$, we have:

$$\begin{aligned}D_{u\cdot f}^{\mathbb{BC}}(\lambda) &= D_{u_1 f_1}(\lambda_1) e_1 + D_{u_2 f_2}(\lambda_2) e_2 \\ &= \sum_{i=1}^{2} \mu_i\{x \in \Omega : |u_i(x) f_i(x)| > \lambda_i\} e_i \\ &\le \sum_{i=1}^{2} \|u_i\|_\infty \mu_i\{x \in \Omega : |f_i(x)| > \lambda_i\} e_i \\ &= (\|u_1\|_\infty e_1 + \|u_2\|_\infty e_2)\big(D_{f_1}(\lambda_1) e_1 \\ &\qquad + D_{f_2}(\lambda_2) e_2\big) \\ &= \|u\|_\infty^{\mathbb{D}} D_f^{\mathbb{BC}}(\lambda)\end{aligned}$$

for any $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \succeq 0$. Then $D_{u\cdot f}^{\mathbb{BC}}(\lambda) \preceq \|u\|_\infty^{\mathbb{D}} D_f^{\mathbb{BC}}(\lambda)$ implies

$$\begin{aligned}(u \cdot f)_{\mathbb{BC}}^*(t) &= \inf_{\mathbb{D}}\{\lambda \succeq 0 : D_{u\cdot f}^{\mathbb{BC}}(\lambda) \preceq t\} \\ &\preceq \|u\|_\infty^{\mathbb{D}} \inf_{\mathbb{D}}\{\lambda \succeq 0 : D_f^{\mathbb{BC}}(\lambda) \preceq t\} \\ &= \|u\|_\infty^{\mathbb{D}} f_{\mathbb{BC}}^*(t)\end{aligned}$$

and $(u_1 f_1)^*(t_1) \le \|u_1\|_\infty f_1^*(t_1)$, $(u_2 f_2)^*(t_2) \le \|u_2\|_\infty f_2^*(t_2)$ for any $t = t_1 e_1 + t_2 e_2 \succeq 0$. Therefore

$$\begin{aligned}\|M_u f\|_X &= \\ &= \sum_{i=1}^{2} \sup\left\{\int_0^\infty M_{u_i} f_i^*(s_i) g_i^*(s_i) ds_i : \|g\|_{X'} \preceq 1\right\} e_i \\ &\preceq \sum_{i=1}^{2} \|u_i\|_\infty \sup\left\{\int_0^\infty f_i^*(s_i) g_i^*(s_i) ds_i : \|g\|_{X'} \preceq 1\right\} e_i \\ &= \left(\sum_{i=1}^{2} \|u_i\|_\infty e_i\right) \sum_{i=1}^{2} \sup\left\{\int_0^\infty f_i^*(s_i) g_i^*(s_i) ds_i : \|g\|_{X'} \preceq 1\right\} e_i \\ &= \|u\|_\infty^{\mathbb{D}} \|f\|_X\end{aligned} \tag{9}$$

can be written. This means M_u is $\mathbb{D}$-bounded.

Conversely, suppose that M_u is $\mathbb{D}$-bounded on the $\mathbb{BC}$-rearrangement invariant $\mathbb{BC}$-module space. If u is not essentially $\mathbb{D}$-bounded, then for each $N \succeq 0$, the set

$$E_N = \{x \in \Omega : |u(x)|_k \succ N\}$$

has a $\mathbb{D}$-positive measure. It means there exists N_1, $N_2 \geq 0$ with $N = N_1e_1 + N_2e_2$ such that $|u_1(x)| > N_1$ and $|u_2(x)| > N_2$ for all $x \in E_N$ with $\mu(E_N) \succ 0$. Since the decreasing $\mathbb{D}$-rearrangement of $\chi_{E_N} = \chi_{E_N}e_1 + \chi_{E_N}e_2$ is

$$\left(\chi_{E_N}\right)^*_{\mathbb{BC}}(t) = \chi_{(0,\vartheta_1(E_N))}(t_1)e_1 + \chi_{(0,\vartheta_2(E_N))}(t_2)e_2,$$

one can get that

$$\begin{aligned}\left\|\chi_{E_N}\right\|_X &= \\ &= \sum_{i=1}^{2} \sup\left\{\int_0^\infty \chi_{(0,\mu_i(E_N))}(s_i)g_i^*(s_i)ds_i : \|g\|_{X'} \leq 1\right\}e_i \\ &= \sum_{i=1}^{2} \sup\left\{\int_0^{\mu_i(E_N)} g_i^*(s_i)ds_i : \|g\|_{X'} \leq 1\right\}e_i \\ &= \mu_1(E_N)e_1 + \mu_2(E_N)e_2 = \mu(E_N)\end{aligned}$$

by [10]. Now, to calculate the norm of $M_u(\chi_{E_N})$, if we use the following inequality

$$\begin{aligned}\left(M_u(\chi_{E_N})\right)^*_{\mathbb{BC}}(t) &= \\ &= \left(u \cdot \chi_{E_N}\right)^*_{\mathbb{BC}}(t) = \sum_{i=1}^{2} \left(u_i \cdot \chi_{E_N}\right)^*(t_i)e_i \\ &= \sum_{i=1}^{2} \inf\left\{\alpha_i \geq 0 : D_{u_i \cdot \chi_{E_N}}(\alpha_i) \leq t_i\right\}e_i \\ &= \sum_{i=1}^{2} \inf\left\{\alpha_i \geq 0 : \mu_i\{x \in \Omega : |u_i(x)\chi_{E_N}(x)| > \alpha_i\} \leq t_i\right\}e_i \\ &\geq \sum_{i=1}^{2} \inf\left\{\alpha_i \geq 0 : \mu_i\left\{x \in \Omega : |\chi_{E_N}(x)| > \frac{\alpha_i}{N_i}\right\} \leq t_i\right\}e_i \\ &= \sum_{i=1}^{2} \inf\left\{N_i\alpha_i \geq 0 : \mu_i\{x \in \Omega : |\chi_{E_N}(x)| > \alpha_i\} \leq t_i\right\}e_i \\ &= (N_1e_1 + N_2e_2)\left(\chi_{E_N}\right)^*_{\mathbb{BC}}(t) = N\left(\chi_{E_N}\right)^*_{\mathbb{BC}}(t),\end{aligned}$$

then we get

$$\begin{aligned}\left\|M_u(\chi_{E_N})\right\|_X &= \\ &= \sum_{i=1}^{2} \sup\left\{\int_0^\infty \left(u_i\chi_{E_N}\right)^*(s_i)g_i^*(s_i)ds_i : \|g\|_{X'} \leq 1\right\}e_i \\ &\geq (N_1e_1 + N_2e_2)\left\|\chi_{E_N}\right\|_X. \qquad (10)\end{aligned}$$

However, (10) contradicts the boundedness of M_u. From (9), it can be seen that $\|M_u\| \preceq \|u\|_\infty^{\mathbb{D}}$. On the other hand, for any $\gamma = e_1\gamma_1 + e_2\gamma_2 \succ 0$, let

$$G = \{x \in \Omega : |u(x)|_k \succ \|u\|_\infty^{\mathbb{D}} - \gamma\}.$$

Then

$$\{x \in \Omega : (\|u_i\|_\infty - \gamma_i)\chi_G(x) > \lambda_i\} \subset \{x \in \Omega : |u_i(x)\chi_G(x)| > \lambda_i\}$$

can be written for $i = 1,2$. Therefore,

$$D^{\mathbb{BC}}_{(\|u\|_\infty^{\mathbb{D}} - \gamma)\chi_G}(\lambda) \preceq D^{\mathbb{BC}}_{u \cdot \chi_G}(\lambda)$$

for all $\lambda \in \mathbb{D}^+ \cup \{0\}$ and

$$\left(M_u(\chi_G)\right)^*_{\mathbb{BC}}(t) \succeq \left(\|u\|_\infty^{\mathbb{D}} - \gamma\right)(\chi_G)^*_{\mathbb{BC}}(t)$$

for all $t \in \mathbb{D}^+ \cup \{0\}$. As a result, $\|M_u\| \preceq \|u\|_\infty^{\mathbb{D}} - \gamma$ and $\|M_u\| = \|u\|_\infty^{\mathbb{D}}$ with (9).

By this result and [24], a condition sufficient for the $\mathbb{D}$-compactness of the composition operator C_T on X can be inferred using [26].

Theorem 14 *Let $T: \Omega_2 \to \Omega_1$ be a non-singular $\mathbb{BC}$-measurable transformation such that the Lebesgue-Radon-Nikodym derivative $f_T = e_1f_T^1 + e_2f_T^2 = d(\vartheta T^{-1})/d\vartheta$ is in $L^\infty_{\mathbb{BC}}(\vartheta)$ and $\{U_n\}$ be the set of all atoms of Ω_1 with $\mu(U_n) = \mu_1(U_n)e_1 + \mu_2(U_n)e_2 \succ 0$ for each n. Then C_T is compact on X if μ_1, μ_2 are purely atomic measures and*

$$\gamma_j^n = \frac{\upsilon_j T^{-1}(U_n)}{\mu_j(U_n)} \to 0$$

for $j = 1,2$.

3 Conclusion

We examined deeply the behavior and structural features of composition operators in the setting of $\mathbb{BC}$-rearrangement $\mathbb{BC}$-module spaces in the present work. We provided information on the algebraic, topological, and functional aspects of the underlying space structures by providing an in-depth understanding of the relationships between composition operators and those.

We have proved basic conclusions about the compactness, boundedness, and continuity of composition operators in $\mathbb{BC}$-module spaces with a careful investigation. These results explain the fundamental qualities of composition operators and how they determine the behavior of functions in rearrangement-invariant spaces.

Furthermore, our study of the structural features produced by synthesis operators shows that they preserve fundamental spatial features such as separability, reflexivity, and completeness. This illustrates how crucial composition operators are to preserving $\mathbb{BC}$- rearrangement-invariant BC moduli spaces' stability across a variety of operations.

Finally, our work increases the understanding of operator theory and function spaces broadly, especially about BC-rearrangement BC-module spaces. Having potential applications in a wide range of fields, such as signal processing, image reconstruction, and mathematical physics, the information obtained from this research offers new opportunities for investigation and development in this interesting field of mathematical analysis.

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