On Some Ways to Increase the Exactness of the Calculating Values of the Required Solutions for Some Mathematical Problems

M. N. IMANOVA[1,2], V. R. IBRAHIMOV[2,3]
[1]Science Development Foundation of the Republic of Azerbaijan,
Baku AZ1025,
AZERBAIJAN

[2]Institute of Control System named after Academician A. Huseynov,
Baku Az1141,
AZERBAIJAN

[3]Computational Mathematics,
Baku State University,
Baku AZ1148,
AZERBAIJAN

Abstract: - The expansion of the application of computational methods for solving many mathematical problems from various fields of natural knowledge does not raise any doubts. One of the promising directions in contemporary sciences is considered to be in areas that are at the intersection of different sciences. Solving such problems is more difficult because different laws from different areas are used. It should be noted that at the intersection of these sciences, there are problems, which can come down to solving ordinary differential equations. Therefore, studies of differential equations have always been considered promising. Based on this, the application of some methods for solving initial problems for first-order ODEs is investigated. For this purpose, scientists studied a numerical solution to the initial problem of the ODE. Here, we have reviewed the study of linear Multistep Methods with constant coefficients. With its help, the order of accuracy of the calculated values is determined. In addition, determines how much accuracy values increase when using Richardson extrapolation methods and also when using linear combinations of various methods. To construct an innovative method is proposed here using advanced methods. It is shown that using these methods it is possible that A-stable methods can be taken as innovative.

Key-Words: - Ordinary Differential Equation (ODE), Local truncation Error, Multistep Method (MM), Richardson Extrapolation (RE), Stability and Degree (S,D), Initial Value Problem, Advanced Methods, Multistep Seconddderivative Methods (MSM).

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1 Introduction

We were seriously engaged in the study of Ordinary Differential Equations after familiarizing ourselves with the work: Constantin Carotheodory, "Calculus of variations and partial differential equations of the first order".

Let us note that many well-known scientists were engaged in the search for a solution to the initial problem of the ODE. They constructed some classes of methods having different properties. Thus creating the opportunity for a wide selection of numerical methods. For this purpose, scientists defined some conceptions for their comparisons. For this purpose, scientists have found some conception by which one can define the boundaries for all the errors received in using methods with constant coefficients (see for example, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]). For the compassion of the known methods let us consider investigating the following problem:

$$z' = \varphi(t, z(t)),\ \ z(t_0) = z_0,\ \ t_0 \le t \le T, \qquad (1)$$

which usually is called the initial-value problem for ODEs of the first order. For the construction numerical methods with the new properties, let us impose some restrictions on the solution of problem (1) and also on the function $\varphi(t,z)$. Let the solution to the problem (1) be a continuous function defined on the segment $[t_0, T]$. We mean that a continuous function $\varphi(t, z)$ is defined in a certain limited domain and, inclusive, has partial derivatives up to p. To find a numerical solution to the problem (1), we divide the segment $[t_0, T]$ into N parts using grid points $t_{i+1} = t_i + h$ $(i = 0,1,2,..)$. And also, here indicate exact values of the solution of the problem (1) by $z(t_i)$, but by the z_i-the corresponding approximate values of the function $z(t)$ at the point t_i.

Note that one of the popular classes of numerical methods is the class of multi-step methods, which can be depicted as follows:

$$\sum_{j=0}^{m} a_j z_{n+j} = h \sum_{j=0}^{m} b_j \varphi_{n+i}, \ n = 0,1,..,N-m, \qquad (2)$$

here $\varphi_j = \varphi(t_j, z_j)$ $(j = 0,1,..., N)$.

Such methods have been studied in the works of many authors,(see for example [3], [4], [5], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]). But fundamentally has been investigated by Dahlquist.

From the outside, [4] has proven that if (2) is a stable method and has a degree p, then $p \le 2[m/2] + 2$ Conception stability and degree, which have been used can be presented as follows:

Definition 1. An integer value is called a power for method (2) if the identity is satisfied:

$$\sum_{i=0}^{m} (\alpha_i z(x+ih) - h\beta\varphi(x+ih)) = O(h)^{p+1}, \ h \to 0. \qquad (3)$$

Definition 2. Note that (2) is called a stable method if the roots of the characteristic polynomial $\rho(\lambda) \equiv \alpha_k \lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_1 \lambda + \alpha_0$ are located in the unit circle, which does not have multiple roots on the boundary. This conception is given in [3] and called the "dispersion". But, [7], have used the concept of "stability". By using results receiving in [4] one can be noted that if method (2) is stable, then $p_{\max} \le k + 2$. The scientists for the calculation of the values of the solution problems (1), have suggested some ways. One of these ways is the known Richardson extrapolation, which in the application to method (2) can be constructed by using the local truncation error.

§1. Some ways to increase of the exactness of the receiving results by using the known methods.

Based on method (2), we assume that in order to construct ways to improve the accuracy of calculated values, that method (2) has a degree p. It is known that the local truncation error for method (2) can be represented as follows:

$$ch^{p+1} z_n^{(p+1)} + O(h^{p+s+2}), \ (s \ge 0). \qquad (4)$$

And now suppose that step-size h to change by the kh. Then formula (4) can be written as:

$$ck^{p+1} h^{p+1} z_{kn}^{p+1} + O(h^{p+s+2}), \ (s \ge 0). \qquad (5)$$

To illustration of Richardson's extrapolation, let us multiply local truncation error (4) by λ, but local truncation error (5) by $\lambda - 1$. Then after summing (4) and (5) receive:

$$ch^{p+1}(\lambda + (1-\lambda)k^{p+1}) Z_n^{(p+1)} + O(h^{p+s+2}). \qquad (6)$$

In usually the values for the k has been taken as the $k = 1/2$ *or* $k = 2$. But here the known cases are generalized by the constant of k.

As is known some authors noted that by using Richardson extrapolation one can be construct more exact methods. Let's show that this is not so. To increase the accuracy of the method it is enough that λ satisfies the following equation:

$$\lambda + (1-\lambda)k^{p+1} = 0. \qquad (7)$$

For this case s=1, here is some constant participated in asymptotic relation (5).

It is obvious that the solution of this equation will be a real number. Thus, after using the value of λ, one can construct a method with constant coefficients. Therefore, as a result, the resulting method must obey the laws from the [4]. Note that the method does not change its structure. In this way receive that, one gets that the function to which the multistep method is applied is changed. Because of this, the calculated values for solving our problem by Richardson extrapolation are more accurate:

$$\bar{z}_{n+m} = \lambda z_{n+m}^{(h)} + (1-\lambda) z_{n+kn}^{(kh)}, \quad n = 0,1,2,..,N-m. \qquad (8)$$

For the $\lambda = 1/2$ required values can be presented as:

$$\bar{z}_{n+m} = (z_{n+m}^{(h)} + z_{n+2m}^{(h/2)})/2.$$

If method (2) is stable and has the degree of p, then receive that $p+s \le 2[k/2]+2$ is holds. Now let's look at finding the values of the solution to problem (1) using a linear combination of some methods. Let's use the following Euler methods to illustrate the advantages of this path

$$\hat{z}_{n+1} = z_n + h\varphi(t_n, z_n); \; z_{n+1} = z_n + h\varphi(t_{n+1}, z_{n+1}). \tag{9}$$

It is not difficult to prove that the Local Truncation Error for these methods can be presented as follows:

$$\hat{R}_n = h^2 z_n''/2 + O(h^3), \; R_n = -h^2 z_n''/2 + O(h^3), \; h \to 0. \tag{10}$$

As follows from this, the half-sum of these local truncation errors will be smaller than the errors have defiant by the asymptotic equality (10). Indeed this is so, for this let's consider the half-sum above the given methods (9), and then we get the following method:

$$(z_{n+1} + \hat{z}_{n+1})/2 = z_n + h(\varphi(t_n, z_n) + \\ + \varphi(t_{n+1}, z_{n+1}))/2, \tag{11}$$

which is the known Trapezoidal role. This method has the degree $p=2$, but methods (9) have, the degree $p=1$. Consequently, the Trapezoidal rule is more exact.

It is easy to define that calculation z_{n+1} is more difficult, than the calculation of the value $\hat{z}_{n+1}$. For the correction of this disadvantage, let us to define the value Z_{n+1} by the following method:

$$\bar{z}_{n+1} = z_n + h\varphi(t_{n+1}, z_n + h\varphi(t_n, z_n)), \tag{12}$$

which is explicit and does not arise any difficulty in the calculation of the value $\bar{z}_{n+1}$ by this formula.
It is easy to show that, method of (12) can presented as the follows:

$$\bar{z}_{n+1} = z_n + h\varphi(t_{n+1}, \bar{z}_{n+1}). \tag{13}$$

Thus, by the described above-mentioned method, receive some predictor-corrector method. The predictor and corrector methods have one and the same degree, which is equal to 1 (one). As was shown above, by using half sum of the values of problem (1) calculated within using predictor and corrector methods, receive the new method, which is more exact than the predictor and corrector methods. In our case, the one-step method of (9) has constructed the one-step method, but in using Richardson extrapolation method remains the same. Thus, in using Richardson extrapolation receives the new function to calculate which applied the using method. We get that when using the Richardson extrapolation each time one can increase the exactness of calculated values at the mesh points. However, in using linear combinations of some multistep methods the degree of exactness must obey the laws from [5]. In this case, if one can receive the results with a higher degree by using a linear combination of some multistep methods, then receive that or the method used has a low order of accuracy or the number of terms in the resulting method increases. And now let us consider the following methods:

$$z_{n+k}^{(1)} = -\sum_{j=0}^{k-1} \bar{\alpha}_j z_{n+j} + h\sum_{j=0}^{k} \bar{\beta}_j \varphi_{n+j}, \\ z_{n+k}^{(2)} = -\sum_{j=0}^{k-1} \hat{\alpha}_j z_{n+j} + h\sum_{j=0}^{k} \beta_j \varphi_{n+j} \tag{14}$$

with the local truncation errors

$$R_n^{(1)} = c_1 z_n^{(p+1)} + O(h^{p+s_1+1}), \\ R_n^{(2)} = c_2 z_n^{(p+1)} + O(h^{p+s_2+1}), \; h \to 0,$$

here $s = \min(s_1, s_2)$.

To get the best results it is enough to use the solution of the following equation:

$$\lambda c_1 + (1-\lambda)c_2 = 0,$$

in the following expression

$$z_{n+k} = \lambda z_{n+k}^{(1)} + (1-\lambda) z_{n+k}^{(2)}. \tag{15}$$

Note that based on formula (15), the resulting value of the solution to problem (1) will be more accurate than the values of $z_{n+k}^{(1)}$ and $z_{n+k}^{(2)}$ It is easy to understand that the values $z_{n+k}^{(1)}$ and $Z_{n+k}^{(2)}$ are calculated with the order of accuracy of p, then value calculated by the formula (15) will be has the order of accuracy of $p+1$. Considering that in this process has used the values, which have been calculated by using a stable method, we get that this scheme gives a positive result. By the above described, one can construct a very simple way to increase the accuracy of the approximate values as the solution of the problem (1). For this aim let us consider the following section.

§2. On some ways for the increasing of exactness of the numerical method (2).

Noted that by the equality of (15) receive that, for more accurate results, here have used linear interpolation. Currently, to improve the accuracy of the solution values to the problem (1), it is proposed to use methods with second derivatives, which are represented as follows:

$$\sum_{i=0}^{k}\alpha_i Z_{n+i} = h\sum_{i=0}^{k}\beta_i Z'_{n+i} + h^2\sum_{i=0}^{k}\gamma_i Z''_{n+i} \ (n=0,1,2,...). \quad (16)$$

It easy to understand that $z''(t)=\varphi'_t+\varphi'_z\cdot\varphi$. In the finding, the values of the computational work are increased, which depends on the calculation of the values $\varphi'_t(t,z)$ *and* $\varphi'_z(t,z)$ at the mesh points t_m $(0\le m\le N)$. With this in mind, it is proposed here to use an extended (or jumping) method, which in some simple form can be represented as follows:

$$\sum_{i=0}^{k-m}\alpha_i z_{n+i} = h\sum_{i=0}^{k}\beta_i\varphi(t_{n+i},z_{n+i}) \ \ (m>0;\ n=0,1,2,..). \quad (17)$$

Similar to the described above, studied by many authors, [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36],[37].

Method (17) was fundamentally studied by in the work [19], [20].who came to the conclusion that if method (17) is stable, then in class (17) there are methods with degree $p\le k+m+1$ for $k\ge 3m$. Obviously, based on Dahlquist's law, we can conclude that if method (2) is stable, then in the class of method (2) there are stable methods with degree $p\le 2[k/2]+2$ (here $k=m$). By a usual comparison, we find that stable methods like (17) are more accurate than (2). As an example, in class (2) there are stable methods of degree $p\le 2[k/2]+2$ for all k. Therefore, for $k=3$ there is a stable method with degree $p_{\max}=4$ $(p_{\max}=k+1)$. But in (17) there is a stable method with degree $p=5$ for $k=3$, represented as:

$$z_{n+2}=(11z_n+8z_{n+1})/19+ \\ +h(10\varphi_n+57\varphi_{n+1}+24\varphi_{n+2}-\varphi_{n+3})/57, \quad (18)$$

Obviously, this method is stable and also has a local truncation error of degree $p=5$, written as

$$R_n=-\frac{11}{3420}h^6 z^{(6)}_{(x_n)}+O(h^7).$$

It is noted that the main disadvantage of these methods is finding the values of the solution to problem (1) in neighboring grid nodes. To solve this problem, we use a predictor-corrector similar to the methods. Let's look at a method like:

$$z_{n+3}=z_{n+2}+h(23\varphi_{n+2}-16\varphi_{n+1}+5\varphi_n)/12. \quad (19)$$

By using method (19) in the formula (18), receive the following method:

$$z_{n+2}=(11z_n+8z_{n+1})/19+h(10\varphi_n+57\varphi_{n+1} \\ +24\varphi_{n+2}-h\varphi(t_{n+3},z_{n+2}+ \\ +h(23\varphi_{n+2}-16\varphi_{n+1}+5\varphi_n)/12)/57. \quad (20)$$

And now let us change z_{n+3} participation in (18) by the following:

$$z_{n+3}=z_{n+1}+h(7\varphi_{n+2}-2\varphi_{n+1}+\varphi_n)/3. \quad (21)$$

In this case, receive the next methods:

$$z_{n+2}=(11z_n+8z_{n+1})/19+h(10\varphi_n+57\varphi_{n+1} \\ +24\varphi_{n+2})/57-h\varphi(t_{n+3},z_{n+1}+ \\ +h(7\varphi_{n+2}-2\varphi_{n+1}+\varphi_n)/3)/57 \quad (22)$$

Obviously, this method is stable and also implicit. Method (20) is A-stable, but the method (22) is stable. To ensure the accuracy of the results, consider the following section.

2 Numerical Results

To illustrate the results, we provide relevant examples:

$$z'=\lambda z,\ z(0)=1,\ 0\le t\le 2.$$

exact solution for which can be presented as: $z(t)=\exp(\lambda t)$. To solve this example let us to us the following couple methods

$$\hat z_{n+2}=3\hat z_{n+1}-2z_n-h\varphi_n, \quad (23)$$

$$z_{n+2}=z_n-h(\hat\varphi_{n+2}-8\varphi_{n+1}-5\varphi_n)/12, \quad (24)$$

$$\hat z_{n+2}=z_{n+1}+h(3\varphi_{n+1}-\varphi_n)/2, \quad (25)$$

$$z_{n+2}=z_n-h(\hat\varphi_{n+2}-8\varphi_{n+1}-5\varphi_n)/12, \quad (26)$$

$$\hat z_{n+2}=-8z_{n+1}+9z_n+h(6\varphi_{n+1}+4\varphi_n), \quad (27)$$

$$z_{n+2}=z_{n+1}+h(9\hat\varphi_{n+2}+18\varphi_{n+1}-3\varphi_n)/24, \quad (28)$$

$$z_{n+2}=(8z_{n+2}+\hat z_{n+2})/9. \quad (29)$$

It is known that the local truncation error for methods (24) and (26) is represented as:

$$R_n = h^3 z_n''' / 24 + O(h^5).$$

The receiving results for method (24),(26), and (27) are tabulated in the following tables:

Table 1. Results for the $h = 0.05$ and $\lambda = 1.$

t_i	Method (24)	Method (26)	Exact value	Method (29)
0. 1 0. 5 2. 0	1.1051988 6 1.9313402 2 more	1.1051702 5 1.6487264 6 7.7698516 8	1.10551709 1 1.64872122 7.76790016	1.1051712 0 1.6487827 3 7.7678623 2

Table 2. Results for the $h = 0.1$ and $\lambda = 1$

t_i	Method (24)	Method (26)	Exact value	Method (29)
0. 2 0. 9 2. 0	1.2216634 3.3829898 8 more	1.2214040 8 2.4599552 2 7.3962984 1	1.2214126 6 2.4596023 2 7.3890552 2	1.2214126 6 24560233 7.3890553 27679016 2

As follows from the Table 2, receive that the results received by the method (29) are unacceptable. For the corrected this situation, let's consider the case, when $\lambda < 0$ the solution is decreasing.

Table 3 . Results for the $h = 0.05$ and $\lambda = -1.$

t_i	Method (24)	Method (26)	Exact value	Method (29)
0. 1 0. 5 2. 0	0.9448141 8 0.4168961 0 more	0.9448374 3 0.6065349 6 0.1353487 4	0.9483745 0.6065309 8 0.1353353 0	0.9463966 0 0.6065325 1 0.1353304 4

Table 4. Results for the $h = 0.1$ and $\lambda = -1$

t_i	Method (24)	Method (26)	Exact value	Method (29)
0. 2 0. 9 2. 0	0.818473 40 0.337485 07 more	0.818722 19 0.367995 56 0.135498 88	0.8187307 6 0.3678795 0 0.1355335 38	0.8187310 7 0.3687812 28 0.1355330 5

By the results of the Table 1, Table 3, Table 4, the results received by the method (29) can be considered as the better.

3 Conclusion

Here are given some ways which usually are used for the increased accuracy of the receiving results by using stable Multistep Methods. Shown that by the selection of predictor methods in the predictor-corrector methods one can receive the method, which behaves like an unstable method. Note that in the predictor-corrector method of (25)-(26), the predictor method is stable, but in the predictor-corrector method of (23) and (24) the predictor method is unstable. Obviously, the predictor-corrector method is convergent if the corrector method is stable. Here, the predictor method was used as a separate unstable method, so the results obtained by method (24) are unacceptable. However, the second predictor method is convergence (the predictor-corrector method is convergence because the predictor-corrector methods are robust). It is known that in the linear combination constructed using methods (27)-(29), unstable methods are involved as predictors, but because of this, the results are better. Similar results are obtained by using Richardson extrapolation. Thus we receive that using linear combinations gives the best results. However, the selected appropriate methods are very important. The reason for the increased accuracy of Richardson extrapolation and the linear combination of various methods is also explored here. Note that for the increased accuracy of the calculated values of solution of the investigated problem, one can use the bilateral methods, this method can be taken as the better, so by using the bilateral methods one can define the availability of the receiving results. We would like to note that here used references with we have also encountered in other popular works. And have given some information about our new articles. Note that this method is interesting and very simple, so we hope that the methods described above will be very useful for a circle of readers and researchers.

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Conflict of Interest

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