Solving the Class of Nonsmooth Nonconvex Fuzzy Optimization Problems via the Absolute Value Exact Fuzzy Penalty Function Method

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Abstract: - In recent years, in optimization theory, there has been a growing use of optimization models of real decision-making processes related to the activities of modern humans, in which the hypotheses are not verifiable in a way typical of classical optimization. This increases the demand for tools that will enable the effective solving of such more real optimization models. Fuzzy optimization problems were developed to model real-world extremum problems with uncertainty, which means that they are not usually well-defined. In this work, we investigate one of such tools, i.e. the absolute value exact fuzzy penalty function method which is applied to solve invex nonsmooth minimization problems with fuzzy objective functions and inequality (crisp) constraints. Namely, we analyze the exactness of the penalization which is the most important property of any such method from a practical point of view. Further, the algorithm of the absolute value exact penalty function method is presented in the context of finding weakly nondominated solutions of the analyzed nonsmooth fuzzy optimization problem and, moreover, its convergence is proven in the considered fuzzy case. Finally, we also simulate the choice of the penalty parameter in the aforesaid algorithm.

Key-Word: - fuzzy optimization, nondifferentiable optimization problem with the fuzzy objective function, Clarke generalized gradient, Karush-Kuhn-Tucker optimality conditions, nondominated solution, absolute value exact penalty function method, exactness of the penalization, invex fuzzy function.

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1 Introduction

Many real-world O.R. systems and processes cannot be modeled easily in deterministic terms since they involve imprecision of data. In fact, the data are often uncertain in nondeterministic models of real-world systems and processes due to, for example, prediction and/or estimation errors, or lack of information (e.g., some extremum problems that arise in economics, industry, engineering applications, commerce, sciences might involve financial returns, differing costs, design parameters of such systems in designing phase are usually under uncertainties, future actions might be unknown at the time of the decision). Hence, most of the real research problems are subject to some form of uncertainty. The reason for this is the fact that some coefficients of the objective and/or the constraint functions in such optimization problems cannot be exactly assessed, due to the fact that they are imprecise, unreliable vague, etc.

Fuzzy optimization is one of the useful and efficient approaches for treating just such real-world decision-making problems under uncertainty. The basic concept of fuzzy decision-making was first proposed by in the paper, [1]. Since then, many authors studied extensively fuzzy mathematical programming problems. Namely, the definition of a convex fuzzy mapping was firstly introduced in the paper, [2]. After that, the convexity notion for fuzzy mapping has been widely used in fuzzy optimization by several authors (see, for example, [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], and others). However, the convexity notion is too restrictive in fuzzy optimization, due to the fact that not all optimization problems modeling real-world O.R. processes with uncertain data are convex. Therefore, several authors have defined and applied generalized convex fuzzy mappings to fuzzy optimization (see, for example, [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], and many others).

One of the well-known approaches in optimization theory for looking for optimal solutions in constrained mathematical programming problems is exact penalty function methods. In the last few decades, many researchers have been focused to find optimal solutions in various types of extremum problems by using exact penalty function methods. The idea behind the aforesaid methods is that, by using chosen exact penalty function, the original problem of a constrained extremum problem can be reduced to an unconstrained optimization problem. Thus, it is possible to avoid the difficulties, that take place in other approaches, at least related to finding feasible points and/or directions. Moreover, in this way, to find optimal solutions of constrained extremum problems the algorithms developed in unconstrained optimization can be applied. The exact penalty function that has been most frequently used by many researchers to solve their constrained optimization problems and, is, therefore, the most popular exact penalty function, is the absolute value exact penalty function, also called the l_1 exact penalty function (see, for example, [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], and others). In [25] and [26] the most important property of the l_1 exact penalty function method, that is, exactness of the penalization, was analyzed for new classes of nonconvex optimization problems. Whereas the aforementioned property was investigated in the paper [40], for the vector l_1 exact penalty function method which they used to solve nondifferentiable invex vector optimization problems. Recently, the classical exact l_1 penalty function method was applied in the paper [27] to solve a nonsmooth constrained interval-valued optimization problem with both equality and inequality constraints and the property of exactness of the penalization was analyzed when this method is applied to solve a nondifferentiable interval-valued mathematical programming problem.

According to the literature, only a few studies have explored the methods for solving nonconvex nondifferentiable fuzzy optimization problems so far, and the present study is one of the first reports to address this problem. In this article, therefore, we use the absolute value exact penalty function method to solve a nonsmooth optimization problem with fuzzy objective function and inequality (crisp) constraints. Then, for the considered fuzzy minimization problem, we construct its associated fuzzy penalized optimization problem with the l_1 exact fuzzy penalty function. Further, in the fuzzy context, we generalize the main property of all exact penalty function methods, i.e. exactness of the penalization. We analyze it, moreover, under appropriate invexity hypotheses in the case when we use the absolute value exact fuzzy penalty function method for solving such nonsmooth fuzzy extremum problems. Namely, we prove that a (weak) Karush-Kuhn-Tucker point of the investigated nondifferentiable fuzzy optimization problem is a (weakly) nondominated solution of its associated penalized fuzzy optimization problem with the fuzzy l_1 exact penalty function for all penalty parameters exceeding the given threshold. We also establish the equivalence between a (weakly) nondominated solution of the considered fuzzy optimization problem and a (weakly) nondominated solution of its associated fuzzy penalized optimization problem with the l_1 exact fuzzy penalty function for sufficiently large penalty parameters. Further, we present an algorithm of the absolute value exact penalty function method which is applied for finding weakly nondominated solutions in the considered nonsmooth optimization problem with fuzzy objective function and inequality constraints. Its convergence is also established in the considered fuzzy case. After that, we analyze the strategy for choosing the penalty parameter in the applied absolute value exact fuzzy penalty function method and we illustrate it by the appropriate examples of constrained fuzzy minimization problems.

2 Notations and Preliminaries

We first present some preliminary notations and present such definitions and results, which will be used in this work. Throughout this paper, R is the set of all real numbers, that is, endowed with the usual topology. A fuzzy subset of R is a function $\tilde{u}: R \to [0,1]$. We usually named this mapping a membership function of a fuzzy number $\tilde{u}$. We now define the α-level set for any fuzzy set $\tilde{u}$ (denoted by $[\tilde{u}]^\alpha$) as follows

$$[\tilde{u}]^\alpha = \begin{cases} \{x \in R: \tilde{u}(x) \geq \alpha\} & if \quad \alpha \in (0,1], \\ cl\big(supp\,(\tilde{u})\big) & if \quad \alpha = 0, \end{cases}$$

where $[\tilde{u}]^0$ is the closure of the support of $\tilde{u}$, that is, $supp\,(\tilde{u}) = \{x \in R: \tilde{u}(x) > \alpha\}$.

Definition 1. [7], [24] A fuzzy number $\tilde{u}$ in R is a fuzzy set on R with the following properties: 1) $\tilde{u}$ is normal, i.e. there exists $x^* \in R$ such that $\tilde{u}(x^*) = 1$, 2) $\tilde{u}$ is quasi-concave, i.e. $\tilde{u}(\lambda x + (1-\lambda)y) \geq min\{\tilde{u}(x), \tilde{u}(y)\}$ for all $x, y \in R$ and any $\lambda \in [0,1]$, 3) $\tilde{u}$ is upper semicontinuous, i.e. $[\tilde{u}]^\alpha = \{x \in R: \tilde{u}(x) \geq \alpha\}$ is a closed subset of R for each $\alpha \in (0,1]$, 4) the 0-level set, i.e. $[u]^0$, is a compact subset of R.

Hence, if a fuzzy set $\tilde{u}$ is such that $[u]^1$ is a singleton, then $\tilde{u}$ is called a fuzzy number, [5], [7].

Let us denote by $\mathcal{F}(R)$ the family of all fuzzy numbers in R. Thus, for every $\tilde{u} \in \mathcal{F}(R)$, $[u]^\alpha$ is a nonempty convex and compact subset of R for each $\alpha \in [0,1]$. Hence, the α-levels of a fuzzy interval can be described by $[u_\alpha^L, u_\alpha^R], u_\alpha^L, u_\alpha^R \in R, u_\alpha^L \leq u_\alpha^R$, for all $\alpha \in [0,1]$. In particular, the fuzzy number $0 \in \mathcal{F}(R)$ is given as follows $\tilde{0}(x) = 1$ if $x = 0$, and $\tilde{0}(x) = 0$, otherwise. Also any $a \in R$ can be regarded as a fuzzy number $\tilde{a} \in \mathcal{F}(R)$ defined by $\tilde{a}(x) = \begin{cases} 1 & if \quad x = a, \\ 0 & if \quad x \neq a. \end{cases}$ Note that a fuzzy number $\tilde{u}$ is often defined in the literature by the end points of the interval $[u_\alpha^L, u_\alpha^R]$ [5], [6], [16], [24], [41] and many others.

Remark 1. The notation $\tilde{1}_{\{a\}}$ was introduced in the paper [41], to represent the crisp number with the value *a*. It is easy to see that $\left(\tilde{1}_{\{a\}}^L\right)_\alpha = \left(\tilde{1}_{\{a\}}^R\right)_\alpha = a$ for all $\alpha \in [0,1]$.

Given two fuzzy numbers $\tilde{u}, \tilde{v} \in \mathcal{F}(R)$ which are represented by their α-level sets as $[\tilde{u}]^\alpha = [u_\alpha^L, u_\alpha^R]$, $[\tilde{v}]^\alpha = [v_\alpha^L, v_\alpha^R]$ for any $\alpha \in [0,1]$, respectively, and $t \in R$. Then, we define the fuzzy addition $\tilde{u} + \tilde{v}$ and the scalar multiplication $t\tilde{u}$ as follows, [6], [7], [10], [20]:

$$\tilde{u} + \tilde{v} = \sup_{y+z=x} \min[\tilde{u}(y), \tilde{v}(z)], \qquad (1)$$

$$t\tilde{u} = \begin{cases} \tilde{u}\left(\frac{x}{t}\right) & if \quad t \neq 0 \\ \tilde{0} & if \quad t = 0 \end{cases}, where \ \tilde{0} \in \mathcal{F}(R). \qquad (2)$$

These operations on fuzzy numbers can be defined in the equivalent way (see, [6], [7]). Namely, for every $\alpha \in [0,1]$,

$$[\tilde{u} + \tilde{v}]^\alpha = [(u+v)_\alpha^L, (u+v)_\alpha^R] = [u_\alpha^L + v_\alpha^L, u_\alpha^R + v_\alpha^R],$$

$$[t\tilde{u}]^\alpha = [(tu)_\alpha^L, (tu)_\alpha^R]$$

$$= [\min\{tu_\alpha^L, tu_\alpha^R\}, \max\{tu_\alpha^L, tu_\alpha^R\}].$$

Definition 2. A special type of a fuzzy number is a triangular number $\tilde{u}$ which is described by three real numbers $u_1 \leq u_2 \leq u_3$ as $\tilde{u} = (u_1, u_2, u_3)$ and its definition is as follows:

$$\tilde{u}(x) = \begin{cases} \frac{x-u_1}{u_2-u_1} & if \quad u_1 \leq x \leq u_2 \\ \frac{u_3-x}{u_3-u_2} & if \quad u_2 \leq x \leq u_3 \\ 0 & otherwise \end{cases} \qquad (3)$$

The α-level set of a triangular fuzzy number is defined by

$$[\tilde{u}]^\alpha = [u_\alpha^L, u_\alpha^R] = [(1-\alpha)u_1 + \alpha u_2, (1-\alpha)u_3 + \alpha u_2] \qquad (4)$$

Definition 3. [41], Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy intervals. If there exists a unique $\tilde{w}$ such that $\tilde{u} = \tilde{v} + \tilde{w}$ (note that the fuzzy addition is commutative), then we call $\tilde{w}$ the Hukuhara difference (H-difference, for short) of $\tilde{u}$ and $\tilde{v}$ and we denote it by $\tilde{u} \ominus_H \tilde{v}$.

Proposition 1. [41] Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy intervals. If the Hukuhara difference $\tilde{w} = \tilde{u} \ominus_H \tilde{v}$ exists, then $\tilde{w}_\alpha^L = u_\alpha^L - \tilde{v}_\alpha^L$ and $\tilde{w}_\alpha^R = u_\alpha^R - \tilde{v}_\alpha^R$ for each $\alpha \in [0,1]$.

Throughout this paper, the following convention for inequalities between two intervals $A = [a^L, a^R]$ and $B = [b^L, b^R]$ in R are used : $A \preccurlyeq B$ if and only if $a^L \leq b^L$ and $a^R \leq b^R$, and $A \prec B$ if and only if $A \preccurlyeq B$ and $A \neq B$. The following two order relations on the space $\mathcal{F}(R)$ are considered and used in this paper. Let $\tilde{u}, \tilde{v} \in \mathcal{F}(R)$ be given two fuzzy intervals described by their α-level sets $[\tilde{u}]^\alpha = [u_\alpha^L, u_\alpha^R]$ and $[\tilde{v}]^\alpha = [v_\alpha^L, v_\alpha^R]$ for each $\alpha \in [0,1]$, respectively.

Definition 4. [24] We write $\tilde{u} \preccurlyeq \tilde{v}$ if and only if $[\tilde{u}]^\alpha \preccurlyeq [\tilde{v}]^\alpha$ for each $\alpha \in [0,1]$, which is equivalent to

$$\begin{cases} u_\alpha^L < v_\alpha^L \\ u_\alpha^R \leq v_\alpha^R \end{cases} \text{or} \begin{cases} u_\alpha^L \leq v_\alpha^L \\ u_\alpha^R < v_\alpha^R \end{cases} \text{or} \begin{cases} u_\alpha^L < v_\alpha^L \\ u_\alpha^R < v_\alpha^R \end{cases} \text{for all } \alpha \in [0,1].$$

Definition 5. [24] We write that $\tilde{u} \prec \tilde{v}$ if and only if $[\tilde{u}]^\alpha \prec [\tilde{v}]^\alpha$ for all $\alpha \in [0,1]$, which is equivalent to

$$\begin{cases} u_\alpha^L < v_\alpha^L \\ u_\alpha^R \leq v_\alpha^R \end{cases} \text{for all } \alpha \in [0,1] \text{ or} \begin{cases} u_\alpha^L \leq v_\alpha^L \\ u_\alpha^R < v_\alpha^R \end{cases} \text{for all } \alpha \in [0,1]$$

$$\text{or} \begin{cases} u_\alpha^L < v_\alpha^L \\ u_\alpha^R < v_\alpha^R \end{cases} \text{for all } \alpha \in [0,1].$$

3 Nondifferentiable Invex Crisp and Fuzzy Functions

Now, we introduce some notations and recall some basic definitions for nondifferentiable crisp functions. It is well-known that a crisp mapping $h: R^n \to R$ is a locally Lipschitz function at a point $x \in R^n$ if there exist scalars $M_x > 0$ and $\varepsilon > 0$ such that the inequality $|h(y) - h(z)| \leqq M_x \|y - z\|$ holds for all $y, z \in x + \varepsilon B$, where B is the open unit ball in R^n, so that $x + \varepsilon B$ is the open ball of radius about *x*. We say that the mapping *h* is a locally Lipschitz function (on R^n) if it is locally Lipschitz at any point of R^n.

Definition 6. [42], Let $h: R^n \to R$ be a locally Lipschitz function at $x^* \in R^n$. The Clarke generalized directional derivative of h at x^* in the direction $d \in R^n$, which is denoted by $h^0(x^*; d)$, is defined by $h^0(x^*; d) = \limsup_{\substack{y \to x^* \\ t \downarrow 0}} \frac{h(y+td)-h(y)}{t}$.

Definition 7. [42], The Clarke generalized subgradient of the locally Lipschitz crisp function $h: R^n \to R$ at $x^* \in R^n$, denoted by $\partial h(x^*)$, is defined by $\partial h(x^*) = \{\xi \in R^n : h^0(x^*; d) \geq \xi^T d, \forall d \in R^n\}$.

From the aforesaid definitions, it follows that, for any $d \in R^n$, $h^0(x^*; d) = max\{\xi^T d : \xi \in \partial h(x^*)\}$, [42].

We recall that the notion of a locally Lipschitz invex function was introduced in [43].

Definition 8. Let $h: R^n \to R$ be a locally Lipschitz crisp function and $x^* \in R^n$ be a given point. If there exists a vector-valued function $\eta: R^n \times R^n \to R^n$ such that the inequality

$$h(x) - h(x^*) \geq \xi^T \eta(x, x^*), \forall \xi \in \partial h(x^*) \ (>) \qquad (5)$$

is fulfilled for all $x \in R^n, (x \neq x^*)$, then f is an invex function (a strictly invex function) at x^* on R^n. If the above inequality is satisfied at any point x^*, then h is an invex function (a strictly invex function) on R^n. If (5) is satisfied on a nonempty subset $S \subset R^n$, then h is a (strictly) invex function on S.

Proposition 2. [42], Let $h: S \to R$ be a locally Lipschitz function on a nonempty set $S \subset R^n$, β be any scalar and x^* be an arbitrary point of S. Then $\partial(\beta h)(x^*) = \beta \partial h(x^*)$.

Proposition 3. [42], Let $h_j: S \to R, j = 1, \dots, p,$ be locally Lipschitz crisp functions on a nonempty open set $S \subset R^n$ and x^* be an arbitrary point of $S \subset R^n$. For any scalars β_j, one has

$$\partial(\textstyle\sum_{j=1}^p \beta_j h_j)(x^*) \subseteq \sum_{j=1}^p \beta_j \partial h_j(x^*).$$

The following result is useful in proving one of the main results in this paper.

Proposition 4. [40], Let $\varphi: S \to R$ be a locally Lipschitz crisp function on S and $x^* \in S$. Further, let $\varphi^+: S \to R$ be defined by $\varphi^+(x) := max\{0, \varphi(x)\}$. If φ is an invex function at $x^* \in S$ on S with respect to the function $\eta: S \times S \to R^n$, then φ^+ is also a locally Lipschitz invex function at $x^* \in S$ on S with respect to η.

Now, we re-call the definition of a fuzzy mapping given, for example, in the paper, [6].

Definition 9. [6], Let S be a nonempty subset of R^n. Then $\tilde{f}: S \to \mathcal{F}(R)$ is said to be a fuzzy mapping. For each $\alpha \in [0,1]$, we associate with $\tilde{f}$ the family of interval-valued functions $\tilde{f}_\alpha: S \to \mathcal{F}(R)$ given by $\tilde{f}_\alpha(x) = [\tilde{f}(x)]^\alpha$. The α-cut of $\tilde{f}$ at $x \in S$, which is a bounded and closed interval for each $\alpha \in [0,1]$, we denote by

$$\tilde{f}_\alpha(x) = [f_\alpha^L(x), f_\alpha^L(x)], \qquad (6)$$

where $f_\alpha^L(x) = min \tilde{f}_\alpha(x)$ and $f_\alpha^R(x) = max \tilde{f}_\alpha(x)$. Thus, $\tilde{f}$ can be represented by two functions f_α^L and f_α^R, which are functions from $S \times [0,1]$ to the set R, f_α^L is a bounded increasing function of α, f_α^R is a bounded decreasing function of α and, moreover, $f_\alpha^L(x) \leq f_\alpha^R(x)$ for all $x \in S$ and each $\alpha \in [0,1]$. Here, the endpoint functions $f_\alpha^L, f_\alpha^R: S \times [0,1] \to R$ are called left- and right-hand functions of $\tilde{f}_\alpha$, respectively.

Now, in a natural way, we generalize the definition of a locally Lipschitz function to the case of a fuzzy mapping.

Definition 10. A fuzzy mapping $\tilde{f}: R^n \to \mathcal{F}(R)$ is said to be locally Lipschitz at a given point $x \in R^n$ if, for each $\alpha \in [0,1]$, its left- and right functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are locally Lipschitz at x.

We now give the definition of the Clarke generalized derivative at $x \in R^n$ of a locally Lipschitz fuzzy function $\tilde{f}$ introduced in [44] as a pair of Clarke generalized derivatives at $x \in R^n$ of its left- and right-hand functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ defined for the fixed α-cut .

Definition 11. The Clarke generalized directional α-derivative of a locally Lipschitz fuzzy function $\tilde{f}: R^n \to \mathcal{F}(R)$ (given by (6) at x for some α-cut $\tilde{f}_\alpha$ in the direction d is defined as the Clarke generalized directional α-derivatives of its left- and right functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ at x in the direction d as follows:

$$\tilde{f}_\alpha^0(x; d) := \left(\limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f_\alpha^L(y+td) - f_\alpha^L(y)}{t}, \right.$$

$$\limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f_\alpha^R(y+td)-f_\alpha^R(y)}{t}\Bigg) = \left(\left(f_\alpha^L\right)^0(x;d), \left(f_\alpha^R\right)^0(x;d)\right).$$

As it follows from the aforesaid definition, the Clarke generalized derivative at $x \in R^n$ of a locally Lipschitz fuzzy function f does not represent an interval.

Definition 12. It is said that a locally Lipschitz fuzzy function $\tilde{f}: R^n \to \mathcal{F}(R)$ is directionally differentiable in the sense of Clarke at x if $\tilde{f}_\alpha^0(x;d)$ exists for each direction d and for all α-cuts.

Now, we give the definition of the Clarke generalized gradient of a locally Lipschitz fuzzy function introduced which was firstly introduced in [44].

Definition 13. The Clarke generalized gradient of a locally Lipschitz fuzzy function $\tilde{f}: R^n \to \mathcal{F}(R)$ on the α-cut is defined as a pair of Clarke generalized gradients of the left- and right-hand functions on this α-cut, that is, the pair $\partial \tilde{f}_\alpha(x) := (\partial f_\alpha^L(x), \partial f_\alpha^R(x))$, where

$$f_\alpha^L(x) := \{\xi_\alpha^L \in R^n : (f_\alpha^L)^0(x;d) \geq (\xi_\alpha^L)^T d, \forall d \in R^n\}$$

and

$$f_\alpha^R(x) := \{\xi_\alpha^R \in R^n : (f_\alpha^R)^0(x;d) \geq (\xi_\alpha^R)^T d, \forall d \in R^n\}$$

.

Remark 2. It follows by Definition 13 that, for each α-cut and any $d \in R^n$, we have

$$\partial \tilde{f}_\alpha(x) = \left(\partial f_\alpha^L(x), \partial f_\alpha^R(x)\right) =$$
$$(max\{(\xi_\alpha^L)^T d : \xi_\alpha^L \in \partial f_\alpha^L(x)\}, max\{(\xi_\alpha^R)^T d : \xi_\alpha^R \in \partial f_\alpha^R(x)\})$$

The the notion of invexity for a differentiable fuzzy function was firstly introduced in [22]. This definition was extended to the case of a locally Lipschitz fuzzy function in [44]. Namely, the concept of invexity for a locally Lipschitz fuzzy function $\tilde{f}$ was defined via invexity of its left-hand and right-hand functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ by using the α-cuts of $\tilde{f}$ given in [24], [41].

Definition 14. [44], Let $\tilde{f}: R^n \to \mathcal{F}(R)$ be a locally Lipschitz function and $u \in R^n$ be a given point. If there exists a vector-valued function $\eta: R^n \times R^n \to R^n$ such that the following inequalities

$$f_\alpha^L(x) - f_\alpha^L(u) \geq (\xi_\alpha^L)^T \eta(x,u), \forall \xi_\alpha^L \in \partial f_\alpha^L(u), \quad (7)$$

$$f_\alpha^R(x) - f_\alpha^R(u) \geq (\xi_\alpha^R)^T \eta(x,u), \forall \xi_\alpha^R \in \partial f_\alpha^R(u) \quad (8)$$

are satisfied for any $x \in R^n, (x \neq u)$, then $\tilde{f}$ is an invex fuzzy function (a strictly invex fuzzy function) at u on R^n. If (7) and (8) are satisfied at any point u, then $\tilde{f}$ is an invex fuzzy function (a strictly invex fuzzy function) on R^n. If (7) and (8) are satisfied on a nonempty subset S of R^n, then $\tilde{f}$ is an invex fuzzy function (a strictly invex fuzzy function) on S.

We now illustrate the concept of invexity for locally Lipschitz fuzzy mappings and, therefore, we present an example of a locally Lipschitz invex fuzzy function.

Example 1. Let the fuzzy function $\tilde{f}: R^2 \to \mathcal{F}(R)$ be defined by $\tilde{f}(x_1,x_2) = \tilde{2}x_1^2 + \tilde{4}x_2$, where $\tilde{2}$ and $\tilde{4}$ are continuous triangular fuzzy numbers which are defined as triples $\tilde{2} = (1,2,4)$ and $\tilde{4} = (1,4,6)$ (see Definition 2). Then, by (4), the α-level sets of both triangular fuzzy numbers are given by $[\tilde{2}]^\alpha = [1+\alpha, 4-2\alpha]$ and $[\tilde{4}]^\alpha = [1+3\alpha, 6-2\alpha]$, respectively. Moreover, by (4) and (6), the α-level cut of the fuzzy function $\tilde{f}$ is given as follows

$$\tilde{f}_\alpha(x_1,x_2) = \begin{cases} [(1+\alpha)x_1^2 + (1+3\alpha)x_2, (4-2\alpha)x_1^2 + (6-2\alpha)x_2] \\ \quad if \quad x_1 \in R, x_2 \geq 0 \\ [(1+\alpha)x_1^2 + (6-2\alpha)x_2, (4-2\alpha)x_1^2 + (1+3\alpha)x_2] \\ \quad if \quad x_1 \in R, x_2 < 0 \end{cases}$$

for each $\alpha \in [0,1]$. Therefore, the left-hand side and right-hand side functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are given by:

$$f_\alpha^L(x_1,x_2) = \begin{cases} (1+\alpha)x_1^2 + (1+3\alpha)x_2 \; if x_1 \in R, x_2 \geq 0 \\ (1+\alpha)x_1^2 + (6-2\alpha)x_2 \; if x_1 \in R, x_2 < 0, \end{cases}$$

$$f_\alpha^R(x_1,x_2) = \begin{cases} (4-2\alpha)x_1^2 + (6-2\alpha)x_2 \; if x_1 \in R, x_2 \geq 0 \\ (4-2\alpha)x_1^2 + (1+3\alpha)x_2 \; if x_1 \in R, x_2 < 0 \end{cases}$$

for each $\alpha \in [0,1]$. Note that $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are not convex functions and so the fuzzy function $\tilde{f}$ is not convex. Moreover, we note that $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are not differentiable at $x = (0,0)$ and, therefore, $\tilde{f}$ is not a level-wise differentiable fuzzy function at x (see Definition 4.2 [40]). The graphs of the left- and right-hand functions of $\tilde{f}_{\alpha=0}, \tilde{f}_{\alpha=\frac{1}{2}}, \tilde{f}_{\alpha=1}$ are given on Figure 1.

It can be shown by definition that $\tilde{f}$ is a locally Lipschitz strictly invex fuzzy function on R^2 with respect to $\eta: R^2 \times R^2 \to R^2$ defined by

$$\eta(y,x) = \begin{cases} \frac{y_1 - x_1}{\frac{1+\alpha}{6-2\alpha}(y_2 - x_2)}, y_1 \in R, y_2 \geq x_2 \geq 0 \vee 0 \geq y_2 > x_2 \\ \frac{y_1 - x_1}{\frac{6-2\alpha}{1+\alpha}(y_2 - x_2)}, y_1 \in R, 0 \geq y_2 > x_2 \vee x_2 \geq y_2 > 0 \end{cases}$$

Since $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are locally Lipschitz functions for every $\alpha \in [0,1]$, by Definition 10, $\tilde{f}$ is a locally Lipschitz fuzzy ampping on R^2. Further, note that, for each $\alpha \in [0,1]$, (7) and (8) are fulfilled for all

$y, x \in R^2$ with respect to η defined above as strict inequalities for the functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$.

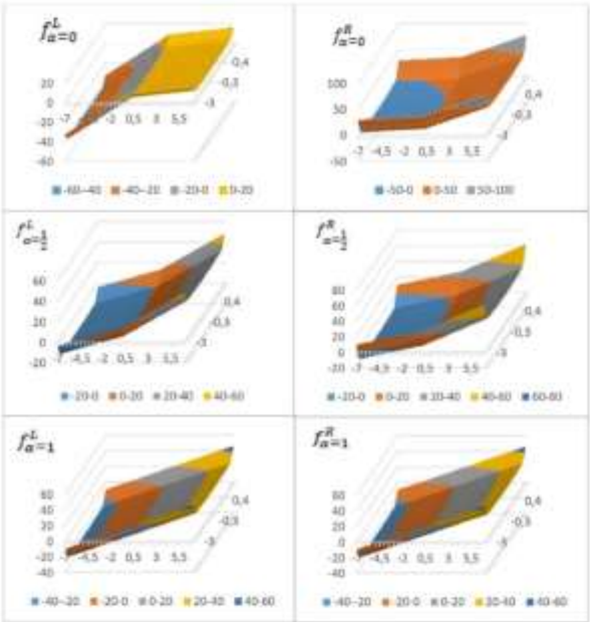

Fig. 1: Graphs of the left- and right-hand functions of $f_{\alpha=0}$, $f_{\alpha=\frac{1}{2}}$, $f_{\alpha=1}$.

Then, by Definition 14, $\tilde{f}$ is a strictly invex fuzzy function on R^2 with respect to η defined above.

Remark 3. Note that there is, in general, more than one vector-valued function η with respect to which a fuzzy function is invex. Indeed, if we consider again the fuzzy function $\tilde{f}(x_1, x_2) = \tilde{2}x_1^2 + \tilde{4}x_2$ which is defined in Example 1, then it can be shown that it is, in fact, invex on R^2 also with other functions $\eta: R^2 \times R^2 \to R^2$. Let us define the vector-valued function η as follows

$$\eta(y,x) = \begin{cases} \frac{y_1 - x_1}{\frac{4-2\alpha}{6-2\alpha}(y_2 - x_2)}, y_1 \in R, y_2 \geq x_2 \geq 0 \vee 0 \geq y_2 > x_2, \\ \frac{y_1 - x_1}{\frac{6-2\alpha}{4-2\alpha}(y_2 - x_2)}, y_1 \in R, 0 > y_2 > x_2 \vee x_2 \geq y_2 > 0, \\ \frac{y_1 - x_1}{\frac{1+\alpha}{6-2\alpha}(y_2 - x_2)}, y_1 \in R, y_2 \geq x_2 = 0, \\ \frac{y_1 - x_1}{\frac{6-2\alpha}{1+\alpha}(y_2 - x_2)}, y_1 \in R, 0 = x_2 > x_2. \end{cases}$$

Thus, the functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are strictly invex on R^2 also with respect to the defined above function η. Hence, by Definition 14, the fuzzy function $\tilde{f}$ is also strictly invex on R^2 with respect to η defined above.

In the work, we assume that only such fuzzy mappings $\tilde{f}: R^n \to \mathcal{F}(R)$ are considered for which their left-hand side and right-hand side functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are locally Lipschitz at a given point x of interest for all $\alpha \in [0,1]$.

4 Nondifferentiable Invex Fuzzy Optimization Problem and its Optimality

In this work, we investigate the following constrained optimization problem with a fuzzy-valued objective function defined as follows:

$$\begin{array}{c} \tilde{f}(x) \to min \\ s.t.\, g_j(x) \leq 0, j \in J = \{1, \dots, m\}, \end{array} \quad (FO)$$

where $\tilde{f}: R^n \to \mathcal{F}(R)$ is a fuzzy function and $g_j: R^n \to R, j \in J$, are real-valued functions defined on R^n. Let $\Omega := \{x \in R^n : g_j(x) \leq 0,\ j \in J\}$ be the set of all feasible solutions of the problem (FO). Now, we denote the set of active inequality constraints at a point $\hat{x} \in \Omega$ by $J(\hat{x}) = \{j \in J : g_j(\hat{x}) = 0\}$. Throughout the article, we shall assume that all functions involved in the fuzzy optimization problem (FO) given above, that is, its fuzzy objective function $\tilde{f}$ and its constraint functions $g_j, j \in J$, are locally Lipschitz on R^n.

In this paper, we use the α-cuts to describe the objective fuzzy function, as it was done in the papers [24] and [41]. Therefore, we shall assume that its left- and right-hand side values of $\tilde{f}$ are given by the functions $f_\alpha^L: R^n \times [0,1] \to R$ and $f_\alpha^R: R^n \times [0,1] \to R$ for each $\alpha \in [0,1]$, respectively.

Now, for the formulated above fuzzy optimization problem (FO), we define its optimal solutions as weakly nondominated and nondominated solutions which have been introduced in the paper [24].

Definition 15. [24], We say that $\hat{x} \in \Omega$ is a weakly nondominated solution in the considered fuzzy optimization problem (FO) if there exists no other $x \in \Omega$ such that $\tilde{f}(x) \prec \tilde{f}(\hat{x})$. In other words, if $\hat{x} \in \Omega$ is a weakly nondominated solution in (FO), then, by Definition 5, there exists no other $x \in \Omega$ such that

$$\begin{cases} f_\alpha^L(x) < f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) \leq f_\alpha^R(\hat{x}) \end{cases} \forall \alpha \in [0,1] \text{ or } \begin{cases} f_\alpha^L(x) \leq f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) < f_\alpha^R(\hat{x}) \end{cases}$$

$$\forall \alpha \in [0,1]$$

$$\text{or } \begin{cases} f_\alpha^L(x) < f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) < f_\alpha^R(\hat{x}) \end{cases} \forall \alpha \in [0,1]. \quad (9)$$

Definition 16. [24] We say that $\hat{x} \in \Omega$ is a nondominated solution in the considered fuzzy optimization problem (FO) if there exists no other

$x \in \Omega$ such that $f(x) \preccurlyeq f(\hat{x})$. This means that, if $\hat{x} \in \Omega$ is a nondominated solution in (FO), by Definition 4, there exists no other $x \in \Omega$ such that

$$\begin{cases} f_\alpha^L(x) < f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) \le f_\alpha^R(\hat{x}) \end{cases} \text{or} \begin{cases} f_\alpha^L(x) \le f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) < f_\alpha^R(\hat{x}) \end{cases}$$
$$\text{or} \begin{cases} f_\alpha^L(x) < f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) < f_\alpha^R(\hat{x}) \end{cases} \forall \alpha \in [0,1]. \tag{10}$$

Remark 4. If we denote by X^{wn} and X^n the sets of weakly nondominated and nondominated solutions in (FO), respectively, then $X^n \subseteq X^{wn}$.

In [44], under invexity hypotheses, optimality conditions of Karush-Kuhn-Tucker type were established for a nonsmooth optimization problem (FO) with fuzzy objective function. We now give the aforesaid Karush-Kuhn-Tucker like optimality conditions for $\hat{x} \in \Omega$ to be a (weakly) nondominated solution in the investigated nonsmooth fuzzy optimization problem (FO).

Theorem 1. [44], Let $\hat{x}$ be a feasible solution in the investigated fuzzy optimization problem (FO). Moreover, assume that there exist $\hat{\vartheta}_1(\alpha) \in R$, $\hat{\vartheta}_2(\alpha) \in R$ and $\hat{v}(\alpha) \in R^m$ for each $\alpha \in [0,1]$ such that the following Karush-Kuhn-Tucker like optimality conditions

$$0 \in \partial\left(\hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x})\right) + \sum_{j=1}^m \hat{v}_j(\alpha)\partial g_j(\hat{x}), \tag{11}$$

$$\hat{v}_j(\alpha) g_j(\hat{x}) = 0,\ j \in J, \tag{12}$$

$$\hat{\vartheta}_1(\alpha) > 0, \hat{\vartheta}_2(\alpha) > 0, \hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha) = 0, \hat{v}(\alpha) \ge 0 \tag{13}$$

hold. Further, assume that the objective function $\tilde{f}$ is an invex fuzzy mapping at $\hat{x}$ on Ω with respect to η and, moreover, each constraint g_j, $j = 1, \dots, m$, is an invex function at $\hat{x}$ on Ω with respect to the same function η, then $\hat{x}$ is a nondominated solution in (FO).

Theorem 2. [44], Let $\hat{x}$ be a feasible solution in the investigated fuzzy optimization problem (FO) and there exist $\hat{\alpha} \in [0,1]$, $\hat{v}(\hat{\alpha}) \in R^r, \hat{v}(\hat{\alpha}) \ge 0$ such that the weak Karush-Kuhn-Tucker like optimality conditions hold.

$$0 \in \partial f_{\hat{\alpha}}^L(\hat{x}) + \sum_{j=1}^m \hat{v}_j(\hat{\alpha})\partial g_j(\hat{x}), \tag{14}$$

$$0 \in \partial f_{\hat{\alpha}}^R(\hat{x}) + \sum_{j=1}^m \hat{v}_j(\hat{\alpha})\partial g_j(\hat{x}), \tag{15}$$

$$\hat{v}_j(\alpha) g_j(\hat{x}) = 0,\ j \in J, \tag{16}$$

Further, assume that $f_{\hat{\alpha}}^L$ and $f_{\hat{\alpha}}^R$ are invex at $\hat{x}$ on Ω with respect to η and, moreover, the functions g_j, $j = 1, \dots, m$, are invex at $\hat{x}$ on Ω with respect to the same function η. Then $\hat{x}$ is a weakly nondominated solution of the fuzzy optimization problem (FO).

Now, we give the necessary optimality conditions of Karush-Kuhn-Tucker type for the considered invex fuzzy optimization problem (FO).

Theorem 3. [44], Let $\hat{x} \in \Omega$ be a weakly nondominated solution in the fuzzy optimization problem (FO). Moreover, assume that the objective function $\tilde{f}$ is an invex fuzzy function at $\hat{x}$ on Ω with respect to η each constraint g_j, $j = 1, \dots, m$, is invex at $\hat{x}$ on Ω with respect to the same function η and the Slater constraint qualification is satisfied for (FO). Then, there exist $\hat{\alpha} \in [0,1]$, $\hat{\vartheta}_1(\hat{\alpha}) \in R$, $\hat{\vartheta}_2(\hat{\alpha}) \in R$ and $\hat{v}(\hat{\alpha}) \in R^m, \hat{v}(\hat{\alpha}) \ge 0$ such that the Karush-Kuhn-Tucker like optimality conditions hold at $\hat{x}$ for (FO).

$$0 \in \partial\left(\hat{\vartheta}_1(\hat{\alpha}) f_{\hat{\alpha}}^L(\hat{x}) + \hat{\vartheta}_2(\hat{\alpha}) f_{\hat{\alpha}}^R(\hat{x})\right) + \sum_{j=1}^m \hat{v}_j(\hat{\alpha})\partial g_j(\hat{x}), \tag{17}$$

$$\hat{v}_j(\hat{\alpha}) g_j(\hat{x}) = 0,\ j \in J, \tag{18}$$

$$\hat{\vartheta}_1(\hat{\alpha}) \ge 0, \hat{\vartheta}_2(\hat{\alpha}) \ge 0, \hat{\vartheta}_1(\hat{\alpha}) + \hat{\vartheta}_2(\hat{\alpha}) = 0, \hat{v}(\hat{\alpha}) \ge 0 \tag{19}$$

Corollary 1. [44] Let $\hat{x} \in \Omega$ be a weakly nondominated solution in the fuzzy optimization problem (FO) and hypotheses of Theorem 3 be fulfilled. Then, there exist $\hat{\alpha} \in [0,1]$, $\hat{\vartheta}_1(\hat{\alpha}) \in R$, $\hat{\vartheta}_2(\hat{\alpha}) \in R$ and $\hat{v}(\hat{\alpha}) \in R^m, \hat{v}(\hat{\alpha}) \ge 0$ satisfying (17)-(18) such that

$$0 \in \hat{\vartheta}_1(\hat{\alpha})\partial f_{\hat{\alpha}}^L(\hat{x}) + \hat{\vartheta}_2(\hat{\alpha})\partial f_{\hat{\alpha}}^R(\hat{x}) + \sum_{j=1}^m \hat{v}_j(\hat{\alpha})\partial g_j(\hat{x}). \tag{20}$$

Throughout this work, it is assumed that the Slater constraint qualification, [28], is fulfilled at any weakly nondominated solution in the investigated fuzzy optimization problem (FO).

Definition 17. The point $\hat{x} \in \Omega$ is said to be Karush-Kuhn-Tucker point (a KKT point, for short) if, for each $\alpha \in [0,1]$, there are Lagrange multipliers $\hat{\vartheta}_1(\alpha) \in R$, $\hat{\vartheta}_2(\alpha) \in R$ and $\hat{v}(\alpha) \in R^m$ such that the Karush-Kuhn-Tucker like optimality conditions (11)-(13) are fulfilled at $\hat{x}$.

Definition 18. The point $\hat{x} \in \Omega$ is said to be a weak Karush-Kuhn-Tucker point (a weak KKT point, for short) if, for some $\hat{\alpha} \in [0,1]$, there are Lagrange multiplier $\hat{v}(\hat{\alpha}) \in R^m$ such that the weak Karush-Kuhn-Tucker like optimality conditions (14)-(16) are fulfilled at $\hat{x}$.

5 Exactness Property of the Absolute Value Exact Penalty Fuzzy function Method for Fuzzy Optimization Problem with Invex Functions

It is known in optimization theory that exact penalty methods are one of approaches which can be applied for finding optimal solutions in constrained extremum problems. Their construction is based on the so-called penalty function whose unconstrained minimizing points are, at the same time, optimal solutions of the constrained optimization problem for all sufficiently large values of the penalty parameter. Hence, an original constrained extremum problem is transformed into a single unconstrained optimization problem in each methods of such a type.

Therefore, if we use any exact penalty function method to solve the given nonlinear constrained optimization problem with a fuzzy objective function, we have to construct in this approach its corresponding unconstrained penalized fuzzy optimization problem as follows

$$\tilde{P}(x,\rho) = \tilde{f}(x) + \tilde{1}_{\{\rho\varphi(x)\}} \to min, \quad (FP(\rho))$$

where $\tilde{f}: R^n \to \mathcal{F}(R)$ is an fuzzy function, p is a suitable penalty function, ρ is a penalty parameter and $\tilde{1}_{\{\rho\varphi(x)\}}$ is a crisp number with value $\rho\varphi(x)$ (see Remark 1). The aforesaid penalized fuzzy optimization problem (FP(ρ)) is constructed in such a way that its fuzzy objective function is the sum of a certain fuzzy "merit" function (which is the counterpart of the fuzzy objective function in the original fuzzy extremum problem) and the penalty term, which is the counterpart of the constraints define its feasible set. The fuzzy merit function is defined as the fuzzy original objective function of the given constrained extremum problem and the penalty term is formulated by multiplying a function designed by the constraints of the aforesaid optimization problem, by a positive parameter ρ. We call the aforesaid parameter ρ the penalty parameter.

Note that the objective function of the unconstrained fuzzy penalized optimization problem is a fuzzy mapping. Note that, by (6), for any arbitrary fixed $\alpha \in [0,1]$, we associate with $\tilde{P}$ the family of interval-valued functions $\tilde{P}_\alpha: R^n \times [0,1] \times R_+ \to \mathcal{F}(R)$ given by $\tilde{P}_\alpha(x,\rho) = [P_\alpha^L(x,\rho), P_\alpha^R(x,\rho)]$ for any $x \in R^n$, where $P_\alpha^L, P_\alpha^R: R^n \times [0,1] \times R_+ \to R$ are real-valued functions. Therefore, for every fixed $\alpha \in [0,1]$, the α-cut of the unconstrained fuzzy penalized optimization problem (FP(ρ)) is defined by:

$$\tilde{P}(x,\rho) = [f_\alpha^L(x) + \rho\varphi(x), \quad f_\alpha^L(x) + \rho\varphi(x)] \to min. \qquad (FP_\alpha(\rho))$$

It is known from the optimization literature, that the property of exactness of the penalization is the most important property from a practical point of view for each exact penalty function method. Now, we extend and generalize in a natural way the definition of this property given in the literature for classical exact penalty function methods to the fuzzy case.

Definition 19. If a threshold value $\bar{\rho} > 0$ exists such that, for every $\rho > \bar{\rho}$,

$$\text{arg (weakly) nondominated } \{\tilde{f}(x) : x \in \Omega\} = \text{arg (weakly) nondominated } \{\tilde{P}(x,\rho) : x \in R^n\},$$

then $\tilde{P}(x,\rho)$ is called a exact penalty fuzzy function and, therefore, we call (FP(ρ)) the penalized fuzzy optimization problem with exact penalty fuzzy function.

Note that the function $\tilde{P}(\cdot,\rho)$ can be interpreted as an exact penalty fuzzy function in such a way that a constrained (weakly) nondominated solution in the original fuzzy optimization problem (FO) can be found by looking for unconstrained (weakly) nondominated solutions of the aforesaid function $\tilde{P}(\cdot,\rho)$, for sufficiently large values of the penalty parameter ρ.

The often used nondifferentiable exact penalty function method to solve nonlinear extremum problems is the absolute value penalty function method, also called in optimization theory the l_1 exact penalty function method. If the aforesaid exact penalty function method is applied to solve (FO), then its formulation is:

$$\tilde{P}(x,\rho) = \tilde{f}(x) + \tilde{1}_{\{\rho \sum_{j=1}^{m} max\{0, g_j(x)\}\}} \to min, \quad (21)$$

where $\tilde{1}_{\{\rho \sum_{j=1}^{m} max\{0, g_j(x)\}\}}$ is a crisp number with the value $\rho \sum_{j=1}^{m} max\{0, g_j(x)\}$. We call (FP($\rho$)) defined above by (21) the fuzzy penalized optimization problem with the l_1 exact fuzzy penalty function. Hence, for any fixed $\alpha \in [0,1]$, we define the α-levels of the l_1 exact fuzzy penalty function for the original nonlinear fuzzy optimization problem (FO) by

$$\tilde{P}(x,\rho) = [f_\alpha^L(x) + \rho \sum_{j=1}^{m} max\{0, g_j(x)\},\; f_\alpha^L(x) + \rho \sum_{j=1}^{m} max\{0, g_j(x)\}], \quad (22)$$

where left-hand side and right-hand side values of $\tilde{P}_\alpha$ are given by $\tilde{P}_\alpha^L(x,\rho) = f_\alpha^L(x) + \rho \sum_{j=1}^{m} max\{0, g_j(x)\}$ and $\tilde{P}_\alpha^R(x,\rho) = f_\alpha^R(x) + \rho \sum_{j=1}^{m} max\{0, g_j(x)\}$, respectively. Hence, we can re-write, for any fixed $\alpha \in [0,1]$, the unconstrained penalized fuzzy optimization problem with the l_1 exact penalty fuzzy function defined by (22) as follows

$$\tilde{P}(x,\rho) = [f_\alpha^L(x) + \rho \sum_{j=1}^{m} max\{0, g_j(x)\},\; f_\alpha^L(x) + \rho \sum_{j=1}^{m} max\{0, g_j(x)\}] \to min. \quad (\text{FP}_\alpha(\rho)) \quad (23)$$

Now, for any inequality constraint function $g_j, j \in J$, we define the function g_j^+ as follows

$$g_j^+(x) = \begin{cases} 0 & for \quad g_j(x) \le 0, \\ g_j(x) & for \quad g_j(x) > 0. \end{cases} \quad (24)$$

Note that the aforesaid function g_j^+ possesses the suitable penalty features which depend on the single inequality constraint function g_j. If we use (24), then, for any fixed $\alpha \in [0,1]$, we can re-formulate the definition of (FP(ρ)) as follows

$$P_\alpha(x,\rho) = [f_\alpha^L(x) + \rho \sum_{j=1}^{m} g_j^+(x),\; f_\alpha^L(x) + \rho \sum_{j=1}^{m} g_j^+(x)] \to min. \quad (\text{FP}_\alpha(\rho)) \quad (25)$$

Now, we establish that a Karush-Kuhn-Tucker point of (FO) is a nondominated solution in (FP(ρ)) for sufficiently large values of penalty parameters ρ greater than the threshold equal to the largest Lagrange multiplier associated to some inequality constraint.

Theorem 4. Let $\hat{x} \in \Omega$ be a Karush-Kuhn-Tucker point of the considered nonsmooth fuzzy optimization problem (FO) and, for each $\alpha \in [0,1]$, the Karush-Kuhn-Tucker like optimality conditions (11)-(13) be fulfilled at $\hat{x}$ with Lagrange multipliers $\hat{\vartheta}_1(\alpha) \in R$, $\hat{\vartheta}_2(\alpha) \in R$ and $\hat{v}_j(\alpha) \in R^m$, $j \in J$. Furthermore, assume that the objective function $\tilde{f}$ is an invex fuzzy mapping at $\hat{x}$ on R^n with respect to η and each inequality constraint g_j, $j \in J$, is an invex function at $\hat{x}$ on R^n with respect to the same function η. If the penalty parameter ρ is assumed to be sufficiently large (namely, let us set the penalty parameter ρ to satisfy the condition $\rho \ge max\{\hat{v}_j(\alpha), j \in J\}$), then $\hat{x}$ is a nondominated solution in the penalized fuzzy optimization problem (FP(ρ)) with the l_1 exact penalty fuzzy function.

Proof. Assume that $\hat{x} \in \Omega$ is a Karush-Kuhn-Tucker point in (FO) and, moreover, for each $\alpha \in [0,1]$, the Karush-Kuhn-Tucker optimality conditions (11)-(13) are satisfied at $\hat{x}$ with Lagrange multipliers $\hat{\vartheta}_1(\alpha) \in R$, $\hat{\vartheta}_2(\alpha) \in R$ and $\hat{v}_j(\alpha) \in R^m$, $j \in J$. By means of contradiction, we suppose that $\hat{x}$ is not a nondominated solution of (FP(ρ)). Thus, by Definition 15, there exists $x^* \in R^n$ such that $\tilde{P}_\alpha(x^*,\rho) \preccurlyeq \tilde{P}_\alpha(\hat{x},\rho)$ for all $\alpha \in [0,1]$. Hence, by Definition 5, the above relation implies

$$\begin{cases} P_\alpha^L(x^*,\rho) < P_\alpha^L(\hat{x},\rho) \\ P_\alpha^R(x^*,\rho) \le P_\alpha^R(\hat{x},\rho) \end{cases} \text{or} \begin{cases} P_\alpha^L(x^*,\rho) \le P_\alpha^L(\hat{x},\rho) \\ P_\alpha^R(x^*,\rho) < P_\alpha^R(\hat{x},\rho) \end{cases} \text{or} \begin{cases} P_\alpha^L(x^*,\rho) < P_\alpha^L(\hat{x},\rho) \\ P_\alpha^R(x^*,\rho) < P_\alpha^R(\hat{x},\rho) \end{cases} \forall \alpha \in [0,1].$$

By the definition of (FP_α(ρ)) (see (25)), it follows that, for all $\alpha \in [0,1]$,

$$\begin{cases} f_\alpha^L(x^*) + \rho \sum_{j=1}^{m} g_j^+(x^*) < f_\alpha^L(\hat{x}) + \rho \sum_{j=1}^{m} g_j^+(\hat{x}) \\ f_\alpha^R(x^*) + \rho \sum_{j=1}^{m} g_j^+(x^*) \le f_\alpha^R(\hat{x}) + \rho \sum_{j=1}^{m} g_j^+(\hat{x}) \end{cases} \text{or}$$
$$\begin{cases} f_\alpha^L(x^*) + \rho \sum_{j=1}^{m} g_j^+(x^*) \le f_\alpha^L(\hat{x}) + \rho \sum_{j=1}^{m} g_j^+(\hat{x}) \\ f_\alpha^R(x^*) + \rho \sum_{j=1}^{m} g_j^+(x^*) < f_\alpha^R(\hat{x}) + \rho \sum_{j=1}^{m} g_j^+(\hat{x}) \end{cases} \text{or}$$
$$\begin{cases} f_\alpha^L(x^*) + \rho \sum_{j=1}^{m} g_j^+(x^*) < f_\alpha^L(\hat{x}) + \rho \sum_{j=1}^{m} g_j^+(\hat{x}) \\ f_\alpha^R(x^*) + \rho \sum_{j=1}^{m} g_j^+(x^*) < f_\alpha^R(\hat{x}) + \rho \sum_{j=1}^{m} g_j^+(\hat{x}). \end{cases}$$

Multiplying the above inequalities by the corresponding Lagrange multipliers $\hat{\vartheta}_1(\alpha) > 0$, $\hat{\vartheta}_2(\alpha) > 0$ associated to the fuzzy objective function, then adding the resulting inequalities and using $\hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha) = 1$, we get

$$\hat{\vartheta}_1(\alpha) f_\alpha^L(x^*) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x^*) + \rho \sum_{j=1}^{m} g_j^+(x^*) <$$

$$\hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x}) + \rho \sum_{j=1}^m g_j^+(\hat{x}) \,. \tag{26}$$

Since $\hat{x} \in \Omega$, by (24), it follows that $\sum_{j=1}^m g_j^+(\hat{x}) = 0$. Thus, (26) gives

$$\hat{\vartheta}_1(\alpha) f_\alpha^L(x^*) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x^*) + \rho \sum_{j=1}^m g_j^+(x^*) < \hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x}) \,.$$

If we use the Karush-Kuhn-Tucker optimality condition (12) together with $\hat{x} \in \Omega$, then we obtain

$$\hat{\vartheta}_1(\alpha) f_\alpha^L(x^*) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x^*) + \rho \sum_{j=1}^m g_j(x^*) < \hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x}) + \rho \sum_{j=1}^m g_j(\hat{x}) \,. \tag{27}$$

Since $\tilde{f}$ is assumed to be an invex fuzzy mapping at $\hat{x}$ on R^n with respect to the function η, therefore, by Definition 14, for each $\alpha \in [0,1]$, the inequalities

$$f_\alpha^L(x^*) - f_\alpha^L(\hat{x}) \geq (\xi_\alpha^L)^T \eta(x^*, \hat{x}), \forall \xi_\alpha^L \in \partial f_\alpha^L(\hat{x}), \tag{28}$$

$$f_\alpha^R(x^*) - f_\alpha^R(\hat{x}) \geq (\xi_\alpha^R)^T \eta(x^*, \hat{x}), \forall \xi_\alpha^R \in \partial f_\alpha^R(\hat{x}) \tag{29}$$

hold. Further, each inequality constraint function $g_j, j \in J$, is invex at $\hat{x}$ on R^n with respect to the same function η. Then, by Definition 14, the following inequalities

$$g_j(x^*) - g_j(\hat{x}) \geq \zeta_j^T \eta(x^*, \hat{x}), \forall \zeta_j \in \partial g_j(\hat{x}), j \in J \tag{30}$$

are satisfied. Multiplying each inequality (28) and (29) by the corresponding Lagrange multiplier and then adding both sides of the resulting inequalities and (30), we get that the inequality

$$\hat{\vartheta}_1(\alpha) f_\alpha^L(x^*) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x^*) + \rho \sum_{j=1}^m g_j(x^*) - \left(\hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x}) + \rho \sum_{j=1}^m g_j(\hat{x})\right) \geq \tag{31}$$
$$\left(\hat{\vartheta}_1(\alpha) \xi_\alpha^L + \hat{\vartheta}_2(\alpha) \xi_\alpha^R + \rho \sum_{j=1}^m \zeta_j\right)^T \eta(x^*, \hat{x})$$

holds. Hence, from the Karush-Kuhn-Tucker optimality condition (11) and Proposition 3, (31) yields that the inequality

$$\hat{\vartheta}_1(\alpha) f_\alpha^L(x^*) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x^*) + \rho \sum_{j=1}^m g_j(x^*) \geq \hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x}) + \rho \sum_{j=1}^m g_j(\hat{x})$$

holds, contradicting (27). Thus, this completes the proof of this theorem.

Corollary 2. Let $\hat{x} \in \Omega$ be a nondominated solution of (FO) and all the hypotheses of Theorem 4 be satisfied. Moreover, if we assume that the penalty parameter ρ is sufficiently large (namely, let us set the penalty parameter ρ is assumed to satisfy the condition $\rho \geq max\{\hat{v}_j, j \in J\}$), then $\hat{x}$ is also a nondominated solution in each associated penalized fuzzy optimization problem (FP(ρ)) with the l_1 exact penalty fuzzy function.

Now, we show that a weak Karush-Kuhn-Tucker point in the original fuzzy minimization problem (FO) is also a weakly nondominated solution in its associated penalized fuzzy optimization problem (FP(ρ)) for sufficiently large ρ.

Theorem 5. Let $\hat{x} \in \Omega$ be a weak Karush-Kuhn-Tucker point of (FO) and the conditions (14)-(16) be satisfied at $\hat{x}$ with Lagrange multipliers $\hat{v}_j(\hat{\alpha}), j \in J$, for some $\hat{\alpha} \in [0,1]$. Furthermore, assume that the functions $f_{\hat{\alpha}}^L$ and $f_{\hat{\alpha}}^R$ are invex at $\hat{x}$ on R^n with respect to η and the constraints g_j, $j = 1, \ldots, m$, are also invex at $\hat{x}$ on R^n with respect to the same function η. If we assume the penalty parameter ρ to be sufficiently large (namely, let us set the penalty parameter ρ to satisfy the condition $\rho \geq max\{\hat{v}_j(\hat{\alpha}), j \in J\}$),), then $\hat{x}$ is a weakly nondominated solution of (FP(ρ)).

Proof. From the assumption, we have that the weak Karush-Kuhn-Tucker optimality conditions (14)-(16) are fulfilled at $\hat{x}$ with Lagrange multipliers $\hat{v}_j(\hat{\alpha})$, $j \in J$, for some $\hat{\alpha} \in [0,1]$. By means of contradiction, suppose that $\hat{x}$ is not a weakly nondominated solution of (FP(ρ)). Therefore, by Definition 15, there exists $x^* \in R^n$ such that $\tilde{P}(x^*, \rho) < \tilde{P}(\hat{x}, \rho)$. In particular, there exists $x^* \in R^n$ such that the system of inequalities

$$P_{\hat{\alpha}}^L(x^*, \rho) < P_{\hat{\alpha}}^L(\hat{x}, \rho) \text{ or } P_{\hat{\alpha}}^R(x^*, \rho) < P_{\hat{\alpha}}^R(\hat{x}, \rho)$$

is satisfied for some $\hat{\alpha} \in [0,1]$. By (25), the above inequalities yield, respectively,

$$f_{\hat{\alpha}}^L(x^*) + \rho \sum_{j=1}^m g_j^+(x^*) < f_{\hat{\alpha}}^L(\hat{x}) + \rho \sum_{j=1}^m g_j^+(\hat{x}) \text{ or}$$
$$f_{\hat{\alpha}}^R(x^*) + \rho \sum_{j=1}^m g_j^+(x^*) < f_{\hat{\alpha}}^R(\hat{x}) + \rho \sum_{j=1}^m g_j^+(\hat{x}). \tag{32}$$

By $\hat{x} \in \Omega$, the inequalities (32) imply, respectively,

$$f_{\hat{\alpha}}^L(x^*) + \rho \sum_{j=1}^m g_j^+(x^*) < f_{\hat{\alpha}}^L(\hat{x}) \text{ or}$$

$$f_{\hat{\alpha}}^{R}(x^*) + \rho \sum_{j=1}^{m} g_j^+(x^*) < f_{\hat{\alpha}}^{R}(\hat{x}). \quad (33)$$

By assumption, $\rho \geq max\{\hat{v}_j(\hat{\alpha}), j \in J\}$),) Thus, the inequalities above give, respectively,

$$f_{\hat{\alpha}}^{L}(x^*) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j^+(x^*) < f_{\hat{\alpha}}^{L}(\hat{x}) \quad or$$

$$f_{\hat{\alpha}}^{R}(x^*) + \rho \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j^+(x^*) < f_{\hat{\alpha}}^{R}(\hat{x}). \quad (34)$$

Using the feasibility of $\hat{x}$ in (FO) and (24) together with the Karush-Kuhn-Tucker optimality condition (16), we get

$$f_{\hat{\alpha}}^{L}(x^*) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j^+(x^*) < f_{\hat{\alpha}}^{L}(\hat{x}) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j^+(\hat{x}) \quad (35)$$

or

$$f_{\hat{\alpha}}^{R}(x^*) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j^+(x^*) < f_{\hat{\alpha}}^{R}(\hat{x}) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j^+(\hat{x}). \quad (36)$$

By hypotheses, the functions $f_{\hat{\alpha}}^{L}$ and $f_{\hat{\alpha}}^{R}$ are invex at $\hat{x}$ on R^n with respect to η and, moreover, the constraint functions g_j, $j = 1, \dots, m$, are also invex at $\hat{x}$ on R^n with respect to the same function η. Hence, by Definitions 14 and 8, respectively, the inequalities hold.

$$f_{\hat{\alpha}}^{L}(x^*) - f_{\hat{\alpha}}^{L}(\hat{x}) \geq (\xi_{\hat{\alpha}}^{L})^T \eta(x^*, \hat{x}), \forall \xi_{\hat{\alpha}}^{L} \in \partial f_{\hat{\alpha}}^{L}(\hat{x}), \quad (37)$$

$$f_{\hat{\alpha}}^{R}(x^*) - f_{\hat{\alpha}}^{R}(\hat{x}) \geq (\xi_{\hat{\alpha}}^{R})^T \eta(x^*, \hat{x}), \forall \xi_{\hat{\alpha}}^{R} \in \partial f_{\hat{\alpha}}^{R}(\hat{x}), \quad (38)$$

$$g_j(x^*) - g_j(\hat{x}) \geq \zeta_j^T \eta(x^*, \hat{x}), \forall \zeta_j \in \partial g_j(\hat{x}), j \in J \quad (39)$$

Now if we multiply the inequalities (37)-(39) by corresponding Lagrange multipliers and then adding both sides of the resulting inequalities, then we obtain that the inequalities

$$f_{\hat{\alpha}}^{L}(x^*) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j(x^*) - f_{\hat{\alpha}}^{L}(\hat{x}) - \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j^+(\hat{x}) \geq \left(\xi_{\hat{\alpha}}^{L} + \rho \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) \zeta_j\right)^T \eta(x^*, \hat{x}) \quad (40)$$

$$f_{\hat{\alpha}}^{R}(x^*) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j(x^*) - f_{\hat{\alpha}}^{R}(\hat{x}) - \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j(\hat{x}) \geq \left(\xi_{\hat{\alpha}}^{R} + \rho \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) \zeta_j\right)^T \eta(x^*, \hat{x}) \quad (41)$$

hold for any $\xi_{\hat{\alpha}}^{L} \in \partial f_{\hat{\alpha}}^{L}(\hat{x})$, $\xi_{\hat{\alpha}}^{R} \in \partial f_{\hat{\alpha}}^{R}(\hat{x})$, $\zeta_j \in \partial g_j(\hat{x})$ $j = 1, \dots, m$. Thus, by the Karush-Kuhn-Tucker optimality conditions (14) and (15), (40)-(41) yield that the inequalities

$$f_{\hat{\alpha}}^{L}(x^*) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j(x^*) \geq f_{\hat{\alpha}}^{L}(\hat{x}) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j^+(\hat{x}),$$

$$f_{\hat{\alpha}}^{R}(x^*) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j(x^*) \geq f_{\hat{\alpha}}^{R}(\hat{x}) + \sum_{j=1}^{m} \hat{v}_j(\hat{\alpha}) g_j(\hat{x})$$

hold, which contradicting (35) or (36). Hence, the proof of this theorem is completed.

Corollary 3. Let $\hat{x} \in \Omega$ be a weakly nondominated solution of (FO) and all the assumptions of Theorem 5 be satisfied. Then $\hat{x}$ is also a weakly nondominated solution of (FP(ρ)).

Now, we establish the converse results to those ones established above. First, we derive some useful results, which we use in proving them.

Proposition 5. Let $\hat{x} \in \Omega$ be a nondominated solution of (FP(ρ)). Then, there is no $x \in \Omega$ such that

$$f(x) \preccurlyeq f(\hat{x}). \quad (42)$$

Proposition 6. Let $\hat{x} \in \Omega$ be a weakly nondominated solution of (FP(ρ)). Then, there is no $x \in \Omega$ such that

$$f(x) \prec f(\hat{x}). \quad (43)$$

Theorem 6. Let Ω be a compact subset of R^n and $\hat{x} \in \Omega$ be a (weakly) nondominated solution of the fuzzy penalized optimization problem (FP(ρ)) with the l_1 exact fuzzy penalty function. Further, assume that the objective function f is an invex fuzzy function at $\hat{x}$ on R^n with respect to η, each inequality constraint g_j, $j \in J$, is invex at $\hat{x}$ on R^n with respect to the same function η. If the penalty parameter ρ is sufficiently large, then $\hat{x}$ is also a (weakly) nondominated solution of the considered fuzzy optimization problem (FO).

Proof. Let $\hat{x} \in \Omega$ be a nondominated solution in the fuzzy penalized optimization problem (FP(ρ)) with the l_1 exact fuzzy penalty function.

Firstly, we assume that $\hat{x} \in \Omega$. Hence, by Proposition 5, it follows that that there does not exist $x \in R^n$ such that (42) is satisfied. Thus, by Definition 16, $\hat{x}$ is a nondominated solution of the considered fuzzy optimization problem (FO).

Now, under the assumptions of this theorem, we shall prove that the case $\hat{x} \notin \Omega$ is impossible. By means of contradiction, suppose that $\hat{x} \notin \Omega$. Since $\hat{x}$ is a nondominated solution in the fuzzy penalized optimization problem $(FP_\alpha(\bar{\rho}))$, there exist $\alpha \in [0,1]$, $\hat{\vartheta}_1(\alpha) \in R$, $\hat{\vartheta}_1(\alpha) > 0$, $\hat{\vartheta}_2(\alpha) \in R$, $\hat{\vartheta}_2(\alpha) > 0$ and $\hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha) = 1$, such that $0 \in \hat{\vartheta}_1(\alpha)\partial P_\alpha^L(\hat{x},\rho) + \hat{\vartheta}_2(\alpha)\partial P_\alpha^R(\hat{x},\rho)$. Using the definition of the absolute value exact fuzzy penalty function, one has

$$0 \in \hat{\vartheta}_1(\alpha)\partial\left(f_\alpha^L(\hat{x}) + \bar{\rho}\sum_{j=1}^m g_j^+(\hat{x})\right) + \hat{\vartheta}_2(\alpha)\partial\left(f_\alpha^R(\hat{x}) + \bar{\rho}\sum_{j=1}^m g_j^+(\hat{x})\right).$$

Since the weights $\hat{\vartheta}_1(\alpha)$ and $\hat{\vartheta}_2(\alpha)$ are nonnegative for each $\alpha \in [0,1]$, therefore, equality holds in Proposition 3. Thus, the above relation yields

$$0 \in \hat{\vartheta}_1(\alpha)\partial f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha)\partial f_\alpha^R(\hat{x}) + \left(\hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha)\right)\partial\left(\bar{\rho}\sum_{j=1}^m g_j^+(\hat{x})\right).$$

Using $\hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha) = 1$ together with Proposition 2, we get

$$0 \in \hat{\vartheta}_1(\alpha)\partial f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha)\partial f_\alpha^R(\hat{x}) + \bar{\rho}\partial\left(\sum_{j=1}^m g_j^+(\hat{x})\right).$$

Thus, by Proposition 3, we have

$$0 \in \hat{\vartheta}_1(\alpha)\partial f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha)\partial f_\alpha^R(\hat{x}) + \bar{\rho}\sum_{j=1}^m \partial g_j^+(\hat{x}). \tag{44}$$

By hypothesis, $\tilde{f}$ is an invex fuzzy function at $\hat{x}$ on R^n (with respect to η). Then, by Definition 14, the functions f_α^L and f_α^R are invex at $\hat{x}$ on R^n with respect to η for each $\alpha \in [0,1]$. Hence, for each $\alpha \in [0,1]$, the inequalities

$$f_\alpha^L(x) - f_\alpha^L(\hat{x}) \geq (\xi_\alpha^L)^T\eta(x,\hat{x}), \forall \xi_\alpha^L \in \partial f_\alpha^L(\hat{x}), \tag{45}$$

$$f_\alpha^R(x) - f_\alpha^R(\hat{x}) \geq (\xi_\alpha^R)^T\eta(x,\hat{x}), \forall \xi_\alpha^R \in \partial f_\alpha^R(\hat{x}) \tag{46}$$

hold for all $x \in R^n$. Further, since each constraint function g_j, $j = 1,\dots,m$, is invex at $\hat{x}$ on R^n with respect to the same function η, by Proposition 4, also the functions $g_j^+, j \in J$, are invex on R^n with respect to the same function η. Hence, by Definition 14, the inequalities

$$g_j^+(x) - g_j^+(\hat{x}) \geq \left(\zeta_j^+\right)^T\eta(x,\hat{x}), \forall \zeta_j^+ \in \partial g_j^+(\hat{x}), j \in J \tag{47}$$

hold for all $x \in R^n$. Multiplying (47) by $\bar{\rho} > 0$, we obtain, for any $\zeta_j^+ \in \partial g_j^+(\hat{x}), j = 1,\dots,m$,

$$\bar{\rho}\sum_{j=1}^m g_j^+(x) - \bar{\rho}\sum_{j=1}^m g_j^+(\hat{x}) \geq \bar{\rho}\sum_{j=1}^m\left(\zeta_j^+\right)^T\eta(x,\hat{x}). \tag{48}$$

Combining (45), (46) and (48), we have that the inequalities

$$f_\alpha^L(x) + \bar{\rho}\sum_{j=1}^m g_j^+(x) - \left(f_\alpha^L(\hat{x}) + \bar{\rho}\sum_{j=1}^m g_j^+(\hat{x})\right) \geq \left(\xi_\alpha^L + \bar{\rho}\sum_{j=1}^m \zeta_j^+\right)^T\eta(x,\hat{x}), \tag{49}$$

$$f_\alpha^R(x) + \bar{\rho}\sum_{j=1}^m g_j^+(x) - \left(f_\alpha^R(\hat{x}) + \bar{\rho}\sum_{j=1}^m g_j^+(\hat{x})\right) \geq \left(\xi_\alpha^R + \bar{\rho}\sum_{j=1}^m \zeta_j^+\right)^T\eta(x,\hat{x}) \tag{50}$$

hold for all $x \in R^n$ and any $\xi_\alpha^L \in \partial f_\alpha^L(\hat{x})$, $\xi_\alpha^R \in \partial f_\alpha^R(\hat{x})$, $\zeta_j^+ \in \partial g_j^+(\hat{x}), j = 1,\dots,m$.

Now, if we multiply (49) and (50) by $\hat{\vartheta}_1(\alpha)$ and $\hat{\vartheta}_2(\alpha)$, respectively, and then we add both sides of them, we get

$$\hat{\vartheta}_1(\alpha)f_\alpha^L(x) + \hat{\vartheta}_2(\alpha)f_\alpha^R(x) + \bar{\rho}\sum_{j=1}^m g_j^+(x) - \left(\hat{\vartheta}_1(\alpha)f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha)f_\alpha^R(\hat{x}) + \bar{\rho}\sum_{j=1}^m g_j^+(\hat{x})\right) \geq$$

$$\left(\hat{\vartheta}_1(\alpha)\xi_\alpha^L + \hat{\vartheta}_2(\alpha)\xi_\alpha^R + \bar{\rho}\left(\hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha)\right)\sum_{j=1}^m \zeta_j^+\right)^T\eta(x,\hat{x})$$

.

Since $\hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha) = 1$, we have that, for all $x \in R^n$ and any $\xi_\alpha^L \in \partial f_\alpha^L(\hat{x})$, $\xi_\alpha^R \in \partial f_\alpha^R(\hat{x})$, $\zeta_j^+ \in \partial g_j^+(\hat{x}), j = 1,\dots,m$,

$$\hat{\vartheta}_1(\alpha)f_\alpha^L(x) + \hat{\vartheta}_2(\alpha)f_\alpha^R(x) + \bar{\rho}\sum_{j=1}^m g_j^+(x) - \left(\hat{\vartheta}_1(\alpha)f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha)f_\alpha^R(\hat{x}) + \bar{\rho}\sum_{j=1}^m g_j^+(\hat{x})\right) \geq \left(\hat{\vartheta}_1(\alpha)\xi_\alpha^L + \hat{\vartheta}_2(\alpha)\xi_\alpha^R + \bar{\rho}\sum_{j=1}^m \zeta_j^+\right)^T\eta(x,\hat{x}). \tag{51}$$

Hence, by (44), (51) implies that the inequality

$$\hat{\vartheta}_1(\alpha)f_\alpha^L(x) + \hat{\vartheta}_2(\alpha)f_\alpha^R(x) + \bar{\rho}\sum_{j=1}^m g_j^+(x) \geq \hat{\vartheta}_1(\alpha)f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha)f_\alpha^R(\hat{x}) + \bar{\rho}\sum_{j=1}^m g_j^+(\hat{x})$$

is satisfied for all $x \in R^n$. By (24), for each $x \in \Omega$, one has $\sum_{j=1}^m g_j^+(x) = 0$. Hence, the above inequality yields that the inequality

$$\hat{\vartheta}_1(\alpha) f_\alpha^L(x) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x) \geq \hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x}) + \bar{\rho} \sum_{j=1}^m g_j^+(\hat{x}) \quad (52)$$

is fulfilled for all $x \in \Omega$. By assumption, $\hat{x}$ is not a feasible solution of the original fuzzy optimization problem (FO). Hence, by (24), one has $\sum_{j=1}^m g_j^+(\hat{x}) > 0$. Then, by the foregoing inequality, (52) gives

$$\bar{\rho} \leq max \left\{ \frac{\hat{\vartheta}_1(\alpha) f_\alpha^L(x) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x)}{\sum_{j=1}^m g_j^+(\hat{x})} \frac{-\hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) - \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x})}{\sum_{j=1}^m g_j^+(\hat{x})} : x \in \Omega \right\}. \quad (53)$$

From the assumption, $\bar{\rho}$ is sufficiently large. Now, we suppose that, for each $\alpha \in [0,1]$, $\bar{\rho}$ is assumed to satisfy

$$\bar{\rho} > max \left\{ \frac{\hat{\vartheta}_1(\alpha) f_\alpha^L(x) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x)}{\sum_{j=1}^m g_j^+(\hat{x})} \frac{-\hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) - \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x})}{\sum_{j=1}^m g_j^+(\hat{x})} : x \in \Omega \right\}. \quad (54)$$

We now prove that, by (53), $\bar{\rho}$ is a finite nonnegative real number. Indeed, by assumption, $\hat{x}$ is a nondominated solution in the penalized fuzzy optimization problem ($FP_\alpha(\bar{\rho})$) with the absolute value exact penalty fuzzy function. Then, by Definition 16, there does not exist $x \in \Omega$ such that $\hat{\vartheta}_1(\alpha) f_\alpha^L(x) + \hat{\vartheta}_2(\alpha) f_\alpha^R(x) - \left(\hat{\vartheta}_1(\alpha) f_\alpha^L(\hat{x}) + \hat{\vartheta}_2(\alpha) f_\alpha^R(\hat{x})\right) < 0$

Hence, by (54), we have that $\bar{\rho} > 0$. From the assumption, Ω is a compact subset of R^n. This implies that $\bar{\rho}$ is a finite real number. Since the inequality (54) contradicts the inequality (53), this gives that the case $\hat{x} \notin \Omega$ is impossible. Thus, $\hat{x}$ is feasible in the orooginal fuzzy optimization problem (FO). This means that, for any $\rho \geq \bar{\rho}$, $\hat{x}$, which is a nondominated solution of (FP($\bar{\rho}$)), is also a nondominated solution of (FO). Thus, the conclusion of the theorem follows from Proposition 5. The proof of this theorem, in the case when $\hat{x}$ is a weakly nondominated solution of the fuzzy penalized optimization problem (FP(ρ)), is similar and the conclusion theorem follows from Proposition 6 in such a case. Thus, this completes the proof of this theorem.

Now, we present one of the main results of this work which follows directly from the results established above.

Theorem 7. Let all the hypotheses of Corollary 2 (Corollary 3, respectively) and Theorem 6 be satisfied. Then $\hat{x}$ is a (weakly) nondominated of the considered fuzzy optimization problem (FO) with the fuzzy objective function if and only if $\hat{x}$ is a (weakly) nondominated of the penalized fuzzy optimization problem (FP(ρ)) with the l_1 exact penalty fuzzy function.

We now present the example of a nonlinear nonconvex fuzzy optimization problem in which its objective function is a nondifferentiable invex fuzzy function and its constraints are invex crisp functions with respect to the same function η. Then, using the l_1 exact penalty method analyzed in this paper, we solve this nonsmooth fuzzy extremum problem in order to illustrate the result formulated in Theorem 7.

Example 2. Consider the nonconvex nonsmooth fuzzy optimization problem with the fuzzy-valued objective function formulated as follows:

$$\tilde{f}(x) = \tilde{2}|1 - e^{-x}| \ominus_H \tilde{1} \rightarrow min$$
$$s.t. \; g_1(x) = e^{-x} - 1 \leq 0, \qquad (FO1)$$
$$g_2(x) = e^{-1} - e^{-x} \leq 0,$$

where $\tilde{1}$ and $\tilde{2}$ are continuous triangular fuzzy numbers. These fuzzy numbers are defined as triples $\tilde{1} = (0,1,2)$ and $\tilde{2} = (0,2,4)$. Hence, by (4), the α-level sets of these triangular fuzzy numbers are $1_\alpha = [\alpha, 2 - \alpha]$ and $2_\alpha = [2\alpha, 4 - 2\alpha]$, respectively. Moreover, we notice that $\Omega = \{x \in R : e^{-x} - 1 \leq 0 \wedge e^{-1} - e^{-x} \leq 0\} = [0,1]$ is the set of all feasible solutions of (FO1) and, moreover, $\hat{x} = 0$ is a feasible solution of (FO1). Further, by (1) and (2), the α-level cut of the fuzzy objective function $\tilde{f}$ is given by $\tilde{f}_\alpha(x) = [2\alpha|1 - e^{-x}| - \alpha, (4 - 2\alpha)|1 - e^{-x}| + \alpha - 2]$ for any $\alpha \in [0,1]$. Clearly, the left- and right-hand side functions $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are not convex and so $\tilde{f}$ is not convex. Since $f_\alpha^L(\cdot)$ and $f_\alpha^R(\cdot)$ are not differentiable at $\hat{x} = 0$, $\tilde{f}$ is not a level-wise differentiable fuzzy function at $\hat{x} = 0$ (see Definition 4.2 [40]). The Karush-Kuhn-Tucker optimality conditions (11)-(13) are fulfilled with Lagrange multipliers $\hat{\vartheta}_1 = \frac{1}{2}$, $\hat{\vartheta}_2 = \frac{1}{2}$, $\hat{v}_1(\alpha) = 0$, $\hat{v}_2(\alpha) = 0$ for each $\alpha \in [0,1]$. Moreover, all functions constituting (FO1) are locally Lipschitz, that is, the objective function is a locally Lipschitz fuzzy function by Definition 10. Further, the functions involved in (FO1) satisfy invexity hypotheses of Corollary 2. Indeed, if we define $\eta : R \times R \rightarrow R$ by $\eta(x, \hat{x}) = e^{-x} - e^{-\hat{x}}$, then the functions $f_\alpha^L(\cdot)$, $f_\alpha^R(\cdot)$, g_1 and g_2 are invex at $\hat{x} = 0$ on R with respect to η. Since we use the l_1 exact penalty function method in solving (FO1), therefore, we have to construct its associated penalized fuzzy optimization problem

(FP1(ρ)) with the l_1 exact fuzzy penalty function defined by:

$$P_\alpha(x,\rho) = [f_\alpha^L(x) + \rho(max\{0, e^{-x}-1\} + max\{0, e^{-1}-e^{-x}\}), f_\alpha^R(x) + \rho(max\{0, e^{-x}-1\} + max\{0, e^{-1}-e^{-x}\})]. \quad (FP1_\alpha(\rho))$$

Since all the assumptions of Corollary 2 are satisfied, $\hat{x}=0$ is a nondominated solution in (FP1(ρ)) for any penalty parameter $\rho > 0$. Further, all hypotheses Theorem 6 are also fulfilled. Thus, if we assume that $\hat{x}=0$ is a nondominated solution in (FP1(ρ)), then it is also a nondominated solution of (FO1). Hence, we have shown under invexity hypotheses the equivalence between nondominated solutions in fuzzy optimization problems (FO1) and (FP1(ρ)) for any penalty parameter $\rho > 0$.

6 The Convergence of the Absolute Value Exact Fuzzy Penalty Function Method

In this section, we present an algorithm for solving the investigated nondifferentiable fuzzy optimization problem (FO) with fuzzy objective function and crisp inequality constraints by using the l_1 exact penalty fuzzy function method and we prove its convergence in the considered fuzzy case.

Therefore, we create the following sequence of the associated fuzzy penalized optimization problems (FP(ρ_k)) for the original nondifferentiable fuzzy extremum problem (FO) as follows:

$$\tilde{P}(x,\rho_k) = \tilde{f}(x) + \tilde{1}_{\{\rho_k \sum_{j=1}^m max\{0, g_j(x)\}\}} \to min, (FP(\rho_k))$$

where $\{\rho_k\}$ is a sequence of penalty parameters with $\rho_k > 0$ and, moreover, $\lim_{k\to\infty} \rho_k = \infty$.

Algorithm (l1EFPFM) of the l_1 exact fuzzy penalty function method:

Given $\rho_0 > 0$, tolerance $\delta > 0$ and starting point $\hat{x}_0$;
FOR $k = 0,1,\dots$
  Starting at $\hat{x}_k$, solve $(FP(\rho_k))$ to find a weakly nondomi-nated solution $\hat{x}_{k+1}$;
  IF $\sum_{j=1}^m g_j^+(\hat{x}_{k+1}) < \delta$, THEN
    STOP with an approximate weakly nondominated solution $\hat{x}_{k+1}$;
  ELSE
    a new penalty parameter $\rho_{k+1} > \rho_k$ should be chosen;
    a new starting point $\hat{x}_{k+1}$ should be chosen;
  END IF;
END FOR;

Before we establish the convergence of the analyzed l_1 exact penalty fuzzy function method which is used to solve the considered nondifferentiable fuzzy optimization problem (FO), we present and prove some useful results.

Lemma 1. i) If $x \in \Omega$, then $\lim_{\rho\to\infty} \rho \sum_{j=1}^m g_j^+(x) = 0$.

ii) If $x \notin \Omega$, then $\lim_{\rho\to\infty} \rho \sum_{j=1}^m g_j^+(x) = \infty$.

Proposition 7. Let $\hat{x}_k$ be a weakly nondominated solution of the penalized fuzzy optimization problem (FP(ρ_k)), $k = 1,2,\dots$, generated by Algorithm (l1EFPFM). If $\{\hat{x}_{k_s}\}$ is a convergent subsequence of $\{\hat{x}_k\}$ and its limit point, i.e. $\hat{x} = \lim_{s\to\infty} \hat{x}_{k_s}$ is a feasible solution of the original fuzzy optimization problem (FO), then $\lim_{s\to\infty} \rho_{k_s} \sum_{j=1}^m g_j^+(\hat{x}_{k_s}) = 0$.

Proof. By means of contradiction, suppose that

$$\lim_{s\to\infty} \rho_{k_s} \sum_{j=1}^m g_j^+(\hat{x}_{k_s}) \neq 0 . \quad (55)$$

Hence, (55) implies that there exists a convergent subsequence $\{\hat{x}_{k_{s_t}}\}$ of $\{\hat{x}_{k_s}\}$ generated by Algorithm (l1EFPFM) such that

$$\lim_{t\to\infty} \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}_{k_{s_t}}) > \varepsilon , \quad (56)$$

where ε is a nonnegative real number. Since $\hat{x}_{k_s}$ is a weakly nondominated solution of $(FP(\rho_{k_{s_t}}))$, by Definition 15, there is no $x \in \Omega$ such that

$$\tilde{P}(x, \rho_{k_{s_t}}) \prec \tilde{P}(\hat{x}_{k_{s_t}}, \rho_{k_{s_t}}), \quad t = 1,2,\dots \quad (57)$$

Hence, (57) implies that there is no $x \in \Omega$ such that

$$\begin{cases} P_\alpha^L(x,\rho_{k_{s_t}}) < P_\alpha^L(\hat{x}_{k_{s_t}},\rho_{k_{s_t}}) \\ P_\alpha^R(x,\rho_{k_{s_t}}) \le P_\alpha^R(\hat{x}_{k_{s_t}},\rho_{k_{s_t}}) \end{cases} \text{for all } \alpha \in [0,1]$$

$$\text{or } \begin{cases} P_\alpha^L(x,\rho_{k_{s_t}}) \le P_\alpha^L(\hat{x}_{k_{s_t}},\rho_{k_{s_t}}) \\ P_\alpha^R(x,\rho_{k_{s_t}}) < P_\alpha^R(\hat{x}_{k_{s_t}},\rho_{k_{s_t}}) \end{cases} \text{for all } \alpha \in [0,1]$$

$$\text{or } \begin{cases} P_\alpha^L(x,\rho_{k_{s_t}}) < P_\alpha^L(\hat{x}_{k_{s_t}},\rho_{k_{s_t}}) \\ P_\alpha^R(x,\rho_{k_{s_t}}) < P_\alpha^R(\hat{x}_{k_{s_t}},\rho_{k_{s_t}}) \end{cases} \text{for all } \alpha \in [0,1].$$

Then, by (23), it follows that, for $s = 1,2,\dots$, there is no $x \in \Omega$ such that,

$$\begin{cases} f_\alpha^L(x) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(x) < f_\alpha^L(\hat{x}_{k_{s_t}}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}_{k_{s_t}}) \\ f_\alpha^R(x) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(x) \le f_\alpha^R(\hat{x}_{k_{s_t}}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}_{k_{s_t}}) \end{cases}$$

for all $\alpha \in [0,1]$ or

$$\begin{cases} f_\alpha^L(x) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(x) \le f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \\ f_\alpha^R(x) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(x) < f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \end{cases}$$

for all $\alpha \in [0,1]$ or

$$\begin{cases} f_\alpha^L(x) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(x) < f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \\ f_\alpha^R(x) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(x) < f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \end{cases}$$

for all $\alpha \in [0,1]$.

By assumption, $\lim_{s\to\infty} \hat{x}_{k_s} = \hat{x} \in \Omega$. Since the above system of inequalities is not satisfied for any $x \in \Omega$, therefore, it is not satisfied also for $\hat{x} \in \Omega$. Thus, we get that the following system of inequalities

$$\begin{cases} f_\alpha^L(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) < f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \\ f_\alpha^R(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) \le f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \end{cases}$$

for all $\alpha \in [0,1]$ or

$$\begin{cases} f_\alpha^L(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) \le f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \\ f_\alpha^R(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) < f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \end{cases}$$

for all $\alpha \in [0,1]$ or

$$\begin{cases} f_\alpha^L(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) < f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \\ f_\alpha^R(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) < f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right) \end{cases}$$

for all $\alpha \in [0,1]$

is not satisfied. Therefore, for each point $\hat{x}_{k_{s_t}}$, we have

$f_\alpha^L(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) \ge f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right)$

or

$f_\alpha^R(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) > f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right)$

for some $\alpha \in [0,1]$,

$f_\alpha^L(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) > f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right)$

or

$f_\alpha^R(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) \ge f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right)$

for some $\alpha \in [0,1]$,

$f_\alpha^L(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) > f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right)$

or

$f_\alpha^R(\hat{x}) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) > f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \rho_{k_{s_t}} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_{s_t}}\right)$

for some $\alpha \in [0,1]$.

Taking limit $t \to \infty$ and using (56), we obtain

$f_\alpha^L(\hat{x}) + \lim_{t\to\infty} \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) > \lim_{t\to\infty} f_\alpha^L\left(\hat{x}_{k_{s_t}}\right) + \varepsilon$ or

$f_\alpha^R(\hat{x}) + \lim_{t\to\infty} \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) > \lim_{t\to\infty} f_\alpha^R\left(\hat{x}_{k_{s_t}}\right) + \varepsilon$

for some $\alpha \in [0,1]$.

We have that $\lim_{t\to\infty} \hat{x}_{k_{s_t}} = \hat{x}$. Since f_α^L and f_α^R are continuous for each $\alpha \in [0,1]$, the above system of inequalities gives

$f_\alpha^L(\hat{x}) + \lim_{t\to\infty} \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) > f_\alpha^L(\hat{x}) + \varepsilon$

or $f_\alpha^R(\hat{x}) + \lim_{t\to\infty} \rho_{k_{s_t}} \sum_{j=1}^m g_j^+(\hat{x}) > f_\alpha^R(\hat{x}) + \varepsilon$

for some $\alpha \in [0,1]$.

Since $\hat{x} \in \Omega$, by Lemma 1, the above inequalities reduce to the inequality $0 > \varepsilon$, which contradicts the fact that ε is a nonnegative real number. This completes the proof of this proposition.

Theorem 8. Let $\hat{x}_k$ be a weakly nondominated solution in $(FP(\rho_k)), k = 1,2,\dots,$ generated by Algorithm (l1EFPFM). If $\{\hat{x}_{k_s}\}$ is a convergent subsequence of $\{\hat{x}_k\}$ and $\hat{x} = \lim_{s\to\infty} \hat{x}_{k_s}$ is a feasible point of (FO), then $\hat{x}$ is its weakly nondominated solution.

Proof. Let $\hat{x} = \lim_{s\to\infty} \hat{x}_{k_s}$. It is known that $\hat{x}_{k_s}$ is a weakly nondominated solution of $\left(FP\left(\rho_{k_s}\right)\right)$. Hence, by Definition 15, there is no $x \in \Omega$ such that

$$\tilde{P}\left(x, \rho_{k_s}\right) \prec \tilde{P}\left(\hat{x}_{k_s}, \rho_{k_s}\right). \quad (58)$$

Hence, (58) gives

$$\begin{cases} P_\alpha^L\left(x, \rho_{k_s}\right) < P_\alpha^L\left(\hat{x}_{k_s}, \rho_{k_s}\right) \\ P_\alpha^R\left(x, \rho_{k_s}\right) \le P_\alpha^R\left(\hat{x}_{k_s}, \rho_{k_s}\right) \end{cases} \text{for all } \alpha \in [0,1]$$

$$\text{or} \begin{cases} P_\alpha^L\left(x, \rho_{k_s}\right) \le P_\alpha^L\left(\hat{x}_{k_s}, \rho_{k_s}\right) \\ P_\alpha^R\left(x, \rho_{k_s}\right) < P_\alpha^R\left(\hat{x}_{k_s}, \rho_{k_s}\right) \end{cases} \text{for all } \alpha \in [0,1]$$

$$\text{or} \begin{cases} P_\alpha^L\left(x, \rho_{k_s}\right) < P_\alpha^L\left(\hat{x}_{k_s}, \rho_{k_s}\right) \\ P_\alpha^R\left(x, \rho_{k_s}\right) < P_\alpha^R\left(\hat{x}_{k_s}, \rho_{k_s}\right) \end{cases} \text{for all } \alpha \in [0,1].$$

Then, by (23), it follows that, for $s = 1,2,\dots,$

$$\begin{cases} f_\alpha^L(x) + \rho_{k_s} \sum_{j=1}^m g_j^+(x) < f_\alpha^L\left(\hat{x}_{k_s}\right) + \rho_{k_s} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_s}\right) \\ f_\alpha^R(x) + \rho_{k_s} \sum_{j=1}^m g_j^+(x) \le f_\alpha^R\left(\hat{x}_{k_s}\right) + \rho_{k_s} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_s}\right) \end{cases}$$

for all $\alpha \in [0,1]$ or

$$\begin{cases} f_\alpha^L(x) + \rho_{k_s} \sum_{j=1}^m g_j^+(x) \le f_\alpha^L\left(\hat{x}_{k_s}\right) + \rho_{k_s} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_s}\right) \\ f_\alpha^R(x) + \rho_{k_s} \sum_{j=1}^m g_j^+(x) < f_\alpha^R\left(\hat{x}_{k_s}\right) + \rho_{k_s} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_s}\right) \end{cases}$$

for all $\alpha \in [0,1]$ or

$$\begin{cases} f_\alpha^L(x) + \rho_{k_s} \sum_{j=1}^m g_j^+(x) < f_\alpha^L\left(\hat{x}_{k_s}\right) + \rho_{k_s} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_s}\right) \\ f_\alpha^R(x) + \rho_{k_s} \sum_{j=1}^m g_j^+(x) < f_\alpha^R\left(\hat{x}_{k_s}\right) + \rho_{k_s} \sum_{j=1}^m g_j^+\left(\hat{x}_{k_s}\right) \end{cases}$$

for all $\alpha \in [0,1]$.

By assumption, $\lim_{s\to\infty} \hat{x}_{k_s} = \hat{x} \in \Omega$. Thus, by Lemma 1 i), it follows that $\lim_{s\to\infty} \rho_{k_s} \sum_{j=1}^m g_j^+(x) = 0$. Since $\hat{x}_{k_s}$,

$s = 1,2,\ldots,$ are weakly nondominated solutions of $(FP(\rho_{k_s}))$, , $s = 1,2,\ldots,$ generated by Algorithm (l1EFPFM), by Proposition 6, we have that $\lim_{s\to\infty} \rho_{k_s} \sum_{j=1}^{m} g_j^+(\hat{x}_{k_s}) = 0$. Taking limit $s \to \infty$ in the above system of inequalities, we get that there is no $x \in \Omega$ such that

$\begin{cases} f_\alpha^L(x) < f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) \le f_\alpha^R(\hat{x}) \end{cases} \forall \alpha \in [0,1]$ or $\begin{cases} f_\alpha^L(x) \le f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) < f_\alpha^R(\hat{x}) \end{cases}$ $\forall \alpha \in [0,1]$

$$\text{or} \begin{cases} f_\alpha^L(x) < f_\alpha^L(\hat{x}) \\ f_\alpha^R(x) < f_\alpha^R(\hat{x}) \end{cases} \forall \alpha \in [0,1].$$

Hence, by Definition 15, $x \in \Omega$ is a weakly nondominated solution of (FO). This completes the proof of this theorem.

7 The Simulation of the Choice of the Penalty Parameter

One of the important factors that can ensure the success of using the fuzzy exact penalty function l_1 method considered in the article for solving fuzzy optimization problems is the strategy for appropriate choosing the penalty parameter. Namely, if the initial value of the penalty parameter ρ_0 is too small in the algorithm, more cycles may be needed in the aforesaid approach to determine its appropriate value. Moreover, the choice of the initial value of ρ_0 also affects the choice of the starting point x_0. In the next examples, we illustrate some difficulties that can be caused by the choice of inappropriate values of the penalty parameter ρ.

Example 3. Consider the following fuzzy optimization problem:

$$\begin{aligned} \tilde{f}(x) &= \tilde{2}x \to min \\ s.t.\ g(x) &= 1 - x \le 0, \end{aligned} \quad (FO2)$$

where $\tilde{2}$ is a continuous triangular fuzzy number. It is given as triple $\tilde{2} = (1,2,4)$. Hence, by (4), the α-level set of this triangular fuzzy number is $[\tilde{2}]^\alpha = [1+\alpha, 4-2\alpha]$. The set of all feasible solutions is $\Omega = \{x \in R : x \ge 1\}$ and $\hat{x} = 1$ is an feasible solution in (FO2).

Now, we use the l_1 exact penalty fuzzy function method in solving the fuzzy optimization problem (F02) considered in this example. Then, by (22), we construct the unconstrained fuzzy optimization problem (FP2). Hence, by (25), for any fixed $\alpha \in [0,1]$, the α-levels of the fuzzy l_1 exact penalty function are as follows:

$$\tilde{P}_\alpha(x,\rho) = [P_\alpha^L(x,\rho), P_\alpha^R(x,\rho)] =$$

$$[f_\alpha^L(x) + \rho max\{0, 1-x\}, f_\alpha^R(x) + \rho max\{0, 1-x\}],$$

where

$$f_\alpha^L(x) = \begin{cases} (4-2\alpha)x & if \quad x<0, \\ (1+\alpha)x & if \quad x \ge 0, \end{cases}$$

$$f_\alpha^R(x) = \begin{cases} (1+\alpha)x & if \quad x<0, \\ (4-2\alpha)x & if \quad x \ge 0. \end{cases}$$

Hence, one has

$$P_\alpha^L(x,\rho) = \begin{cases} (4-2\alpha-\rho)x+\rho & if \quad x<0, \\ (1+\alpha-\rho)x+\rho & if \quad 0 \le x < 1, \\ (1+\alpha)x & if \quad x \ge 1, \end{cases}$$

$$P_\alpha^R(x,\rho) = \begin{cases} (1-\alpha-\rho)x+\rho & if \quad x<0, \\ (4-2\alpha-\rho)x+\rho & if \quad 0 \le x < 1, \\ (4-2\alpha)x & if \quad x \ge 1. \end{cases}$$

Now, we consider two cases:

1) $\rho = 0.5$

In this case, $\tilde{P}_\alpha(x,0.5) = [P_\alpha^L(x,0.5), P_\alpha^R(x,0.5)]$, where

$$P_\alpha^L(x,0.5) = \begin{cases} \left(\frac{7}{2}-2\alpha\right)x+\frac{1}{2} & if \quad x<0, \\ \left(\frac{1}{2}+\alpha\right)x+\frac{1}{2} & if \quad 0 \le x < 1, \\ (1+\alpha)x & if \quad x \ge 1, \end{cases}$$

$$P_\alpha^R(x,0.5) = \begin{cases} \left(\frac{1}{2}+\alpha\right)x+\frac{1}{2} & if \quad x<0, \\ \left(\frac{7}{2}-2\alpha\right)x+\frac{1}{2} & if \quad 0 \le x < 1, \\ (4-2\alpha)x & if \quad x \ge 1. \end{cases}$$

The graphs of $P_\alpha^L(\cdot,0.5)$ and $P_\alpha^R(\cdot,0.5)$ for chosen α-cuts are presented on Figure 2.

2) $\rho = 4$.

In this case, $\tilde{P}_\alpha(x,4) = [P_\alpha^L(x,4), P_\alpha^R(x,4)]$, where

$$P_\alpha^L(x,4) = \begin{cases} -2\alpha x+4 & if \quad x<0, \\ (-3+\alpha)x+4 & if \quad 0 \le x < 1, \\ (1+\alpha)x & if \quad x \ge 1, \end{cases}$$

$$P_\alpha^R(x,4) = \begin{cases} (-3+\alpha)x+4 & if \quad x<0, \\ -2\alpha x+4 & if \quad 0 \le x < 1, \\ (4-2\alpha)x & if \quad x \ge 1. \end{cases}$$

The graphs of $P_\alpha^L(\cdot,4)$ and $P_\alpha^R(\cdot,4)$ for chosen α-cuts are presented on Figure 3.

It is not difficult to note that there is the significantly better case than the previous one. Namely, if $\rho = 4$ and the current iterate x_k is any real number (including the initial point x_0), then, for the almost any α-cuts $\alpha \in [0,1]$, all implementations of the absolute value exact penalty fuzzy function method will give a step that moves to the solution $x = 1$ (maybe except the 0-cut, namely $P_{\alpha=0}^L(\cdot,4)$ if $\hat{x}_k \le 0$ and $P_{\alpha=0}^R(\cdot,4)$ if $0 \le \hat{x}_k < 1$.

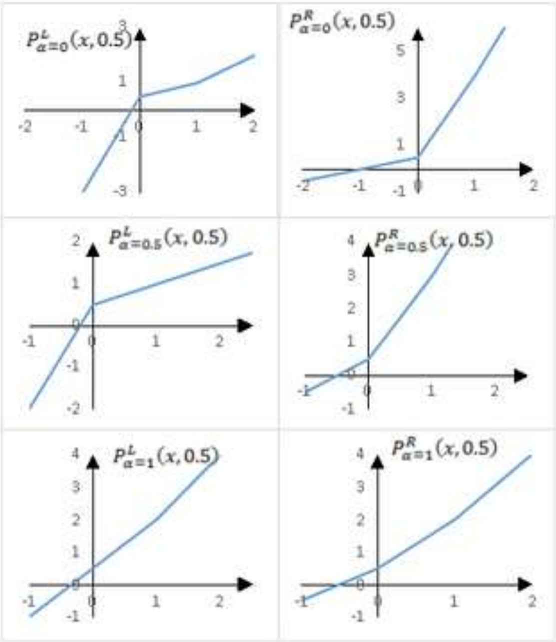

Fig. 2: The graphs of $P^L_\alpha(\cdot,0.5)$ and $P^R_\alpha(\cdot,0.5)$

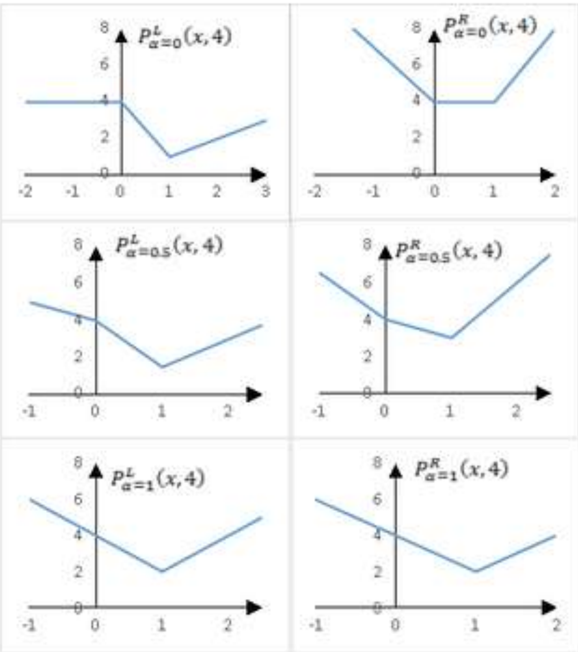

Fig. 3: The graphs of $P^L_\alpha(\cdot,4)$ and $P^R_\alpha(\cdot,4)$.

As it follows even from these above cases, the value of the penalty parameter is crucial to obtain a correct solution in the applied l_1 exact penalty fuzzy function method to solve fuzzy optimization problems. However, also the value of α is important in the considered penalty method.

For example, for $\rho = 1$, for larger α-cuts (that is, $\alpha \in (1/2,1]$), and any $x_k \in (0,\infty)$, the value of $P^L_{\alpha=1}(\cdot,4)$ in next iterates tends to a solution (moreover, the method behaved similarly for even greater values of the penalty parameter, for which the functions $P^L_\alpha(\cdot,\rho)$, $P^R_\alpha(\cdot,\rho)$ tends to a solution, what is more, for all α-cuts. Therefore, the initial x_0 depends on the values of the penalty parameter and also on α-cuts.

From which it follows the aforesaid result? We consider the Karush-Kuhn-Tucker necessary optimality conditions at $\hat{x} = 1$ for the analyzed fuzzy optimization problem. Then, we have the following relation:

$$\hat{\vartheta}_1(\alpha)(1-\alpha) + \hat{\vartheta}_2(\alpha)(4-2\alpha) - \hat{v}_1(\alpha) = 0.$$

Note that Lagrange multipliers depend on the value α. In fact, we consider the following cases of α-cuts (where we normalize Lagrange $\hat{\vartheta}_1(\alpha), \hat{\vartheta}_2(\alpha)$ to satisfy $\hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha) = 1$):

1) $\alpha = 0$. Then, we have from the above equation:
$$\hat{v}_1(0) = \hat{\vartheta}_1(0) + 4\hat{\vartheta}_2(0) \in [1,4].$$
2) $\alpha = 0.5$. Then, we have from the above equation:
$$\hat{v}_1(0.5) = \frac{3}{2}\hat{\vartheta}_1(0.5) + 3\hat{\vartheta}_2(0.5) \in \left[\frac{3}{2},3\right].$$
3) $\alpha = 1$. Then, we have from the above equation:
$$\hat{v}_1(1) = 2\hat{\vartheta}_1(1) + 2\hat{\vartheta}_2(1) = 2.$$

As it follows from the above, the maximum value of Lagrange multiplier $\hat{v}_1(\alpha)$ decreases for larger α-cuts. Hence, for all penalty parameters ρ greater than the threshold equal to the largest Lagrange multiplier associated to the constraint g, we can obtain the solution (in such a case, threshold is equal to $\hat{v}_1(\alpha)$ since there is only one constraint in the analyzed fuzzy optimization problem). Hence, we conclude that that if the value of α is larger then, in general, the aforesaid threshold is smaller.

In the previous example, we considered such a fuzzy optimization problem, for which there exists a threshold of the penalty parameter ρ such that, for any penalty parameter greater than the aforesaid threshold, all implementations of the l_1 exact fuzzy penalty function method will give a step that moves to the solution $\hat{x}$ starting from any initial point (no maybe, except for the case of 0-cut). Moreover, such a behavior will be repeated in the algorithm and, thus, it will produce increasingly better iterates, until the penalty parameter ρ is not decreased below some threshold value. However, there are also such cases, in which the aforesaid threshold of the penalty parameter may not exist. So, there are such cases, in which, even if we know an appropriate value of the penalty parameter ρ for a given solution $\hat{x}$, this value may cause the appearance of iterations that move away from the correct solution or it may be an insufficient at the starting point. The next example of a fuzzy optimization problem shows that it is not possible to prescribe in advance a value of the penalty parameter that is adequate at every iteration.

Example 4. Consider the following fuzzy optimization problem:

$$\begin{aligned} \tilde{f}(x) &= \widetilde{0.5}x^3 \rightarrow min \\ s.t.\, g(x) &= -1 - x \leq 0, \end{aligned} \quad (FO3)$$

where $\widetilde{0.5}$ is a continuous triangular fuzzy number, which is defined as triple $\widetilde{0.5} = (0,0.5,1)$. Then, by (4), the α-level set of this triangular fuzzy number is $[\widetilde{0.5}]^\alpha = \left[\frac{1}{2}\alpha, 1 - \frac{1}{2}\alpha\right]$. The set of all feasible solutions is $\Omega = \{x \in R : x \geq -1\}$ and the solution $\hat{x} = -1$.

We now apply the l_1 exact penalty fuzzy function method in solving (FO3) considered in this example. Then, by (22), we construct the unconstrained fuzzy optimization problem (FP3(ρ)). Hence, by (25), for any fixed $\alpha \in [0,1]$, the α-levels of the fuzzy l_1 exact penalty function are as follows:

$$\tilde{P}_\alpha(x,\rho) = [P_\alpha^L(x,\rho), P_\alpha^R(x,\rho)] =$$

$$[f_\alpha^L(x) + \rho max\{0, -1 - x\}, f_\alpha^R(x) + \rho max\{0, -1 - x\}]$$

,

where

$$f_\alpha^L(x) = \begin{cases} \left(1 - \frac{1}{2}\alpha\right)x^3 & if \quad x < 0, \\ \frac{1}{2}\alpha x^3 & if \quad x \geq 0, \end{cases}$$

$$f_\alpha^R(x) = \begin{cases} \frac{1}{2}\alpha x^3 & if \quad x < 0, \\ \left(1 - \frac{1}{2}\alpha\right)x^3 & if \quad x \geq 0. \end{cases}$$

Hence, one has

$$P_\alpha^L(x,\rho) = \begin{cases} \left(1 - \frac{1}{2}\alpha\right)x^3 + \rho(-1-x) & if \quad x < 0, \\ \left(1 - \frac{1}{2}\alpha\right)x^3 & if \quad 0 \leq x < 1, \\ \frac{1}{2}\alpha x^3 & if \quad x \geq 1, \end{cases}$$

$$P_\alpha^R(x,\rho) = \begin{cases} \left(1 - \frac{1}{2}\alpha\right)x^3 + \rho(-1-x) & if \quad x < 0, \\ \frac{1}{2}\alpha x^3 & if \quad 0 \leq x < 1, \\ \left(1 - \frac{1}{2}\alpha\right)x^3 & if \quad x \geq 1. \end{cases}$$

Note that by the Karush-Kuhn-Tucker necessary optimality conditions, one has

$$\hat{v}_1(\alpha) = 3\hat{\vartheta}_1(\alpha)\left(1 - \tfrac{1}{2}\alpha\right) + \tfrac{3}{2}\hat{\vartheta}_2(\alpha)\alpha.$$

Then, if we normalize Lagrange multipliers $\hat{\vartheta}_1(\alpha), \hat{\vartheta}_2(\alpha)$ (to satisfy the condition $\hat{\vartheta}_1(\alpha) + \hat{\vartheta}_2(\alpha) = 1$) associated to α-cuts of the objective function, then we note that $\hat{v}_1(\alpha) \in \left[\frac{3}{2}\alpha, 3\left(1 - \frac{1}{2}\alpha\right)\right]$. Therefore, the threshold of the penalty parameter ρ, for which there is the equivalence between nondominated solutions in (F03) and (FP3(ρ)) just for all penalty parameters, satisfies the condition $\rho(\alpha) \geq 3\left(1 - \frac{1}{2}\alpha\right)$. Now, we illustrate this result and, therefore, we consider two sample values of ρ.

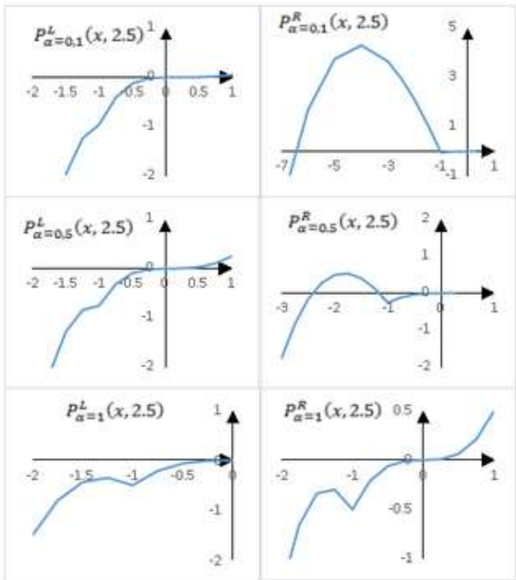

Fig. 4: The graphs of $P_\alpha^L(\cdot,2.5)$ and $P_\alpha^R(\cdot,2.5)$ dla $\alpha = 0$, $\alpha = \frac{1}{2}$, $\alpha = 1$

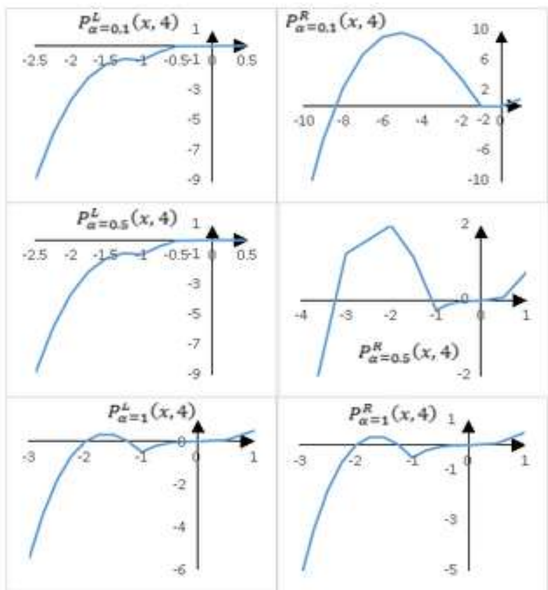

Fig. 5: The graphs of $P_\alpha^L(\cdot,4)$ and $P_\alpha^R(\cdot,4)$ dla $\alpha = 0$, $\alpha = \frac{1}{2}$, $\alpha = 1$

Now, we consider two cases:

Firstly, we consider the case, in which the penalty parameter $\rho = 2.5$.

Note that the graphs on Figure 4 confirm that if α is larger then $\hat{x} = -1$ is a local undominated solution of $\tilde{P}(\cdot, \rho)$ provided that the penalty parameter satisfies the condition given above. However, $P_\alpha^L(\cdot, 2.5)$ and $P_\alpha^R(\cdot, 2.5)$ are unbounded below as $\hat{x} \to -\infty$ (Figure 4). Therefore, for all α-cut and for every penalty parameter ρ, there is a starting point x_0, such that there doesn't exist decreasing path in both $P_\alpha^L(\cdot, 2.5)$ and $P_\alpha^R(\cdot, 2.5)$ from the aforesaid starting point x_0 to the solution $\hat{x} = -1$.

Now, we consider the case, in which the penalty parameter $\rho = 4$.

Note that we know the value of the penalty parameter ρ in the considered case, which is analyzed for the given solution $\hat{x} = -1$. However, this value is inadequate at any starting point x_0. In fact, for any $x_0 < -1{,}5$, the values of $P_\alpha^L(\cdot, 4)$ in next iterates will tend to $-\infty$. In other words, each next iterate $\hat{x}_k \in (-\infty, -1.5)$ moves away from the solution $\hat{x} = -1$. Analogously, if we take a sufficiently small starting point x_0, then the values of $P_\alpha^R(\cdot, 4)$ in next iterates x_k will tend to $-\infty$ for each $\alpha \in [0{,}1]$. In other words, each next iterate $\hat{x}_k \in (-\infty, -1.5)$ for some $z \in R$ with $\rho = 4$, moves away from the solution $\hat{x} = -1$. As it follows from the graphs on Figure 5, such a set is different for various α-cuts.

There is the following question – why there is no equivalence between nondominated solutions in (FP3) and (FP3(ρ)), even if we take the penalty parameter ρ greater than the threshold? This follows from the fact that none of the functions f_α^L and f_α^R is invex with respect to any function η (see [45]). Hence, the assumption that the functions f_α^L and f_α^R are invex for all $\alpha \in [0{,}1]$ is not fulfilled. Therefore, the objective fuzzy function $\tilde{f}$ is not invex with any function η. Thus, this example illustrates the case in which not all the functions involved in the investigated fuzzy optimization problem are invex. In such a case, there is practically no starting point x_0 at which the exact method of the fuzzy penalty function l_1 could start successfully searching for the correct solution (since next iterate may move to $-\infty$ and the functions $P_\alpha^L(\cdot, \rho)$ and $P_\alpha^R(\cdot, \rho)$ may tend also to $-\infty$ for all $\alpha \in [0{,}1]$).

8 Conclusions

We have mentioned in Introduction that there are many works in the literature on fuzzy optimization problems in which fundamental results from optimization theory have been established for such mathematical programming problems. However, there are still open problems in the iterature regarding the introduction of new methods for solving such non-deterministic extreme problems in optimization theory. In this work, the absolute value exact penalty fuzzy function method has been applied for the first time to solve a new class of mathematical programming problems, which are nonconvex and nonsmooth optimization problems with fuzzy objective functions. Therefore, for the considered minimization problem with fuzzy objective function and inequality constraints, the formulation of the corresponding fuzzy penalized fuzzy optimization problem with the l_1 exact fuzzy penalty function has been presented. Then, the main from the practical point of view property of the l_1 exact fuzzy penalty function method, i.e. exactness of the penalization, has been defined and analyzed in the considered fuzzy case. The equivalence between (weakly) nondominated solutions in the analyzed constrained fuzzy minimization problem and its corresponding penalized fuzzy optimization problem has been proven under appropriate invexity hypotheses. Also the threshold of the penalty parameter has been given. Then, it has been proven that the aforesaid equivalence holds if the penalty parameter in the penalized fuzzy optimization problem exceeds this threshold. However, if the functions constituting the considered fuzzy optimization problems are not invex, this threshold may not exist. This result has been investigated and illustrated by an appropriate example of such fuzzy optimization problems. In other words, the approach for the choice of the penalty parameter has been analyzed in the paper and the analysis has been made both theoretically and practically. Thus, it has been shown that the l_1 exact fuzzy penalty function method is applicable also for solving a larger class of nonsmooth optimization problems with fuzzy objective functions than convex ones. Further, the algorithm for the l_1 exact fuzzy penalty function method, which is applied for finding weakly nondominated solutions of the considered nondifferentiable fuzzy optimization problem has been given. Also the convergence results have been obtained for the algorithm presented in this work. Moreover, the simulation of the choice of the initial penalty parameter in the aforesaid algorithm has been performed. Hence, it can be concluded here that the l_1 exact fuzzy penalty function method, originally designed for deterministic constrained extremum problems, can also be applied for solving fuzzy optimization problems.

Although we have focused on solving scalar fuzzy extremum problems with fuzzy objective functions and inequality constraints by applying the absolute value exact penalty fuzzy function, however, we believe that the established results are also applicable for such not well-defined operations research problems which are modeled by nondeterministic optimization problems of other types. Therefore, there remain some interesting questions for further research. Namely, it would be interesting to investigate whether it is possible to prove analogous results for various types of fuzzy extremum problems. This question will be investigated in our subsequent works.

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