Analytical Approaches for Computing Exact Solutions to System of Volterra Integro-Differential Equations

NIDAL ANAKIRA[1,2,*], ADEL ALMALKI[3], M. J. MOHAMMED[4], SAFWAT HAMAD[4], OSAMA OQILAT[5], ALA AMOURAH[1]

[1]Department of Mathematics, Faculty of Education and Arts,
Sohar University,Sohar 3111,
SULTANATE OF OMAN

[2]Jadara University Research Center,
Jadara University,
JORDAN

[3] Department of Mathematics, Al-Qunfudhah University College,
Umm Al-Qura University,
Mecca,
SAUDI ARBIA

[4]Department of Mathematics, College of Science,
University of Anbar Ramadi,
IRAQ

[5]Department of Basic Sciences, Faculty of Arts and Science,
Al-Ahliyya Amman University,
Amman 19328,
JORDAN

Corresponding Author

Abstract: - This paper presents a modified technique that utilizes the homotopy-perturbation method (HPM) to solve a system of integro-differential equation of Volterra kind. By providing practical examples and conducting numerical simulations, we showcase the effectiveness and efficiency of this modification in solving these systems encountered in various scientific fields. Furthermore, we compare the performance of the HPM with the exact solution, emphasizing its advantages in terms of accuracy, convergence, and computational efficiency.

Key-Words: - Volterra integro-differential equations; Numerical approximation; HPM; Series solution; Laplace transformation; Padé approximants.

Received: August 29, 2023. Revised: April 11, 2024. Accepted: May 8, 2024. Published: June 27, 2024.

1 Introduction

Integro-differential equations are commonly encountered in engineering and scientific disciplines. Scientific and engineering disciplines frequently use integral-differential equations. A wide range of physical phenomena, including wind ripples in deserts, dropwise consideration in nonhydrodynamics, and the formation of glass, are explained by them, [1], [2] and [3]. In addition, many equations equation have important applications in theoretical physics and as mathematical representations of viscoelasticity. Numerical solutions are essential for comprehending complicated dynamical systems in physics, biology, and economics, among other disciplines.Volterra integro-differential equations, which incorporate integrals and derivatives, describe these systems. These equations are difficult to find analytical solutions for, which is why approximating their solutions numerically is a common practice.

Different numerical techniques have been developed by many scholars to solve systems of linear integro-differential equations. Numerous scholars have devised numerous numerical techniques to resolve linear integro-differential equation systems. Specifically, systems of Volterra integro-differential equations are solved using the reconstruction of the variational iteration approach, [4]. This method produces results that are more accurate than the homotopy perturbation method, according to comparisons between the two. Researchers in their work, [5], used the homotopy perturbation method to estimate the solution of Volterra integro-differential equation systems. The optimal homotopy asymptotic method was employed to find the solutions of a system of Volterra integro-differential equations, [6].

Moreover, authors in their research, [7], used the Sinc collocation and Chebyshev wavelet methods to solve linear Volterra integro-differential equation systems. Chebyshev polynomials [8], the single-term Walsh series technique, [9], the differential transform method, [10], the power series method, [11], the homotopy perturbation method, [12], the homotopy analysis method, [13], and the modified Adomian decomposition method, [14], are other numerical techniques that are frequently used to solve such systems. However, the presence of integrals requires special attention to efficiently handle the integral terms. Additionally, in general, the stability and convergence of numerical schemes, [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], are crucial factors in obtaining accurate results.

In this work, we introduce a new improvement to the solution of the HPM procedure, specifically for solving systems of Volterra integro-differential equations. Our method applies to any given problem, offering accurate approximate solutions that approach the exact solution as the number of approximation terms increases. It is important to note that the accuracy of our method depends on the order of the approximation used, which may require additional computational effort and time, especially for nonlinear problems. Therefore, researchers are continuously striving to develop or modify numerical techniques to achieve higher accuracy or exact solutions.

The main objective of this paper is to enhance the accuracy of the HPM by employing an alternative approach. This approach involves modifying the series solution of the HPM by applying the Laplace transformation to the truncated HPM solution. Subsequently, the transformed series is converted into a meromorphic function using Padé approximants. Finally, we apply the inverse Laplace transformation to obtain the desired solution for the given problem. This method is straightforward and yields precise results with high performance, without requiring significant effort. The structure of this paper is organized as: Section 2 introduces the fundamental concept of the HPM, along with a brief explanation of the Pade approximants. In Section 3, numerical examples are presented to demonstrate the effectiveness of the discussed procedure in obtaining the analytic solution of systems of Volterra integro differential equations. The results highlight that accurate solutions can be obtained with only a few terms. The final section summarizes the conclusions of this work.

2 Fundamental Idea of HPM Procedure

To demonstrate the fundamental concept of the HPM procedure, [27], [28], [29] and [30], consider:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \qquad (1)$$

Given that A is a general integral operator, B is a boundary operator, $f(r)$is a known analytic function, and Γ is the boundary of the domain Ω. The operator A can be divided into two parts: L is linear, while N is nonlinear. Thus, the Eq. (1) can be rewritten as follows:

$$L(u) - N(u) - f(r) = 0. \qquad (2)$$

Now, we construct $v : \Omega[0, 1] \rightarrow$R which satisfies

$$H(v;\, p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0. \qquad (3)$$

or

$$H(v;\, p) = (1 - p)[L(v) - L(v_0)] + p[A(v_0) - F(r)] = 0. \qquad (4)$$

where $r \epsilon \Omega$, $p \epsilon [0, 1]$ that is parameter, and $v_0(x)$is an initial approximation of Eq. (1). Hence

$$(v;\, 0) = L(u) - L(v_0) = 0, \quad H(v;\, 1) = A(v) - F(r) = 0. \qquad (5)$$

and the process of changing p from to 0 to 1, and $H(v;\, p)$ from $L(u) - L(v_0)$ to $A(v) - F(r)$ which is called deformation in topology, this, where $L(u) - L(v_0)$ and $A(v) - F(r)$ are called homotopic. Since $0 \leq p \leq 1$, considered a small parameter, we assume the solution of Eqs. (4) or (5) expressed in the following:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \tag{6}$$

when $p \to 1$, Eq. (6) becomes the approximate solution of Eq. (1). i.e., $u(x) = \lim_{p\to 1} v(x) = v_0 + v_1 + v_2 + v_3 + \ldots$ (7)

3 Padè Approximation

The function $u(x)$ is defined by the Padé approximation, [25], [31], [32], [33] and [34].

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)},$$

where the highest degree polynomial for L is $P_L(x)$ and the highest degree polynomial for M is $Q_M(x)$. Regarding the formal power series

$$u(x) = \sum_{i=1}^{\infty} a_i x^i.$$

We can find the coefficients of the polynomials by using the following equation:
$P_L(x)$ and $Q_M(x)$.

$$u(t) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}). \tag{8}$$

When the denominator and numerator's functions $\frac{P_L(x)}{Q_M(x)}$ is multiplied by a constant that is not zero, the fractional values stay the same, such that we can set up the normalization requirement as:

$$\mathrm{Q_M}(0) = 1. \tag{9}$$

It should be noted that the polynomial for functions $P_L(x)$ and $Q_M(x)$ has no public factors. If the coefficients of the polynomial functions $Q_M(x)$ and $P_L(x)$ are expressed as:

$$\begin{gathered} P_L(t) = P_0 + P_1 t + P_2 t^2 + \cdots + P_L t^L \\ Q_M(t) = q_0 + q_1 t + q_2 t^2 + \cdots + q_M t^M \end{gathered} \tag{10}$$

We can derive the following linear systems of coefficients by multiplying Eq. (8) by $Q_M(x)$ to be:

$$\left\{\begin{array}{c} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0 \\ \cdot \\ \cdot \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{array}\right\}, \tag{11}$$

$$\left\{\begin{array}{c} a_0 = P_0 \\ a_1 + a_0 q_1 = P_1 \\ a_2 + a_1 q_1 + a_0 q_2 = P_2 \\ \cdot \\ \cdot \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = P_L \end{array}\right\}, \tag{12}$$

These equations will be solved using Eq. (11), which represents a set of linear formulas for the unidentified variables. Once the $q's$ are identified, we can derive an explicit formula for the unknown p's, which will provide the solution to the problem.

$$\left[\frac{L}{M}\right] = \frac{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_L & a_{L+1} & \cdot & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} X^j & \sum_{j=M-1}^{L} a_{j-M+1} X^j & \cdots & \sum_{j=0}^{L} a_j X^j \end{bmatrix}}{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ X^M & X^{M-1} & \cdots & 1 \end{bmatrix}}, \tag{13}$$

4 Applications of HPM

The purpose is to demonstrate the effectiveness and reliability of our modified procedure.

Example **1**. Given the system of integro-differential equations of Volterra type, [34], [35], [36], and [37].

$$u_1'(t) = 1 + t + t^2 - u_2(t) - \int_0^t \big(u_1(x) + u_2(x)\big)dx,$$

$$u_2'(t) = 1 - t + u_1(t) - \int_0^t \big(u_1(x) - u_2(x)\big)dx, \tag{14}$$

Subject to $u_1(0) = 1, u_2(0) = -1$, and exact solutions $u = (u_1(t), u_2(t)) = (t + e^t, t - e^t)$. Based on the algorithm presented in Section 2, we will now proceed to construct the following homotopy equation.

$$(1-q)[\tfrac{dv(t;p)}{dt} = (h;q)\left[\frac{dv(t;p)}{dt} - 1 - t - t^2 + v_2(t;p) + \int_0^t (u_1(x) + u_2(x))dx\right],$$

$$(1-p)[\tfrac{dv(t;p)}{dt} =$$

$$(h;q)\left[\frac{dv(t;p)}{dt}+1+t-v_1(t;p)+\int_0^t(u_1(x)-u_2(x))dx\right], \quad (15)$$

The zeroth-order problem is expressed in Eqs. (16), as follows:

$$u'_{1,0}(x)=1,$$
$$u'_{2,0}(x)=-1, \quad (16)$$

which has the solution

$$u_{1,1}(x)=1,$$
$$u_{2,1}(x)=-1, \quad (17)$$

Based on Eqs. (15) the first-order problem is given in the form of

$$u'_{1,1}(t)=2+t+t^2,$$
$$u'_{2,1}(t)=-3t, \quad (18)$$

under the conditions $u_1(0)=0,\ u_2(0)=0$. Therefore, it has the following solution.

$$u_{1,1}(t)=2t+\frac{t^2}{2}+\frac{t^3}{3}$$
$$\mathrm{u}_{2,1}(\mathrm{t})=-\frac{3\mathrm{t}^2}{2} \quad (19)$$

The following formula defines the second-order problem.

$$u'_{1,2}(t)=\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{12},$$
$$u'_{2,2}(t)=2t-\frac{t^2}{2}-\frac{t^3}{3}-\frac{t^4}{12}, \quad (20)$$

using the conditions $u_1(0)=0,\ u_2(0)=0$. We have the following solution

$$u_{1,2}(t)=\frac{t^3}{6}+\frac{t^4}{12}-\frac{t^5}{60},$$
$$u_{2,2}(t)=t^2-\frac{t^3}{6}-\frac{t^4}{12}-\frac{t^5}{60}, \quad (21)$$

Based on the HPM procedure, we have the 5th -order HPM approximate solution

$$\tilde{u}_1(t)=1+2t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{2520}-\frac{t^9}{90720}+\frac{t^{10}}{907200}+\frac{t^{11}}{4989600}, \quad (22)$$

$$\tilde{u}_2(t)=-1-\frac{t^2}{2}-\frac{t^3}{6}-\frac{t^4}{24}-\frac{t^5}{120}-\frac{t^6}{240}+\frac{t^8}{5040}-\frac{t^9}{90720}-\frac{t^{10}}{302400}, \quad (23)$$

This leads to $u=(u_1(t),u_2(t))=(t+e^t,t-e^t)$. as $\lim_{n\to\infty}\tilde{u}_n(t)$. Table 1 and Table 2 depicts numerical results for HPM Procedure and the exact one. The HPM is well-known for its simplicity and versatility. It allows the calculation of approximate solutions to differential equations without requiring linearization or restrictive assumptions. To enhance the accuracy of the HPM solution, we will use the MHPM that builds upon the capabilities of the HPM by addressing its limitations in terms of the number of terms, convergence rate, and computational complexity. We achieve this by employing Pade approximation, Laplace transformation, and ultimately the inverse Laplace transformation, as described below:

$$L(\tilde{u}_1(t))=\frac{1}{s^7}+\frac{1}{s^6}+\frac{1}{s^5}+\frac{1}{s^4}+\frac{1}{s^3}+\frac{2}{s^2}+\frac{1}{s},$$
$$L(\tilde{u}_2(t))=-\frac{3}{s^7}-\frac{1}{s^6}-\frac{1}{s^5}-\frac{1}{s^4}-\frac{1}{s^3}-\frac{1}{s}. \quad (24)$$

Use $s=\frac{1}{x}$, leads to

$$L(\tilde{u}_1(t))=z+2z^2+z^3+z^4+z^5+z^6+z^7,$$
$$L(\tilde{u}_2(t))=-z-z^3-z^4-z^5-z^6-3z^7. \quad (25)$$

The Pade approximates of order $\left[\frac{3}{3}\right]$ in term of $x=\frac{1}{s}$, gives

$$\left[\frac{3}{3}\right]=-\frac{1}{(1-\frac{1}{s})s^3}+\frac{1}{(1-\frac{1}{s})s^2}+\frac{1}{(1-\frac{1}{s})s}.$$
$$\left[\frac{3}{3}\right]=-\frac{1}{\left(1-\frac{1}{s}\right)s^3}+\frac{1}{\left(1-\frac{1}{s}\right)s^2}-\frac{1}{\left(1-\frac{1}{s}\right)s}, \quad (26)$$

The modified approximation solution $u=(u_1(t),u_2(t))=(t+e^t,t-e^t)$., is obtained by applying the inverse Laplace transform to the $\left[\frac{3}{3}\right]$ Pade approximate.

Example 4. 2 Considering the system of differential equations of Volterra integro type, [38], [39] and [40].

$$u''_1(t)=-sint-t^2-1+\int_0^t\big(u_1(x)+u_2(x)\big)dx,$$
$$u''_2(t)=1-cost-2sint+\int_0^t\big(u_1(x)-u_2(x)\big)dx, \quad (27)$$

Subject to $u_1(0)=u'_1(0)=1, u_2(0)=0$, $u'_2(0)=2$, and exact solutions
$$u=(u_1(t),u_2(t))=(t+cost,t+sint). \quad (28)$$

Based on the algorithm presented in Section 2, we will now proceed to construct the following homotopy equations.

$$(1-p)[\frac{dv(t;p)}{dt} = (h;q)\left[\frac{dv^2(t;p)}{dt^2} + Sint + t^2 + 1 - \int_0^t (u_1(x)+u_2(x))dx\right],$$

$$(1-p)[\frac{dv(t;p)}{dt} = (h;q)\left[\frac{dv^2(t;p)}{dt^2} + cost + 2sint - 1 - \int_0^t (u_1(x)-u_2(x))dx\right], \tag{29}$$

Following the same process in example one, we have the 5th -order HPM approximate solution

$$\tilde{u}_1(t) = 1 + t - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \frac{t^8}{40320} - \frac{t^{10}}{3628800} + \frac{t^{11}}{39916800} + \frac{t^{12}}{479001600} - \frac{t^{13}}{6227020800} + \frac{t^{14}}{43589145600} - \frac{t^{16}}{10461394944000} + \frac{t^{17}}{59281238016000} + \frac{t^{18}}{1067062284288000} - \frac{t^{19}}{20274183401472000} + \frac{t^{21}}{12772735542927360000}, \tag{30}$$

$$\tilde{u}_2(t) = 2t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} + \frac{t^{11}}{39916800} - \frac{t^{12}}{479001600} - \frac{t^{13}}{6227020800} + \frac{t^{15}}{653837184000} + \frac{t^{17}}{59281238016000} - \frac{t^{18}}{1067062284288000} - \frac{t^{19}}{20274183401472000} + \frac{t^{20}}{608225502044160000} + \frac{t^{21}}{6386367771463680000} - \frac{t^{22}}{28100018194401920000}. \tag{31}$$

This leads to the exact solution $u = (u_1(t), u_2(t)) = (t + cost, t + sint)$, as $\lim_{n\to\infty} \tilde{u}_i(t), i = 1,2.$

Table 3 and Table 4 depicts numerical results for HPM procedure and the exact solutions. We observed that accuracy depends on the order of the approximations. To obtain more accurate results, we will modify the HPM solutions. We will achieve this by employing the Laplace transformation on the initial terms of the HPM series solutions, using the Pade approximants, and finally applying the inverse Laplace transformation as depicted below.

$$L(\tilde{u}_1(t)) = \frac{1}{s^9} - \frac{1}{s^7} + \frac{1}{s^5} - \frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{s},$$
$$L(\tilde{u}_2(t)) \frac{1}{s^{12}} + \frac{1}{s^{10}} - \frac{1}{s^8} + \frac{1}{s^6} - \frac{1}{s^4} + \frac{2}{s^2}, \tag{32}$$

Use $s = \frac{1}{z}$, leads to

$$L(\tilde{u}_1(t)) = z + z^2 - z^3 + z^5 - z^7 + z^9,$$
$$L(\tilde{u}_2(t)) 2z^2 - z^4 + z^6 - z^8 + z^{10} + z^{12}, \tag{33}$$

Using of $t = \frac{1}{s}$, Then, Pade approximates of order $\left[\frac{4}{4}\right]$, yield to

$$\left[\frac{4}{4}\right] = \frac{1}{(1+\frac{1}{s^2})s^4} + \frac{1}{(1+\frac{1}{s^2})s^2} + \frac{1}{(1+\frac{1}{s^2})s},$$
$$\left[\frac{4}{4}\right] = \frac{1}{(1+\frac{1}{s^2})s^4} + \frac{2}{(1+\frac{1}{s^2})s^2}, \tag{34}$$

The exact solutions $u = (u_1(t), u_2(t)) = (\mathrm{t} + \cos\mathrm{t}, \mathrm{t} + \sin\mathrm{t})$., are obtained by applying the inverse Laplace transform to the $\left[\frac{4}{4}\right]$ Pade approximate.

Table 1. Numerical result of example 4.1

x	Exact Solution $u_1(t) = t + e^t$	Approximate Solution	HPM Absolute Error
0.0	1.0	1.0	0.0
0.2	1.4214027582	1.4214027606	2.47×10^{-9}
0.4	1.8918246976	1.8918250029	3.05×10^{-7}
0.6	2.4221188004	2.4221238049	5.00×10^{-6}
0.8	3.0255409285	3.0255766320	3.57×10^{-5}
1.0	3.7182818285	3.7184426607	1.61×10^{-4}

Table 2. Numerical result of example 4.1

x	Exact Solution $u_1(t) = t - e^t$	Approximate Solution	HPM Absolute Error
0.0	−1.000000000	−1.000000000	0.0
0.2	−1.0214027582	−1.0214029328	1.75×10^{-7}
0.4	−1.0918246976	−1.0918356065	1.09×10^{-5}
0.6	−1.2221188003	−1.2222391985	1.20×10^{-4}
0.8	−1.4255409285	−1.4261914798	6.51×10^{-4}
1.0	−1.7182818284	−1.7206492504	2.37×10^{-3}

Table 3. Numerical result of example 4.2

x	Exact Solution $u_1(t) = t + cost$	Approximate Solution	HPM Absolute Error
0.0	1.0	1.0	0.0
0.2	1.1800665778	1.1800665778	4.44×10^{-16}
0.4	1.3210609940	1.3210609940	1.05×10^{-12}
0.6	1.4253356150	1.42533561490	9.07×10^{-11}
0.8	1.4967067115	1.49670670935	2.14×10^{-9}
1.0	1.5403023307	1.54030230587	2.49×10^{-9}

Table 4. Numerical result of example 4.2

x	Exact Solution $u_1(t) = t + sint$	Approximate Solution	HPM Absolute Error
0.0	1.0	1.0	0.0
0.2	0.3986693308	0.3986693308	9.99×10^{-16}
0.4	0.7894183423	0.7894183423	2.06×10^{-12}
0.6	1.1646424734	1.1646424736	1.77×10^{-10}
0.8	1.5173560909	1.5173560950	4.14×10^{-9}
1.0	1.8414709848	1.8414710325	4.77×10^{-8}

5 Conclusion

In this research study, we propose a new procedure based on the HPM for solving a system of Volterra integro-differential equations. This procedure is not only effective and reliable, but it also offers a distinct advantage over other methods. Its ability to provide accurate solutions for challenging systems highlights its potential as a valuable tool for researchers and practitioners seeking to understand and analyze dynamic phenomena governed by these systems. Through illustrative examples and comparisons with numerical results reported in the literature, we observed that this procedure can achieve the exact analytical solution by utilizing only a few terms of the truncated series solution derived from the HPM solutions. Consequently, we conclude that this procedure represents a potent approach and a promising tool for resolving not only this particular class of differential equations but also various other types of differential equations.

References:

[1] Aminikhah H. A new analytical method for solving systems of linear integro-differential equations. *Journal of King Saud University-Science*. 2011 Oct 1;23(4):349-53, https://doi.org/10.1016/j.jksus.2010.07.016.

[2] Eslahchi, M. R., Mehdi Dehghan, and Maryam Parvizi. "Application of the collocation method for solving nonlinear fractional integro-differential equations." *Journal of Computational and Applied Mathematics*, 257 (2014):105-128, https://doi.org/10.1016/j.cam.2013.07.044.

[3] Wazwaz, Abdul-Majid. "The Volterra integro-differential forms of the singular Flierl-Petviashvili and the Lane-Emden equations with boundary conditions." *Rom. J. Phys*., 58, no. 7, 2013, pp. 1-9.

[4] Hesameddini, Esmail, and Azam Rahimi. "A new numerical scheme for solving systems of integro-differential equations." *Computational Methods for Differential Equations*, 1, no. 2 (2013): 108-119.

[5] Biazar, J., H. Ghazvini, and M. Eslami. "He's homotopy perturbation method for systems of integro-differential equations." *Chaos, Solitons & Fractals*, 39, no. 3 (2009): 1253-1258, https://doi.org/10.1016/j.chaos.2007.06.001.

[6] Jameel, Ali, N. R. Anakira, A. K. Alomari, and Noraziah H. Man. "Solution and analysis of the fuzzy Volterra integral equations via homotopy analysis method." *Computer Modeling in Engineering & Sciences*, 127, no. 3 (2021): 875-899, https://doi.org/10.32604/cmes.2021.014460.

[7] Issa, Ahmad, Naji Qatanani, and Adnan Daraghmeh. "Approximation Techniques for Solving Linear Systems of Volterra Integro-Differential Equations." *Journal of Applied Mathematics*, 2020, no. 1 (2020): 1-13, https://doi.org/10.1155/2020/2360487.

[8] Aminikhah, Hossein, and Sahar Hosseini. "Numerical solution of linear system of integro-differential equations by using Chebyshev wavelet method." *Mathematical Sciences Letters*, 4, no. 1 (2015): 45-50,

[9] Sekar, R. Chandra Guru, and K. Murugesan. "System of linear second order Volterra integro-differential equations using Single Term Walsh Series technique." *Applied Mathematics and Computation*, 273 (2016): 484-492, https://doi.org/10.1016/j.amc.2015.09.092.

[10] Behiry SH. Nonlinear integro-differential equations by differential transform method with Adomian polynomials. Australian *Journal of Basic and Applied Sciences*. 2013;7(7):128-39.

[11] Gachpazan M. Numerical Scheme to Solve Integro-Differentia Equations Systeml. *Journal of Advanced Research in Scientific Computing*. 2009 Jul 12;1., 11-21.

[12] Raftari B. Numerical solutions of the linear Volterra integro-differential equations: Homotopy perturbation method and finite difference method. *World Applied Sciences Journal*. 2010;9(07-12).

[13] Hetmaniok, Edyta, Iwona Nowak, Damian Słota, and Roman Wituła. "Convergence and error estimation of homotopy analysis method for some type of nonlinear and linear integral equations." *Journal of Numerical Mathematics*, 23, no. 4 (2015): 331-344, https://doi.org/10.1515/jnma-2015-0022.

[14] Ibrahim, Rabha W., M. Zaini Ahmad, and M. Jasim Mohammed. "Periodicity of a time-delay system of fractional order joining n-deviating arguments." *Systems Science & Control Engineering* 4, no. 1 (2016): 209-214, https://doi.org/10.1080/21642583.2016.1228488.

[15] Zhang X, Tang B, He Y. Homotopy analysis method for higher-order fractional integro-differential equations. *Computers & Mathematics with Applications*. 2011 Oct 1;62(8):3194-203, https://doi.org/10.1016/j.camwa.2011.08.032.

[16] Özkan O. Numerical implementation of differential transformations method for integro-differential equations. *International Journal of Computer Mathematics*. 2010 Oct 1, 87 (12):2786-97, https://doi.org/10.1080/00207160902795627.

[17] Mohamed, Mohamed S., Khaled A. Gepreel, Muteb R. Alharthi, and Refah A. Alotabi. "Homotopy analysis transform method for integro-differential equations." *General Mathematics Notes*, 32, no. 1 (2016): 32,

[18] Ibrahim, Rabha W., M. Z. Ahmad, and M. Jasim Mohammed. "Symmetric-periodic solutions for some types of generalized neutral equations." *Mathematical Sciences*, 10 (2016): 219-226, https://doi.org/10.1007/s40096-016-0196-7.

[19] Jameel, Ali, N. R. Anakira, A. K. Alomari, Ishak Hashim, and M. A. Shakhatreh. "Numerical solution of n'th order fuzzy initial value problems by six stages." *Journal of Nonlinear Science Applications*, 9, no. 2 (2016): 627-640.

[20] Rao KN, Chakraverty S. Nonlinear fractional integro-differential equations by using the homotopy perturbation method. In *Computation and Modeling for Fractional Order Systems*, 2024 Jan 1 (pp. 103-111). Academic Press, https://doi.org/10.1016/B978-0-44-315404-1.00011-4.

[21] Jameel, A. F., Nidal Anakira, A. K. Alomari, Ishak Hashim, and Shaher Momani. "A new approximation method for solving fuzzy heat equations." *Journal of Computational and Theoretical Nanoscience*, 13, no. 11 (2016): 7825-7832, https://doi.org/10.1166/jctn.2016.5784.

[22] Jameel, A. F., N. R. Anakira, M. M. Rashidi, A. K. Alomari, A. Saaban, and M. A. Shakhatreh. "Differential Transformation Method for Solving High Order Fuzzy Initial Value Problems." *Italian Journal of Pure and Applied Mathematics*, 39 (2018): 194-208.

[23] He JH. Homotopy perturbation method: a new nonlinear analytical technique. *Applied Mathematics and computation*. 2003 Feb 15;135(1):73-9,

[24] Mirzaee F, Alipour S. An efficient cubic B-spline and bicubic B-spline collocation method for numerical solutions of multidimensional nonlinear stochastic quadratic integral equations. *Mathematical Methods in the Applied Sciences*. 2020 Jan 15;43(1):384-97, https://doi.org/10.1002/mma.5890.

[25] Mirzaee F, Alipour S. Bicubic B-spline functions to solve linear two-dimensional weakly singular stochastic integral equation. *Iranian Journal of Science and Technology, Transactions A: Science*. 2021 Jun;45(3):965-72,

[26] Ramm AG. A collocation method for solving integral equations. *International Journal of Computing Science and Mathematics*. 2009 Jan 1;2(3):222-228.

[27] He, J. H. (1999). Homotopy perturbation technique. Computer methods Mathematical and computer Modelling, 2008 in *Applied Mechanics and Engineering*, 178(3-4), 257-262,

[28] Shakeri F, Dehghan M. Solution of delay differential equations via a homotopy perturbation method. *Mathematical and computer Modelling*. 2008 Aug 1;48(3-4):486-98, https://doi.org/10.1016/j.mcm.2007.09.016.

[29] Rabbani M. New homotopy perturbation method to solve non-linear problems. *J. Math. Comput. Sci.*, 2013;7(1):272-5.

[30] Ganji DD. The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer. *Physics letters A.*, 2006 Jul 10;355(4-5):337-41, https://doi.org/10.1016/j.physleta.2006.02.056

[31] Al-Ahmad S, Sulaiman IM, Nawi MM, Mamat M, Ahmad MZ. Analytical solution of systems of Volterra integro-differential equations using modified differential transform method. *J. Math. Comput. Sci.*, 2022;26:1-9, doi: 10.22436/jmcs.026.01.01.

[32] Al-Ahmad S, Sulaiman IM, Mamat M, Kamfa K. Solutions of classes of differential equations using modified differential transform method. *J. Math. Comput. Sci.*, 2020 Jul 9;10(6):2360-82,

https://doi.org/10.28919/jmcs/4855.

[33] Baker, G.A., *Essentials of Padé approximants*. 1975: Academic Press.

[34] Arikoglu A, Ozkol I. Solutions of integral and integro-differential equation systems by using differential transform method. *Computers & Mathematics with Applications*. 2008 Nov 1;56(9):2411-7, https://doi.org/10.1016/j.camwa.2008.05.017.

[35] Berenguer MI, Garralda-Guillem AI, Galán MR. An approximation method for solving systems of Volterra integro-differential equations. *Applied Numerical Mathematics*. 2013 May 1;67:126-35, https://doi.org/10.1016/j.apnum.2011.03.007.

[36] Mirzaee F, Bimesl S. A new Euler matrix method for solving systems of linear Volterra integral equations with variable coefficients. *Journal of the Egyptian Mathematical Society*. 2014 Jul 1;22(2):238-48, https://doi.org/10.1016/j.joems.2013.06.016.

[37] Biazar J, Ghazvini H. He's homotopy perturbation method for solving systems of Volterra integral equations of the second kind. *Chaos, Solitons & Fractals*. 2009 Jan 30;39(2):770-777, https://doi.org/10.1016/j.chaos.2007.01.108.

[38] Agarwal P, Akbar M, Nawaz R, Jleli M. Solutions of system of Volterra integro-differential equations using optimal homotopy asymptotic method. *Mathematical Methods in the Applied Sciences*. 2021 Feb;44(3):2671-81, https://doi.org/10.1002/mma.6783.

[39] Dehghan M, Shakourifar M, Hamidi A. The solution of linear and nonlinear systems of Volterra functional equations using Adomian–Pade technique. *Chaos, Solitons & Fractals*. 2009 Mar 15;39(5):2509-21, https://doi.org/10.1016/j.chaos.2007.07.028.

[40] Behzadi SS. The use of iterative methods to solve two-dimensional nonlinear Volterra-Fredholm integro-differential equations. *Communications in Numerical Analysis*. 2012; 2012:1-20.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study

Conflict of Interest

The author has no conflicts of interest to declare.