A study of fuzzy prime near-rings involving fuzzy semigroup ideals

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Abstract: In this paper, our main objective is to introduce the notion of a fuzzy semigroup ideal by using Yuan and Lee's definition of fuzzy group based on fuzzy binary operations. Also, some of its basic properties are studied analogously to the known results in the case of semigroup ideals defined in the framework of ordinary near-rings.

Key-Words: Fuzzy group, Fuzzy near-rings, Semigroup ideals, Prime near-rings, Binary operations, Commutativity.

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1 Introduction

In [1], gave the definition of the fuzzy binary operation using the notion of a fuzzy subset of a fuzzy set introduced by Zadeh in his famous paper, [2], published in 1965. Taking advantage of this definition, extensive work has been published by several researchers (see, [3], [4], [5], for further references). In 2004, a new definition of fuzzy group was created by the two researchers, [6], they also presented the notion of commutativity of a fuzzy group and some of its basic principles. After these studies, in [7], created the concept of fuzzy ring based on the definition of [6], of a fuzzy group, and they obtained interesting results on this subject. Motivated by the classical theory of near-rings, we refer the reader to [8], and the work of [7], in which the two operations $*$ and $\circ$ are two mappings constructed from the fuzzy binary operations T and L as given in the section of preliminary, we succeed to define the notion of fuzzy near-ring as follows:

Definition 1. *For any nonempty set X with two fuzzy binary operations T and L is said fuzzy left near-ring if the following assertions hold:*

i) *(X, T) is a fuzzy group not necessarily commutative,*

ii) *$\forall a, b, c, x_1, x_2 \in X$, we have $(a*(b*c))(x_1) > \theta$ and $((a*b)*c)(x_2) > \theta \implies x_1 = x_2$,*

iii) *$\forall a, b, c, x_1, x_2 \in X$, we have $(a*(b\circ c))(x_1) > \theta$ and $((a*b)\circ(a*c))(x_2) > \theta \implies x_1 = x_2$,*

where $\theta \in [0, 1)$ is a fixed number. Further, if we replace the last condition by:

iv) *$\forall a, b, c, x_1, x_2 \in X$, $(b \circ c) * a)(x_1) > \theta$ and $((b * a) \circ (c * a))(x_2) > \theta \implies x_1 = x_2$,*

then (X, T, L) is called right fuzzy near-ring. And (X, T, L) is said to be a fuzzy ring, when (X, T, L) is a left and right fuzzy near-ring and (X, T) is abelain. Moreover, according to, [7, Definition 9], *(X, T, L) is said to be a commutative fuzzy ring if $(a * b)(u) > \theta \Leftrightarrow (b * a)(u) > \theta$ for all $a, b \in X$.*

Noting that (X, T, L) is called a prime fuzzy near-ring, if it has the property that $((x * y) * z)(e) > \theta$ for all $x, y, z \in X$ implies that $x = e$ or $z = e$. Also, $Z_F(X) = \{x \in X \,/\, L(x, y, z) > \theta \iff L(y, x, z) > \theta,\ \forall y, z \in X\}$ denote the fuzzy multiplicative center of X.

2 Preliminary results

In this section, we will formulate some basic definitions and results that will be essential for the rest of this paper.

Definition 2. [9, Definition 2.1] *Let X be a nonempty set and T be a fuzzy subset of $X \times X \times X$ and $\theta \in [0, 1)$ is a fixed number. T is called a*

fuzzy binary operation on X if the following conditions hold:
(C_1) $\forall\, x, y \in X, \exists\, z \in X$ *such that* $T(x, y, z) > \theta$.
(C_2) $\forall\, x, y, t_1, t_2 \in X, T(x, y, t_1) > \theta$ *and* $T(x, y, t_2) > \theta$ *implies* $t_1 = t_2$.

Let T and L be two fuzzy binary operations on X, then we can defined the following mappings:

$$\begin{array}{rcl} \circ : \mathbb{F}(X) \times \mathbb{F}(X) & \longrightarrow & \mathbb{F}(X) \qquad \textit{and} \\ (\mu, v) & \longmapsto & \mu \circ v \\ * : \mathbb{F}(X) \times \mathbb{F}(X) & \longrightarrow & \mathbb{F}(X) \\ (\mu, v) & \longmapsto & \mu * v \end{array},$$

where $\mathbb{F}(X) = \{\mu \,|\, \mu : X \longrightarrow [0, 1]\}$, and for all $\mu, v \in \mathbb{F}(X)$, we have

$$\begin{cases} (\mu * v)(z) = \bigvee_{x,y \in X} \Big(\mu(x) \wedge v(y) \wedge L(x, y, z)\Big), \\ (\mu \circ v)(z) = \bigvee_{x,y \in X} \Big(\mu(x) \wedge v(y) \wedge T(x, y, z)\Big). \end{cases}$$

Let $x, y \in X$, $\mu = \{x\}$ and $v = \{y\}$, and let $\mu \circ v$ and $\mu * v$ be denoted by $x \circ y$ and $x * y$, respectively. Then, we have for all $z, t \in X$

$$(x \circ y)(z) = T(x, y, z), \tag{1}$$

$$(x * y)(z) = L(x, y, z), \tag{2}$$

$$((x \circ y) \circ z)(t) = \bigvee_{h \in X} \Big(T(x, y, h) \wedge T(h, z, t)\Big), \tag{3}$$

$$(x \circ (y \circ z))(t) = \bigvee_{h \in X} \Big(T(y, z, h) \wedge T(x, h, t)\Big), \tag{4}$$

$$((x * y) * z)(t) = \bigvee_{h \in X} \Big(L(x, y, h) \wedge L(h, z, t)\Big), \tag{5}$$

$$(x * (y * z))(t) = \bigvee_{h \in X} \Big(L(y, z, h) \wedge L(x, h, t)\Big), \tag{6}$$

$$(x * (y \circ z))(t) = \bigvee_{h \in X} \Big(T(y, z, h) \wedge L(x, h, t)\Big), \tag{7}$$

$$((x * y) \circ (x * z))(t) = \bigvee_{d,h \in X} \Big(L(x, y, d) \wedge L(x, z, h) \wedge T(d, h, t)\Big). \tag{8}$$

Definition 3. [9, Definition 2.2] *Let X be a nonempty set and T a fuzzy binary operation on X. Then (X, T) is called a fuzzy group if the following conditions hold:*

1. *$\forall a, b, c, c_1, c_2 \in X$, $((a \circ b) \circ c)(c_1) > \theta$ and $(a \circ (b \circ c))(c_2) > \theta \implies c_1 = c_2$.*
2. *$\exists\, e \in X$ such that for all $x \in X$, $(e \circ x)(x) > \theta$ and $(x \circ e)(x) > \theta$. e is called the identity element of (X, T).*
3. *$\forall x \in X$, $\exists\, y \in X$ such that $(x \ o \ y)(e) > \theta$ and $(y \ o \ x)(e) > \theta$. y is called the inverse element of x and denoted by x^{-1}.*

Lemma 1. [6, Proposition 2.1] *Let (X, T) be a fuzzy group, then*

1) $(x \circ y)(a) > \theta$ *and* $(x \circ z)(a) > \theta \implies y = z$;
2) $(a \circ x)(y) > \theta$ *and* $(b \circ x)(y) > \theta \implies a = b$;
3) $(a \circ b)(c) > \theta$ *and* $(b^{-1} \circ a^{-1})(d) > \theta \implies d = c^{-1}$;
4) $(a \circ a)(a) > \theta \implies a = e$;
5) $(a^{-1})^{-1} = a$.

Definition 4. [7, Definition 6] *Let (X, T) be a fuzzy group. (X, T) is called abelian fuzzy group if we have, for all $x, y, z \in X$,*

$$T(x, y, z) > \theta \iff T(y, x, z) > \theta.$$

Lemma 2. [9, Theorem 3.1 & Theorem 3.3]
1- Let (X, T, L) be a left fuzzy near-ring, then

$Z_F(T) =$
$\{x \in X \mid \forall y \in X, ((x * y) \circ (y * x^{-1}))(e) > \theta\}$.

2- Let (X, T, L) be a right fuzzy near-ring, then

$Z_F(T) =$
$\{x \in X \mid \forall y \in X, ((x * y) \circ (y^{-1} * x))(e) > \theta\}$.

Lemma 3. [9, Proposition 3.1 & Proposition 3.2]
1- Let (X, T, L) be a left fuzzy near-ring, then

$$\forall x, y, z \in X, \quad \big(((x*y)*z) \circ ((x*y)*z^{-1})\big)(e) > \theta.$$

2- Let (X, T, L) be a right fuzzy near-ring, then

$$\forall x, y, z \in X, \quad \big((x*(y*z)) \circ (x^{-1}*(y*z))\big)(e) > \theta.$$

Lemma 4. [9, lemma 3.1 & lemma 3.2]
1- Let (X, T, L) be a left fuzzy near-ring, then

$$\forall k, x \in X, \quad \big((k * x) * (k * x^{-1})\big)(e) > \theta.$$

2- Let (X, T, L) be a right fuzzy near-ring, then

$$\forall x, k \in X, \quad \big((k * x) * (k^{-1} * x)\big)(e) > \theta.$$

3 Main results

In this section, we define the notion of fuzzy left semigroup ideal (resp. fuzzy right semigroup ideal) of a fuzzy near-ring and we prove some of their basic properties which are analogous to those of the classical semigroup theory in the case of near-rings. Also, we show that under some conditions on fuzzy left semigroup (resp. fuzzy right semigroup ideal), the fuzzy near-rings must be a fuzzy commutative rings.

Definition 5. *Let (X,T,L) be a fuzzy near-ring and I a nonempty subset of X, then I is called*

1. *A fuzzy left semigroup ideal of X if $(x*s)(t)>\theta$ implies $t\in I$, $\forall x\in X, \forall s\in I$.*
2. *A fuzzy right semigroup ideal of X if $(s*x)(t)>\theta$ implies $t\in I$, $\forall x\in X, \forall s\in I$.*
3. *A fuzzy semigroup ideal of X if is both right and left fuzzy semigroup ideal. Moreover, I is said to be non trivial if $I\neq\{e\}$.*

Lemma 5. *Let (X,T,L) be a fuzzy prime near-ring, I a nontrivial fuzzy semigroup ideal of X and let $x\in X$.*

i) If for all $y\in I$, $(x*y)(e)>\theta$ then $x=e$,

ii) If for all $y\in I$, $(y*x)(e)>\theta$ then $x=e$.

Proof. $i)$ Assume that for all $y\in I$, $(x*y)(e)>\theta$. Letting $z\in X, s\in I$, then there exits $t\in X$ satisfies $(z*s)(t)=L(z,s,t)>\theta$. Since I is a fuzzy semigroup ideal of X, it follows that $t\in I$. In particular, putting $y=t$ in our assumption, we get

$$L(x,t,e)=(x*t)(e)>\theta. \tag{9}$$

Consequently,

$$\big(x*\big(z*s\big)\big)(e)\geq L(z,s,t)\wedge L(x,t,e)>\theta. \tag{10}$$

In view of the fuzzy primeness of (X,T,L), the last result shows that $x=e$ or $s=e$. Taking into account that I is nontrivial, we can consider $s\neq e$ and therefore $x=e$.

$ii)$ Consider $z\in X$ and $s\in I$, there exits $t\in X$ such that $(s*z)(t)=L(s,z,t)>\theta$. Using the same argument as used above, we infer that

$$\big(\big(s*z\big)*x\big)(e)\geq L(s,z,t)\wedge L(t,x,e)>\theta. \tag{11}$$

By the fuzzy primeness of (X,T,L) and I is nontrivial, we get the required result. □

In the case of classical near-rings, in [10], showed in the case of a 3-prime near-ring $\mathcal{N}$, if $x\mathcal{I}y=\{0\}$ then $x=0$ or $y=0$, where $\mathcal{I}$ is a semigroup ideal of $\mathcal{N}$. The following theorem treats this result in the case of a fuzzy semigroup ideal.

Theorem 1. *Let (X,T,L) be a fuzzy prime near-ring, I a nontrivial fuzzy semigroup ideal of X and let $x,y\in X$. Then,*

$$\forall r\in I, \big(\big(x*r\big)*y\big)(e)>\theta \implies \quad x=e \;\text{ or }\; y=e.$$

Proof. Suppose that $\forall r\in I$, $\big(\big(x*r\big)*y\big)(e)>\theta$. Let $(z,s)\in X\times I$ and taking $t\in X$ satisfying $(z*s)(t)>\theta$. Using the fact that I is a fuzzy semigroup ideal of X, we conclude that $t\in I$ and hence,

$$\big(\big(x*t\big)*y\big)(e)>\theta. \tag{12}$$

Let $h\in X$ such that $L(x,t,h)>\theta$, then equation (12) proves that $L(h,y,e)>\theta$. Also,

$$\big(x*(z*s)\big)(h)\geq L(z,s,t)\wedge L(x,t,h)>\theta, \tag{13}$$

Let $v,l\in X$ such that $L(x,z,v)>\theta$ and $L(v,s,l)>\theta$, then

$$\big((x*z)*s\big)(l)\geq L(x,z,v)\wedge L(v,s,l)>\theta. \tag{14}$$

Definition 1 (ii) together (13) and (14) show that $l=h$, so that $L(v,s,h)>\theta$, which implies that

$$\big((v*s)*y)\big)(e)\geq L(v,s,h)\wedge L(h,y,e)>\theta. \tag{15}$$

Let k and n be two elements of X which are satisfying $L(s,y,k)>\theta$ and $L(v,k,n)>\theta$. So,

$$\big(v*(s*y)\big)(n)\geq L(s,y,k)\wedge L(v,k,n)>\theta. \tag{16}$$

Once again Definition 1 $ii)$ forces $n=e$ which gives $L(v,k,e)>\theta$. Consequently,

$$\big((x*z)*k\big)(e)\geq L(x,z,v)\wedge L(v,k,e)>\theta. \tag{17}$$

In virtue of the fuzzy primeness of (X,T,L), the latter result shows that $x=e$ or $k=e$. Now, suppose that $k=e$, then $L(s,y,e)>\theta$ which means that $(s*y)(e)>\theta$. Since s is an arbitrary element of I, then Lemma 5 (ii) assures that $y=e$ and consequently,

$$x=e \text{ or } \; y=e.$$

□

Lemma 6. *Let (X,T) be a fuzzy group. If for all $x,y\in X$, $\exists\, t\in X$ such that*

$$\big((x\circ x)\circ(y\circ y)\big)(t)>\theta \textit{ and}$$
$$\big((x\circ y)\circ(x\circ y)\big)(t)>\theta,$$

then (X,T) is abelian.

Proof. Let $a, b, c \in X$ such that $T(a, b, c) > \theta$. Our main is to prove that $T(b, a, c) > \theta$, for this, choosing $h \in X$ which is satisfying $T(b, a, h) > \theta$ and proving that $h = c$.
From our hypotheses, there exists $t \in X$ such that

$$\big((a \circ a) \circ (b \circ b)\big)(t) > \theta, \tag{18}$$

and

$$\big((a \circ b) \circ (a \circ b)\big)(t) > \theta. \tag{19}$$

Taking $t_1, t_2 \in X$ such that $T(a, a, t_1) > \theta$ and $T(b, b, t_2) > \theta$, thus (18) and (19) give respectively $T(t_1, t_2, t) > \theta$ and $T(c, c, t) > \theta$. Then,

$$\big(t_1 \circ (b \circ b)\big)(t) \geq T(b, b, t_2) \wedge T(t_1, t_2, t) > \theta, \tag{20}$$

and

$$\big(c \circ (a \circ b)\big)(t) \geq T(a, b, c) \wedge T(c, c, t) > \theta. \tag{21}$$

Now, let $x_1, x_2, y_1, y_2 \in X$ satisfy $T(t_1, b, x_1) > \theta$, $T(x_1, b, y_1) > \theta$, $T(c, a, x_2) > \theta$ and $T(x_2, b, y_2) > \theta$. Then,

$$\big((t_1 \circ b) \circ b\big)(y_1) > \theta \text{ and } \big((c \circ a) \circ b\big)(y_2) > \theta,$$

which, because of (20) and (21) together Definition 3, implies that $y_1 = t$ and $y_2 = t$. It follows that $T(x_1, b, t) > \theta$ and $T(x_2, b, t) > \theta$, hence in view of Lemma 1(2) we get $x_1 = x_2$. Consequently, $T(t_1, b, x_1) > \theta$ and $T(c, a, x_1) > \theta$. So that,

$$\big((a \circ a) \circ b\big)(x_1) \geq T(a, a, t_1) \wedge T(t_1, b, x_1) > \theta, \tag{22}$$

and

$$\big((a \circ b) \circ a\big)(x_1) \geq T(a, b, c) \wedge T(c, a, x_1) > \theta. \tag{23}$$

Let $v, k \in X$ such that $T(a, c, v) > \theta$ and $T(a, h, k) > \theta$. Then,

$$\begin{aligned} &\big(a \circ (a \circ b)\big)(v) \geq T(a, b, c) \wedge T(a, c, v) > \theta \\ &\textit{and} \\ &\big(a \circ (b \circ a)\big)(k) \geq T(b, a, h) \wedge T(a, h, k) > \theta. \end{aligned} \tag{24}$$

Combining (22), (23), (24) and using Definition 3, we obtain $v = x_1$ and $h = x_1$ and therefore, $T(a, c, x_1) > \theta$ and $T(a, h, x_1) > \theta$. Once again, using Lemma 1(1) we conclude that $c = h$. So that, $T(b, a, c) > \theta$.
Conversely, assuming that $T(b, a, c) > \theta$ and using similar arguments as used above to prove that $T(a, b, c) > \theta$. Thus, (X, T) is a commutative fuzzy group. □

Theorem 2. *Let (X, T, L) be a fuzzy near-ring. If there exists $z \in Z_F(X)^*$ such that $T(z, z, r) > \theta \implies r \in Z_F(X)$ for all $r \in X$, then (X, T) is abelian.*

Proof. Suppose that (X, T, L) is a left fuzzy near-ring and let $x, y \in X$. Consider z the element of our hypothesis; by Definition 2 there exists $t \in X$ such that $T(z, z, t) > \theta$ which, according to our hypothesis, implies that $t \in Z_F(X)$.
Now, let $v, h \in X$ such that $T(x, y, v) > \theta$ and $L(t, v, h) > \theta$. Then,

$$\big(t * (x \circ y)\big)(h) \geq T(x, y, v) \wedge L(t, v, h) > \theta. \tag{25}$$

Taking $h_1, h_2, h' \in X$ such that $L(t, x, h_1) > \theta$, $L(t, y, h_2) > \theta$ and $T(h_1, h_2, h') > \theta$. Then,

$$\begin{aligned} &\big((t * x) \circ (t * y)\big)(h') \geq \\ &\qquad L(t, x, h_1) \wedge L(t, y, h_2) \wedge T(h_1, h_2, h'). \end{aligned} \tag{26}$$

From Definition 1 (iii), (25) and (26) give $h' = h$, so that

$$T(h_1, h_2, h) > \theta. \tag{27}$$

Also, as $t \in Z_F(X)$, we have $L(x, t, h_1) > \theta$ and $L(y, t, h_2) > \theta$, then
$\big(x * (z \circ z)\big)(h_1) \geq T(z, z, t) \wedge L(x, t, h_1) > \theta$ and
$\big(y * (z \circ z)\big)(h_2) \geq T(z, z, t) \wedge L(y, t, h_2) > \theta$.
Choosing $v_1, v_2, v_3, v_4 \in X$ such that $L(x, z, v_1) > \theta$, $L(y, z, v_2) > \theta$, $T(v_1, v_1, v_3) > \theta$ and $T(v_2, v_2, v_4) > \theta$, we infer that

$$\big((x*z) \circ (x*z)\big)(v_3) > \theta \text{ and } \big((y*z) \circ (y*z)\big)(v_4) > \theta. \tag{28}$$

Consequently, in virtue of Definition 1 (iii), we conclude that $v_3 = h_1$ and $v_4 = h_2$ which implies that $T(v_1, v_1, h_1) > \theta$ and $L(v_2, v_2, h_2) > \theta$. Once again, since $z \in Z_F(X)^*$, we have $L(z, x, v_1) > \theta$ and $L(z, y, v_2) > \theta$, thus

$$\big((z*x) \circ (z*x)\big)(h_1) > \theta \text{ and } \big((z*y) \circ (z*y)\big)(h_2) > \theta. \tag{29}$$

Next, choose $\ell_1, \ell_2, \ell_3, \ell_4$ such that $T(x, x, \ell_1) > \theta$, $T(y, y, \ell_2) > \theta$, $L(z, \ell_1, \ell_3) > \theta$ and $L(z, \ell_2, \ell_4) > \theta$, which yields

$$\big(z * (x \circ x)\big)(\ell_3) > \theta \text{ and } \big(z * (y \circ y)\big)(\ell_4) > \theta. \tag{30}$$

Invoking Definition 1 (iii), we arrive at $\ell_3 = h_1$ and $\ell_4 = h_2$. So that, $L(z, \ell_1, h_1) > \theta$ and $L(z, \ell_2, h_2) > \theta$. Because of (27), we get

$$\begin{aligned} &\big((z * \ell_1) \circ (z * \ell_2)\big)(h) \geq \\ &\qquad L(z, \ell_1, h_1) \wedge L(z, \ell_2, h_2) \wedge T(h_1, h_2, h) > \theta. \end{aligned} \tag{31}$$

Taking $x_1, h^{''} \in X$ satisfy $T(\ell_1, \ell_2, x_1) > \theta$ and $L(z, x_1, h^{''}) > \theta$, we obtain

$$\big(z * (\ell_1 \circ \ell_2)\big)(h^{''}) > \theta, \tag{32}$$

which, because of Definition 1 (iii), implies that $h = h^{''}$ and therefore,

$$L(z, x_1, h) > \theta. \tag{33}$$

Similarly, since $L(v, t, h) > \theta$, then we have

$$\big(v * (z \circ z)\big)(h) \geq T(z, z, t) \wedge L(v, t, h) > \theta. \tag{34}$$

Let $t_1, t_2 \in X$ such that $L(v, z, t_1) > \theta$ and $T(t_1, t_1, t_2) > \theta$, which implies that

$$\begin{gathered}\big((v * z) \circ (v * z)\big)(t_2) \geq \\ L(v, z, t_1) \wedge L(v, z, t_1) \wedge T(t_1, t_1, t_2) > \theta.\end{gathered} \tag{35}$$

Definition 1 (iii) assures that $t_2 = h$ and thus, $T(t_1, t_1, h) > \theta$.
In virtue of $z \in Z_F(X)^*$ and $L(v, z, t_1) > \theta$, we get $L(z, v, t_1) > \theta$, then $\big((z * v) \circ (z * v)\big)(h) > \theta$. Taking $h^{'''}, x_2 \in X$ verify $T(v, v, x_2) > \theta$ and $L(z, x_2, h^{'''}) > \theta$, then $\big(z * (v \circ v)\big)(h^{'''}) > \theta$ which, by Definition 1, guarantees that $h^{'''} = h$, and thus

$$L(z, x_2, h) > \theta. \tag{36}$$

Now, from Lemma 4 (1.), we have

$$\big((z * x_2) \circ (z * x_2^{-1})\big)(e) > \theta. \tag{37}$$

Let $m \in X$ such that $L(z, x_2^{-1}, m) > \theta$ and combining (36) and (37), we find that $T(h, m, e) > \theta$. Hence, because of (33) we get

$$\begin{gathered}\big((z * x_1) \circ (z * x_2^{-1})\big)(e) \geq \\ L(z, x_1, h) \wedge L(z, x_2^{-1}, m) \wedge T(h, m, e) > \theta.\end{gathered} \tag{38}$$

Let $y_1, y_2 \in X$ such that $T(x_1, x_2^{-1}, y_1) > \theta$ and $L(z, y_1, y_2) > \theta$. It follows that

$$\begin{gathered}\big(z * (x_1 \circ x_2^{-1})\big)(y_2) \geq \\ T(x_1, x_2^{-1}, y_1) \wedge L(z, y_1, y_2) > \theta.\end{gathered} \tag{39}$$

Once again by Definition 1 (iii), (38) and (39) shows that $y_2 = e$ and thus $L(z, y_1, e) > \theta$.
Let k an arbitrary element of X, we have

$$\big(k * (z * y_1)\big)(e) \geq L(z, y_1, e) \wedge L(k, e, e) > \theta. \tag{40}$$

Taking $s, c \in X$ such that $L(k, z, s) > \theta$ and $L(s, y_1, c) > \theta$, which proves that

$$\big((k * z) * y_1\big)(c) > \theta. \tag{41}$$

Because of Definition 1 (ii), the last two results show that $c = e$, which allowed us to conclude that $L(s, y_1, e) > \theta$.
In view of $z \in Z_F(X)^*$ and $L(k, z, s) > \theta$, we have $L(z, k, s) > \theta$ and hence, for all $k \in X$

$$\big((z * k) * y_1\big)(e) \geq L(z, k, s) \wedge L(s, y_1, e) > \theta. \tag{42}$$

In the light of the primeness of (X, T, L) and $z \in Z_F(X)^*$, the last relation shows that $y_1 = e$, and thus

$$T(x_1, x_2^{-1}, e) > \theta. \tag{43}$$

Once again, from Lemma 4 (1.), we have

$$\big((z * x_2^{-1}) \circ (z * x_2)\big)(e) > \theta. \tag{44}$$

By reasoning in the same way as above, we arrive at

$$T(x_2^{-1}, x_1, e) > \theta. \tag{45}$$

Now, from Definition 3 and (43) and (45), we obtain $x_1 = (x_2^{-1})^{-1}$ and by lemma 1 (5), we arrive at $x_1 = x_2$. Thus,

$$\begin{gathered}\big((x \circ x) \circ (y \circ y)\big)(x_1) \geq \\ L(x, x, \ell_1) \wedge L(y, y, \ell_2) \wedge T(\ell_1, \ell_2, x_1) > \theta,\end{gathered} \tag{46}$$

and

$$\begin{gathered}\big((x \circ y) \circ (x \circ y)\big)(x_1) \geq \\ T(x, y, v) \wedge T(x, y, v) \wedge T(v, v, x_1) > \theta.\end{gathered} \tag{47}$$

Consequently, (X, T) is an abelian fuzzy group by Lemma 6. This ends the prove of our Theorem.

Remark 1. *The results in the previous Theorem remain valid for right fuzzy near-rings with the obvious changes by using the second case of Lemma 4.*

□

Theorem 3. *Let (X, T, L) be a fuzzy prime near-ring, I be a nontrivial fuzzy semigroup ideal of (X, T, L) and $x \in X$. If for all $u \in I$, there exists $t \in X$ such that $(u * x)(t) > \theta$ and $(x * u)(t) > \theta$, then $x \in Z_F(X)$.*

Proof. Suppose that (X, T, L) is a fuzzy prime left near-ring. By Lemma 2 (1), it suffices to prove that $\forall y \in X, ((x * y) \circ (y * x^{-1}))(e) > \theta$. For this, let $y \in X$ and $u \in I$.
By the definition of a fuzzy binary operation, there are $\ell_1, \ell_2, \ell \in X$ such that $L(x, y, \ell_1) > \theta$, $L(y, x^{-1}, \ell_2) > \theta$ and $T(\ell_1, \ell_2, \ell) > \theta$ which give $((x * y) \circ (y * x^{-1}))(\ell) > \theta$. Our goal is to show that $\ell = e$.
Taking $h \in X$ such that $(u * y)(h) = L(u, y, h) >$

θ, in virtue of I is a fuzzy semigroup ideal of X, we find that $h \in I$. Taking into account our hypotheses, there exists $t \in X$ satisfying $(h * x)(t) > \theta$ and $(x * h)(t) > \theta$. Then,

$$\big((u*y)*x\big)(t) \geq L(u,y,h) \wedge L(h,x,t) > \theta, \quad (48)$$

and

$$\big(x*(u*y)\big)(t) \geq L(u,y,h) \wedge L(x,h,t) > \theta. \quad (49)$$

Also, by our hypotheses, there exists an element $t_1 \in X$ such that $(x*u)(t_1) = L(x,u,t_1) > \theta$ and $(u*x)(t_1) = L(u,x,t_1) > \theta$. Choosing $h_1 \in X$ satisfying $L(t_1,y,h_1) > \theta$. Then,

$$\big((x*u)*y\big)(h_1) \geq L(x,u,t_1) \wedge L(t_1,y,h_1) > \theta. \quad (50)$$

According to the second condition of Definition 1, the two relations (49) and (50) affirm that $h_1 = t$, then $L(t_1,y,t) > \theta$. So that,

$$\big((u*x)*y\big)(t) \geq L(u,x,t_1) \wedge L(t_1,y,t) > \theta, \quad (51)$$

Now, taking $h_2 \in X$ satisfying $L(u,\ell_1,h_2) > \theta$, we get

$$\big(u*(x*y)\big)(h_2) \geq L(x,y,\ell_1) \wedge L(u,\ell_1,h_2) > \theta. \quad (52)$$

Once again, in view of Definition 1 (ii), (51) and (52) give $h_2 = t$ and hence $L(u,\ell_1,t) > \theta$. From Lemma 3 (1), we have

$$\Big(\big((u*y)*x\big) \circ \big((u*y)*x^{-1}\big)\Big)(e) > \theta. \quad (53)$$

Let $v, t_2 \in X$ such that $L(u,y,t_2) > \theta$ and $L(t_2,x^{-1},v) > \theta$, which implies that

$$((u*y)*x^{-1})(v) \geq L(u,y,t_2) \wedge L(t_2,x^{-1},v) > \theta. \quad (54)$$

Using (48), (53) and (54), we infer that $T(t,v,e) > \theta$. Choose $h_3 \in X$ such that $L(u,\ell_2,h_3) > \theta$, then

$$\big(u*(y*x^{-1})\big)(h_3) \geq L(y,x^{-1},\ell_2) \wedge L(u,\ell_2,h_3) > \theta. \quad (55)$$

From (54) and (55), because of Definition 1 (ii), we conclude that $h_3 = v$ which implies that $L(u,\ell_2,v) > \theta$. Consequently,

$$\big((u*\ell_1)\circ(u*\ell_2)\big)(e) \geq L(u,\ell_1,t) \wedge L(u,\ell_2,v) \wedge T(t,v,e) > \theta. \quad (56)$$

Let $k \in X$ such that $L(u,\ell,k) > \theta$, it follows that

$$\big(u*(\ell_1 \circ \ell_2)\big)(k) \geq L(\ell_1,\ell_1,\ell) \wedge L(u,\ell,k) > \theta. \quad (57)$$

According to Definition 1 (iii), (56) and (57) assure $k = e$ and then $(u * \ell)(e) = L(u,\ell,e) > \theta$. Once again Lemma 5 shows that $\ell = e$, and hence $x \in Z_F(X)$. This proves the desired result. □

Remark 2. *If we consider that (X,T,L) is a fuzzy right near-ring, we can follow the same arguments as those used previously, taking into account the second condition in Lemmas 2 and 3.*

As an application of Theorems 2 and 3, we get the following result.

Theorem 4. *Let (X,T,L) be a fuzzy near-ring. If $Z_F(X)$ contains a nontrivial fuzzy semigroup ideal I, then (X,T,L) is a fuzzy commutative ring.*

Proof. We divide the proof into four essential steps.

• In the first part, we assume that (X,T,L) is a fuzzy left near-ring, and we prove that property (iv) of Definition 1 is satisfied.

Firstly, showing that $\forall x,y,t_1,t_2 \in X$ and $z \in I$, if $\big((x\circ y)*z\big)(t_1) > \theta$ and $\big((x*z)\circ(y*z)\big)(t_2) > \theta$, then $t_1 = t_2$.

In fact, let $x,y,t_1,t_2 \in X$ and $z \in I$ satisfy $\big((x\circ y)*z\big)(t_1) > \theta$ and $\big((x*z)\circ(y*z)\big)(t_2) > \theta$. Choosing $h_1 \in X$ such that $T(x,y,h_1) > \theta$ and using the fact that $\big((x\circ y)*z\big)(t_1) > \theta$, we obtain $L(h_1,z,t_1) > \theta$ and in view of $z \in I \subseteq Z_F(X)$, we find that $L(z,h_1,t_1) > \theta$.

Similarly, let $h_2,h_3 \in X$ such that $L(x,z,h_2) > \theta$ and $L(y,z,h_3) > \theta$ and since $\big((x*z)\circ(y*z)\big)(t_2) > \theta$, we conclude that $T(h_2,h_3,t_2) > \theta$.

Also, as $z \in I \subseteq Z_F(X)$, we have $L(z,x,h_2) > \theta$ and $L(z,y,h_3) > \theta$. Then,

$$\big((z*x)\circ(z*y)\big)(t_2) \geq L(z,x,h_2) \wedge L(z,y,h_3) \wedge T(h_2,h_3,t_2) > \theta, \quad (58)$$

and

$$\big(z*(x\circ y)\big)(t_1) \geq T(x,y,h_1) \wedge L(z,h_1,t_1) > \theta. \quad (59)$$

Invoking Definition 1 (iii), the last two relations give $t_1 = t_2$.

• Secondly, checking that $\forall x,y,t,t_1,t_2 \in X$ if $\big((x\circ y)*t\big)(t_1) > \theta$ and $\big((x*t)\circ(y*t)\big)(t_2) > \theta$, then $t_1 = t_2$.

For this purpose, let $x,y,t,t_1,t_2 \in X$ such that $\big((x\circ y)*t\big)(t_1) > \theta$ and $\big((x*t)\circ(y*t)\big)(t_2) > \theta$. Let $z \in I$ and taking $s \in X$ such that $(t*z)(s) > \theta$, by defining I, we get $s \in I$.

Let $\ell,h_1,h_2,h,v \in X$ such that $T(x,y,\ell) > \theta$, $L(x,s,h_1) > \theta$, $L(y,s,h_2) > \theta$, $L(\ell,s,h) > \theta$ and

$T(h_1, h_2, v) > \theta$. Then,

$$\big((x \circ y) * s\big)(h) \geq T(x, y, \ell) \wedge L(\ell, s, h) > \theta, \quad (60)$$

and

$$\big((x * s) \circ (y * s)\big)(v) \geq L(x, s, h_1) \wedge L(y, s, h_2) \wedge T(h_1, h_2, v) > \theta. \quad (61)$$

The previous step guarantees $h = v$, so that $T(h_1, h_2, h) > \theta$.
Also, we have

$$(\ell * (t * z))(h) \geq L(t, z, s) \wedge L(\ell, s, h) > \theta, \quad (62)$$

and from $T(x, y, \ell) > \theta$ together $\big((x \circ y) * t\big)(t_1) > \theta$, we get $L(\ell, t, t_1) > \theta$.
Now, taking $k \in X$ such that $L(t_1, z, k) > \theta$. It follows that,

$$\big((\ell * t) * z\big)(k) \geq L(\ell, t, t_1) \wedge L(t_1, z, k) > \theta. \quad (63)$$

Applying Definition 1 (ii), for (62) and (63), we get $k = h$ and hence,

$$L(t_1, z, h) > \theta. \quad (64)$$

On the other hand, we have

$$\big(x * (t * z)\big)(h_1) \geq L(t, z, s) \wedge L(x, s, h_1) > \theta \quad (65)$$

and

$$\big(y * (t * z)\big)(h_2) \geq L(t, z, s) \wedge L(y, s, h_2) > \theta. \quad (66)$$

Considering $\ell_1, \ell_2, m, n \in X$ satisfy $L(x, t, \ell_1) > \theta$, $L(y, t, \ell_2) > \theta$, $L(\ell_1, z, m) > \theta$ and $L(\ell_2, z, n) > \theta$. Then, we can see that $\big((x * t) * z\big)(m) > \theta$ and $\big((y * t) * z\big)(n) > \theta$. Combining the last two results with (65) and (66), respectively, and apply Definition 1 (ii), we arrive at $m = h_1$ and $n = h_2$ which give $L(\ell_1, z, h_1) > \theta$ and $L(\ell_2, z, h_2) > \theta$. Accordingly,

$$\big((\ell_1 * z) \circ (\ell_2 * z)\big)(h) \geq L(\ell_1, z, h_1) \wedge L(\ell_2, z, h_2) \wedge T(h_1, h_2, h) > \theta. \quad (67)$$

As well, from $L(x, t, \ell_1) > \theta$, $L(y, t, \ell_2) > \theta$ and $\big((x * t) \circ (y * t)\big)(t_2) > \theta$, we get $T(\ell_1, \ell_2, t_2) > \theta$.
Let $h' \in X$ such that $L(t_2, z, h') > \theta$, then

$$\big((\ell_1 \circ \ell_2) * z)\big)(h') \geq L(t_2, z, h') \wedge T(\ell_1, \ell_2, t_2) > \theta. \quad (68)$$

Since $z \in I$, then in view of the first part, (67) and (68) assure that $h = h'$ which implies that

$$L(t_2, z, h) > \theta. \quad (69)$$

Let $t \in X$ such that $L(e, z, t) > \theta$, then

$$\big((e \circ e) * z\big)(t) \geq T(e, e, e) \wedge L(e, z, t) > \theta. \quad (70)$$

Consider $v \in X$ such that $T(t, t, v) > \theta$, we have

$$\big((e * z) \circ (e * z)\big)(v) \geq L(e, z, t) \wedge L(e, z, t) \wedge T(t, t, v) > \theta. \quad (71)$$

Using the conclusion of the first part, we obtain $v = t$ which means that $T(t, t, t) > \theta$ and hence $t = e$ by Lemma 1 (4). Thus, $L(e, z, e) > \theta$.
Now, let $t_3 \in X$ such that $L(\ell, t^{-1}, t_3) > \theta$ which means $\big((x \circ y) * t^{-1}\big)(t_3) > \theta$. From lemma 4 (i), we have $\big(\big(\ell * t\big) \circ \big(\ell * t^{-1}\big)\big)(e) > \theta$. Using our hypotheses that $L(\ell, t, t_1) > \theta$ and $L(\ell, t^{-1}, t_3) > \theta$, we find that

$$(t_1 \circ t_3)(e) = T(t_1, t_3, e) > \theta, \quad (72)$$

that is,

$$\big((t_1 \circ t_3) * z\big)(e) \geq T(t_1, t_3, e) \wedge L(e, z, e) > \theta. \quad (73)$$

Let $s_1, p \in X$ such that $L(t_3, z, s_1) > \theta$ and $T(h, s_1, p) > \theta$ and invoking (64), we obtain

$$\big((t_1 * z) \circ (t_3 * z)\big)(p) \geq L(t_1, z, h) \wedge L(t_3, z, s_1) \wedge T(h, s_1, p) > \theta. \quad (74)$$

In view of the preceding step, we conclude that $p = e$, and therefore $T(h, s_1, e) > \theta$.
By using (69), we get

$$\big((t_2 * z) \circ (t_3 * z\big)(e) \geq L(t_2, z, h) \wedge L(t_3, z, s_1) \wedge T(h, s_1, e) > \theta, \quad (75)$$

Now, choosing $v_1, v_2 \in X$ such that $T(t_2, t_3, v_1) > \theta$ and $L(v_1, z, v_2) > \theta$, then

$$\big((t_2 \circ t_3) * z\big)(v_2) > \theta. \quad (76)$$

Once again from the previous step, we get $v_2 = e$, so that $(v_1 * z)(e) > \theta$ which, because of Lemma 5 (i), gives $v_1 = e$. And therefore,

$$(t_2 \circ t_3)(e) = T(t_2, t_3, e) > \theta. \quad (77)$$

Applyiny Lemma 1 (2) to (72) and (77), we find that $t_1 = t_2$.

We can use the same arguments to show the property (iii) of Definition 1 when (X, T, L) is a fuzzy right near-ring.

• Thirdly, showing that (X, T) is commutative. The fact that I is nontrivial ideal assures

the existence of $x, y, t \in I$, with $t \neq e$, such that $(x * y)(t) > \theta$. Indeed, suppose that for all $x, y, t \in I$, where $t \neq e$, we have $(x * y)(t) \leq \theta$, then inevitably we will have for all $x, y \in X$, $(x*y)(e) > \theta$, and thus Lemma 5 forces that $x = e$ which means that $I = \{e\}$, leading to a contradiction with the fact that I is a nontrivial ideal of X. Consequently, there exist $x, y, t \in I$ such that $(x * y)(t) > \theta$ and $t \neq e$. Since $t \in I \subseteq Z_F(X)$ and $t \neq e$, then $t \in (Z_F(X))^*$.
Let $v \in X$ such that $T(t, t, v) > \theta$. Our goal is to show that $v \in Z_F(X)$. We have

$$\big((x * y) \circ (x * y)\big)(v) \geq L(x, y, t) \wedge L(x, y, t) \wedge T(t, t, v) > \theta. \tag{78}$$

Now, letting $h, k \in X$ satisfy $T(y, y, h) > \theta$ and $L(x, h, k) > \theta$, then

$$\big(x * (y \circ y)\big)(k) \geq T(y, y, h) \wedge L(x, h, k) > \theta. \tag{79}$$

Definition 1 (iii) assures that $k = v$ and therefore $L(x, h, v) > \theta$. Taking into account that $x \in I$ and $h \in X$ we conclude that $v \in I$, so that $v \in Z_F(X)$ and hence (X, T) is commutative by Theorem 2.
• Finally, to complete the proof of this theorem, we show that (X, T, L) is commutative. For this, let $x \in X$ and $u \in I$, then there exists $t \in X$ such that $L(x, u, t) > \theta$. In view of $u \in I \subseteq Z_F(X)$, we obtain $L(u, x, t) > \theta$. Then,
$\forall\ u \in I,\ \exists\ t \in X$ such that $L(x, u, t) > \theta$ and $L(u, x, t) > \theta$.
By application of Theorem 3, we find $x \in Z_F(X)$ and hence, (X, T, L) is a commutative fuzzy ring. □

4 Conclusion

In this paper, a new type of fuzzy semigroup ideal was created and some of its related properties were studied analogously to the ordinary semigroup ideal. Also, using this new definition, we proved that under some other conditions, a fuzzy near ring must be a fuzzy commutative ring. The practical applications of our study will be the subject of future research.

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