Study of a Diseased Volterra Type Population Model featuring Prey Refuge and Fear Influence

N. MOHANA SORUBHA SUNDARI[1], S.P.GEETHA[2,*]
[1]Department of Mathematics,
Chikkaiah Naicker College, Erode 638 004,
INDIA

[2,*]Department of Mathematics,
Vellalar College for Women, Erode-638012,
INDIA

[*] Corresponding author

Abstract: - In order to study the local stability characteristics of a predator-prey dynamical model, this work proposes a Volterra-type model that takes into account the fear influence of prey resulting from predator domination. Because of an outbreak of disease in the prey species, the prey gets classified as either healthy or diseased. Both predator and prey species compete for their resources. In addition, the prey sought refuge against the predator. All these factors are addressed when setting up the mathematical model. The biological validity of the model is ensured by testing its boundedness. The equilibrium points have been identified. The short-term behavior of the system is analyzed at all equilibrium points. Routh Hurwitz conditions are employed to examine the local stability property.

Key-Words: - Predator-prey model, Equilibrium points, Routh Hurwitz conditions, Fear effect, Prey refuge, Local stability.

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1 Introduction

The framework existing among the living and non-living organisms in the environmental structure exhibit nonlinear feature. To investigate their interactions, their behaviours are captured and mathematically described mainly in the form of differential equations. Many predator-prey species demonstrate an unexpected diversity of dynamical behavioural patterns, which has sparked a boom in the design of mathematical models of ecosystems.

The first predator-prey model has been developed by Alfred James Lotka and Vito Volterra. The Lotka-Volterra system of equations has an extensive record which originated before a century. These equations expressed an association between two or more species. From then on, various types of model equations have been developed, modified, and extended extensively incorporating many traits of the species under study. In [1], the authors focused on the predator-prey system, striving to provide a cutting edge over view of recent models incorporating the Allee effect, fear effect, cannibalism and immigration, and juxtaposed the qualitative outcomes achieved for each element with a special focus on equilibria, both local and global stability and the presence of limit cycles. Anderson and May (1981) were the first to explore a population model with infection. Since then, many ecoepidemiological models have been studied incorporating disease in prey / predator or both species with various modes of disease transmission. Considering the Holling-type interaction, [2], proved that a judicious selection of general Holling parameters, disease management can be achieved by regulating the interacting function within the ecosystem. The authors in [3], studied a prey-predator model where the disease spreads only among predators, transmitted horizontally through contact between infected and susceptible individuals. An epidemic model that integrates vertical and horizontal transmission of infection employing a nonlinear incidence rate was investigated, [4].

By offering refuge, the ecosystem provides some kind of defense to the prey from predators. The influence of prey refuge was explored on the dynamic behavior of the model using the Lotka-Volterra framework that features a Holling type III functional response, [5]. Prey refuges have highly complicated consequences on the dynamics of population, [6]. The fear factor may alter the normal behavior of the prey, which in turn has a significant impact on the population model, [7], [8], [9], [10].

Fear induced by the risk of predation decreases prey birth rates. Also limit cycle observation revealed that predation fear can both stabilise and destabilise the ecosystem, [11]. With an increasing fear effect, the final density of the prey species may approach zero, driving them to extinction. The fear phenomena negatively impact prey survival, potentially a significant factor in their extinction, [12], differing from Wang's result, [13]. The findings of [14], suggested that increasing the prey refuge or Allee effect enhances the dynamic complexity of the system. Moreover, while the fear effect or the Allee effect does not have an impact on the density of the predator, it can reduce the predator population at a positive equilibrium, [14]. Fear can cause backward bifurcation and chaos by supressing prey growth and disease transmission leading to a significant reduction in the infection rate, [15]. Ecologically, prey adapt to fear beyond a critical threshold, which is essential to sustain the ecosystem. Fear not only stabilizes the system, but also regulates disease and diminishes predator population, [16]. Increasing fear level enhances system stability by eradicating periodic solutions and reducing predator population at the coexisting equilibrium point without leading to predator extinction. Also, prey refuge significantly contributes to predator persistence, [17]. The fear factor serves to stabilize the dynamics of the system, [18]. Despite its study, the influence of predator fear on prey with the inclusion of disease in the dynamical model has not yet been fully addressed, [19], [20], [21].

The objective of the present work is to formulate and study a predator-prey model that integrates predators' fear effects on prey together with prey affected by disease and with a Volterra-type functional response. The sections in this paper are ordered as follows. Section 2 presents the mathematical population model. Section 3 discusses the positivity of the solution and the boundedness. In section 4, all the equilibrium points are determined. Section 5 analysis the local stability behavior at the equilibrium points. The last section is the conclusion.

2 Mathematical Model

The proposed mathematical dynamical model consists of the prey density X and the predator density z at any time t. As a result of infection in the prey group, they are classified as susceptible prey x and infected prey y. Only susceptible prey reproduces. There is intraspecific competition in the prey and also in the predator species. The predator preys on susceptible and infected prey. Without predator, the prey species increases logistically. The prey is the only source of food, and in conditions of nonavailability of prey, the predator dies. With these assumptions the model takes the form:

$$\dot{x} = \frac{\alpha}{1+fz}x - \delta_1 x^2 - \delta_2 xy - d_1 x - \beta_1 xy - c_1(1-m)xz = F_1(x,y,z), \quad (1)$$

$$\dot{y} = \beta_1 xy - \delta_3 y^2 - \delta_4 xy - d_2 y - c_2(1-m)yz = F_2(x,y,z), \quad (2)$$

$$\dot{z} = \mu_1 c_1(1-m)xz + \mu_2 c_2(1-m)yz - \delta_5 z^2 - d_3 z = F_3(x,y,z), \quad (3)$$

with initial conditions

$$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0. \quad (4)$$

All the parameters are presumed positive.

Nomenclature

x	susceptible prey density
y	infected prey density
z	predator density
f	fear rate of the prey
α	growth rate of prey
β_1	disease transmission rate
δ_1	competition within the susceptible prey
δ_2	competition between the susceptible and infected prey
δ_3	competition within the infected prey
δ_4	competition between the susceptible and infected prey
δ_5	competition within the predator
c_1	catchability rate of susceptible prey
c_2	catchability rate of the infected prey
μ_1	conversion rate of susceptible prey
μ_2	conversion rate of infected prey
$m \in [0,1)$	constant proportion of prey taking refuge

3 Positivity and Boundedness

We prove that the system given (1)-(3) with (4) is well posed mathematically in the positive quadrant

$$\Omega = \{(x,y,z)/x \geq 0, y \geq 0, z \geq 0\}$$

and solutions exists for all positive time. The variables x, y, z represent biological species and have the domain Ω in R_+^3. The R.H.S of system (1)-(3) is continuously differentiable and locally Lipschitz in the first quadrant Ω. Hence, there are solutions for the initial value problem (1)-(3) with non-negative initial conditions.

Theorem 3.1 *For every solution of system (1)-(3) that starts in the positive quadrant, the solutions are uniformly bounded.*

Proof: Considering (x, y, z) as the solution of (1)-(3) with (4).
Let $S = x + y + z$.
Taking the time derivative of S, we get

$$\begin{aligned}\dot{S} &= \dot{x} + \dot{y} + \dot{z} \\ &= \frac{\alpha}{1+fz}x - \delta_1 x^2 - (d_1 x + d_2 y + d_3 z) \\ &\quad -\delta_2 xy - \delta_4 xy - \delta_3 y^2 - \delta_5 z^2 \\ &\quad -c_1(1-m)xz(1-\mu_1) \\ &\quad -c_2(1-m)yz(1-\mu_2)\end{aligned}$$

Let $\eta = min\{d_1, d_2, d_3\}$,
After simple algebraic simplification,

$$\begin{aligned}\dot{S} &\le \alpha x - \delta_1 x^2 - \eta S - (\delta_2 - \delta_4)xy - \delta_3 y^2 \\ &\quad -\delta_5 z^2 - (1-m)xzc_1(1-\mu_1) \\ &\quad -(1-m)yzc_2(1-\mu_2)\end{aligned}$$

If $\mu_1 < 1, \mu_2 < 1$, then the above equation becomes,

$$\begin{aligned}\dot{S} &\le \alpha x - \delta_1 x^2 - \eta S \\ \dot{S} + \eta S &\le -\delta_1\left(x^2 - \frac{\alpha x}{\delta_1}\right)\end{aligned}$$

Rearranging and writing as perfect squares,

$$\dot{S} + \eta S \le -\delta_1\left(x - \frac{\alpha}{2\delta_1}\right)^2 + \frac{\alpha^2}{4\delta_1}$$

Let $\frac{\alpha^2}{4\delta_1} = N$, then $\dot{S} + \eta S \le N$
By a theorem of differential inequality,

$$\underbrace{lim}_{t\to\infty} supS(t) \le \frac{N}{\eta}, \quad \forall t > 0$$

Hence, the proof is complete.

4 Determination of Equilibrium Points

Here the equilibrium points of system (1)-(3) are determined.

1. $f_0(0,0,0)$ is the trivial equilibrium point.
2. $f_1(\overline{x},0,0)$ is the infection free and predator free equilibrium point where

$$\overline{x} = \frac{\alpha - d_1}{\delta_1}, \alpha - d_1 > 0$$

3. $f_2(\overline{\overline{x}},\overline{\overline{y}},0)$-the predator free equilibrium point exists only with the existence of positive solution to the below equations:

$$\begin{aligned}\alpha\overline{\overline{x}} - \delta_1\overline{\overline{x}}^2 - \delta_2\overline{\overline{xy}} - d_1\overline{\overline{x}} - \beta_1\overline{\overline{xy}} &= 0, \\ \beta_1\overline{\overline{xy}} - \delta_3\overline{\overline{y}}^2 - \delta_4\overline{\overline{xy}} - d_2\overline{\overline{y}} &= 0.\end{aligned}$$

From the above equation, we get,

$$\overline{\overline{y}} = \frac{(\beta_1 - \delta_4)\overline{\overline{x}} - d_2}{\delta_3},$$

Then,

$$\overline{\overline{x}} = \frac{(\alpha - d_1)\delta_3 + d_2(\delta_2 + \beta_1)}{\delta_1\delta_3 + (\delta_2 + \beta_1)(\beta_1 - \delta_4)}.$$

This equilibrium point exists if

$$\overline{\overline{x}} > \frac{d_2}{\beta_1 - \delta_4}, \quad (\beta_1 - \delta_4) > 0 \quad \text{and} \quad (\alpha - d_1) > 0.$$

4. $f_3(\widetilde{x},0,\widetilde{z})$ - the equilibrium point without disease exists only with the existence of positive solution to the following equations:

$$\frac{\alpha}{1+f\widetilde{z}} - \delta_1\widetilde{x} - d_1 - c_1(1-m)\widetilde{z} = 0, \quad (5)$$

$$\mu_1 c_1(1-m)\widetilde{x} - \delta_5\widetilde{z} - d_3 = 0. \quad (6)$$

From (6), we get,

$$\widetilde{x} = \frac{\delta_5\widetilde{z} + d_3}{\mu_1 c_1(1-m)} \quad (7)$$

Substituting (7) in (5) and letting $A = \frac{\delta_1}{\mu_1 c_1(1-m)}$, we obtain

$$\begin{aligned}&[c_1(1-m)f + A\delta_5 f]\widetilde{z}^2 \\ &+ [c_1(1-m) + A\delta_5 + (Ad_3 + d_1)f]\widetilde{z} \\ &+ (Ad_3 + d_1) - \alpha = 0 \quad (8)\end{aligned}$$

Utilizing sign rule of Descarte's if $(Ad_3 + d_1) < \alpha$, then (8) has a unique positive root.

5. $f_4\left(\widetilde{\widetilde{x}},\widetilde{\widetilde{y}},\widetilde{\widetilde{z}}\right)$ - the interior equilibrium point exists only with the existence of positive solution to the below equations:

$$\frac{\alpha}{1+f\widetilde{\widetilde{z}}} - \delta_1\widetilde{\widetilde{x}} - \delta_2\widetilde{\widetilde{y}} - d_1 - \beta_1\widetilde{\widetilde{x}} - c_1(1-m)\widetilde{\widetilde{z}} = 0, \quad (9)$$

$$\beta_1\widetilde{\widetilde{x}} - \delta_3\widetilde{\widetilde{y}} - \delta_4\widetilde{\widetilde{x}} - d_2 - c_2(1-m)\widetilde{\widetilde{z}} = 0, \quad (10)$$

$$\mu_1 c_1(1-m)\widetilde{\widetilde{x}} + \mu_2 c_2(1-m)\widetilde{\widetilde{y}} - \delta_5\widetilde{\widetilde{z}} - d_3 = 0. \quad (11)$$

From (11) we have

$$\widetilde{\widetilde{z}} = \frac{\mu_1 c_1(1-m)\widetilde{\widetilde{x}} + \mu_2 c_2(1-m)\widetilde{\widetilde{y}} - d_3}{\delta_5}$$

Let $\frac{\mu_1 c_1(1-m)}{\delta_5} = w_1$; $\frac{\mu_2 c_2(1-m)}{\delta_5} = w_2$; $\frac{d_3}{\delta_5} = w_3$.
Then,

$$\widetilde{\widetilde{z}} = w_1\widetilde{\widetilde{x}} + w_2\widetilde{\widetilde{y}} - w_3 \quad (12)$$

$$\text{exists if} \quad w_1\widetilde{\widetilde{x}} + w_2\widetilde{\widetilde{y}} > w_3 \quad (13)$$

Substituting $\widetilde{\widetilde{z}}$ in (10), we get,

$$\widetilde{\widetilde{y}} = \frac{[(\beta_1-\delta_4)-c_2(1-m)w_1]\widetilde{\widetilde{x}}+[c_2(1-m)w_3-d_2]}{c_2(1-m)w_2+\delta_3}.$$

Let $(\beta_1 - \delta_4) - c_2(1-m)w_1 = A_1$;
$c_2(1-m)w_3 - d_2 = A_2$;
$c_2(1-m)w_2 + \delta_3 = A_3$.

Therefore, $\widetilde{\widetilde{y}} = \frac{A_1\widetilde{\widetilde{x}}+A_2}{A_3}$ exists, provided that $A_1\widetilde{\widetilde{x}} + A_2 > 0$.

(12) can be written as

$$\widetilde{\widetilde{z}} = \left[w_1 + \frac{w_2 A_1}{A_3}\right]\widetilde{\widetilde{x}} + \frac{w_2 A_2}{A_3} - w_3.$$

Substitute $\widetilde{\widetilde{y}}$ and $\widetilde{\widetilde{z}}$ in (9), we get

$$\begin{aligned}&-[\delta_1 f\left(w_1 + \tfrac{w_2A_1}{A_3}\right) + \tfrac{(\delta_2+\beta_1)fA_1}{A_3}\left(w_1 + \tfrac{w_2A_1}{A_3}\right)\\ &+c_1 f(1-m)\left(w_1+\tfrac{w_2A_1}{A_3}\right)^2]\widetilde{\widetilde{x}}^2\\ &-[\delta_1 + \tfrac{(\delta_2+\beta_1)A_1}{A_3} + \delta_1 f\left(\tfrac{w_2A_2}{A_3} - w_3\right)\\ &+\tfrac{(\delta_2+\beta_1)fA_1}{A_3}\left(\tfrac{w_2A_2}{A_3}-w_3\right)\\ &+\tfrac{(\delta_2+\beta_1)fA_2}{A_3}\left(\tfrac{w_2A_1}{A_3}+w_1\right)\\ &+(d_1 f + c_1(1-m))\left(\tfrac{w_2A_1}{A_3}+w_1\right)\\ &+2c_1 f(1-m)\left(\tfrac{w_2A_1}{A_3}+w_1\right)\left(\tfrac{w_2A_2}{A_3}-w_3\right)]\widetilde{\widetilde{x}}\\ &+\alpha - [\tfrac{(\delta_2+\beta_1)A_2}{A_3} + d_1 + \tfrac{(\delta_2+\beta_1)fA_2}{A_3}\left(\tfrac{w_2A_2}{A_3}-w_3\right)\\ &+(d_1 f + c_1(1-m))\left(\tfrac{w_2A_2}{A_3}-w_3\right)\\ &+c_1 f(1-m)\left(\tfrac{w_2A_2}{A_3}-w_3\right)^2] = 0\end{aligned}$$

Using the sign rule of Descarte's, $\widetilde{\widetilde{x}}$ has a unique positive root if

$$\begin{aligned}&[\tfrac{(\delta_2+\beta_1)A_2}{A_3} + d_1 + \tfrac{(\delta_2+\beta_1)fA_2}{A_3}\left(\tfrac{w_2A_2}{A_3}-w_3\right)\\ &+(d_1 f + c_1(1-m))\left(\tfrac{w_2A_2}{A_3}-w_3\right)\\ &+c_1 f(1-m)\left(\tfrac{w_2A_2}{A_3}-w_3\right)^2] < \alpha.\end{aligned}$$

5 Analysis for Local Stability Behavior

Determining the local stability of the equilibrium points involves analyzing the behavior of the system near each equilibrium point. This analysis typically involves linearizing the system of differential equations around each equilibrium point and examining the eigenvalues of the resulting Jacobian matrix. The sign of the real parts of the eigenvalues indicates whether the equilibrium point is stable, unstable, or semi-stable. The necessary criteria for the system (1)-(3) to be stable locally at the equilibrium points are determined in this section using the Routh Hurwitz criterion, [22].

Theorem 5.1 *The trivial equilibrium point $f_0(0,0,0)$ is locally asymptotically stable for system (1)-(3) if $d_1 > \alpha$.*

Proof:
The Jacobin matrix at $f_0(0,0,0)$ is:

$$J_{(0,0,0)} = \begin{bmatrix} \alpha - d_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}$$

If $d_1 > \alpha$, then all the roots of the characteristic equation of $J_{(0,0,0)}$ are negative and thus the trivial equilibrium point is locally asymptotically stable.

Theorem 5.2 *The equilibrium point $f_1(\overline{x},0,0)$ is locally asymptotically stable for the system (1)-(3) if $\beta_1 - \delta_4 < \frac{d_2\delta_1}{\alpha-d_1}$ and $(\alpha-d_1)\mu_1 c_1(1-m) < d_3\delta_1$.*

Proof:
The Jacobin matrix at $f_1(\overline{x},0,0)$ is:
$J_{(\overline{x},0,0)} =$

$$\begin{bmatrix} -(\alpha-d_1) & -\frac{(\delta_2-\beta_1)(\alpha-d_1)}{\delta_1} & -\frac{(\alpha-d_1)[\alpha f+c_1(1-m)]}{\delta_1} \\ 0 & \frac{(\beta_1-\delta_4)(\alpha-d_1)}{\delta_1} - d_2 & 0 \\ 0 & 0 & \frac{(\alpha-d_1)\mu_1 c_1(1-m)}{\delta_1} - d_3 \end{bmatrix}$$

If $\beta_1 - \delta_4 < \frac{d_2\delta_1}{\alpha-d_1}$ and $(\alpha-d_1)\mu_1 c_1(1-m) < d_3\delta_1$, then all the roots of the characteristic equation of $J_{(\overline{x},0,0)}$ are negative. So, the equilibrium point $f_1(\overline{x},0,0)$ is locally asymptotically stable.

Theorem 5.3 *The equilibrium point $f_2\left(\overline{\overline{x}},\overline{\overline{y}},0\right)$ is locally asymptotically stable for the system (1)-(3) if it satisfies the condition*

$$(\alpha - d_1) < 2\delta_1\overline{\overline{x}} + (\delta_2+\beta_1)\overline{\overline{y}}, \quad (14)$$

$$(\beta_1-\delta_4)\overline{\overline{x}} < 2\delta_3\overline{\overline{y}} + d_2, \quad (15)$$

$$\mu_1 c_1(1-m)\overline{\overline{x}} + \mu_2 c_2(1-m)\overline{\overline{y}} < d_3. \quad (16)$$

Proof:
The Jacobin matrix at $f_2\left(\overline{\overline{x}},\overline{\overline{y}},0\right)$ is:

$$J_{(\overline{\overline{x}},\overline{\overline{y}},0)} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

where

$c_{11} = (\alpha - d_1) - 2\delta_1\overline{\overline{x}} - (\delta_2 + \beta_1)\overline{\overline{y}};$
$c_{12} = -(\delta_2 + \beta_1)\overline{\overline{x}};$
$c_{13} = -[\alpha f + c_1(1-m)]\overline{\overline{x}};$
$c_{21} = (\beta_1 - \delta_4)\overline{\overline{y}};$
$c_{22} = (\beta_1 - \delta_4)\overline{\overline{x}} - 2\delta_3\overline{\overline{y}} - d_2;$
$c_{23} = -c_2(1-m)\overline{\overline{y}};$
$c_{31} = 0; \quad c_{32} = 0;$
$c_{33} = \mu_1 c_1(1-m)\overline{\overline{x}} + \mu_2 c_2(1-m)\overline{\overline{y}} - d_3.$

The characteristic equation of $J_{(\overline{\overline{x}},\overline{\overline{y}},0)}$ is

$$(c_{33} - \lambda)[\lambda^2 - (c_{11} + c_{22})\lambda + c_{11}c_{22} - c_{12}c_{21}] = 0. \quad (17)$$

$$\text{ie., } (c_{33} - \lambda) = 0, \quad (18)$$

$$\text{and } \lambda^2 - (c_{11} + c_{22})\lambda + c_{11}c_{22} - c_{12}c_{21} = 0. \quad (19)$$

From (18) $\lambda = c_{33} < 0$ provided $\mu_1 c_1(1-m)\overline{\overline{x}} + \mu_2 c_2(1-m)\overline{\overline{y}} < d_3$.

Using Routh-Hurwitz criterion (19) has negative roots only if conditions (14) and (15) are satisfied. Henceforth, the theorem follows.

Theorem 5.4 *The equilibrium point $f_3(\widetilde{x}, 0, \widetilde{z})$ is locally asymptotically stable for the system (1)-(3) if it satisfies the condition:*

$$\frac{1}{1 + f\widetilde{z}} < 2\delta_1\widetilde{x} + d_1 + c_1(1-m)\widetilde{z}, \quad (20)$$

$$\widetilde{x} < min\left\{\frac{d_2 + c_2(1-m)\widetilde{z}}{\beta_1 - \delta_4}, \frac{2\delta_5\widetilde{z} + d_3}{\mu_1 c_1(1-m)}\right\}. \quad (21)$$

Proof:
The Jacobin matrix at $f_3(\widetilde{x}, 0, \widetilde{z})$ is given by

$$J_{(\widetilde{x},0,\widetilde{z})} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

where
$c_{11} = \frac{\alpha}{1+f\widetilde{z}} - 2\delta_1\widetilde{x} - c_1(1-m)\widetilde{z};$
$c_{12} = -(\delta_2 + \beta_1)\widetilde{x} < 0;$
$c_{13} = -\left[\frac{\alpha f}{(1+f\widetilde{z})^2} + c_1(1-m)\right]\widetilde{x} < 0;$
$c_{21} = 0;\ c_{22} = (\beta_1 - \delta_4)\widetilde{x} - d_2 - c_2(1-m)\widetilde{z};$
$c_{23} = 0;\ c_{31} = \mu_1 c_1(1-m)\widetilde{z} > 0;$
$c_{32} = \mu_2 c_2(1-m)\widetilde{z} > 0;$
$c_{33} = \mu_1 c_1(1-m)\widetilde{x} - 2\delta_5\widetilde{z} - d_3.$

The characteristic equation of $J_{(\widetilde{x},0,\widetilde{z})}$ is

$$(c_{22} - \lambda)[\lambda^2 - (c_{11} + c_{33})\lambda + c_{11}c_{33} - c_{13}c_{31}] = 0. \quad (22)$$

$$\text{ie., } (c_{22} - \lambda) = 0, \quad (23)$$

$$\text{and } \lambda^2 - (c_{11} + c_{33})\lambda + c_{11}c_{33} - c_{13}c_{31} = 0. \quad (24)$$

From (22) $\lambda = c_{22} < 0$ if condition (21) is met.
By Routh-Hurwitz criterion (24) has negative roots only if conditions (20) and (21) are satisfied.
Henceforth, the theorem follows.

Theorem 5.5 *The equilibrium point $f_4\left(\widetilde{\widetilde{x}}, \widetilde{\widetilde{y}}, \widetilde{\widetilde{z}}\right)$ is locally asymptotically stable for the system (1)-(3) if it satisfies the condition:*

$$\frac{\alpha}{1 + f\widetilde{\widetilde{z}}} < 2\delta_1\widetilde{\widetilde{x}} + d_1 + (\delta_2 + \beta_1)\widetilde{\widetilde{y}} + c_1(1-m)\widetilde{\widetilde{z}}, \quad (25)$$

$$(\beta_1 - \delta_4)\widetilde{\widetilde{x}} < 2\delta_3\widetilde{\widetilde{y}} + d_2 + c_2(1-m)\widetilde{\widetilde{z}}, \quad (26)$$

$$\mu_1 c_1(1-m)\widetilde{\widetilde{x}} + \mu_2 c_2(1-m)\widetilde{\widetilde{y}} < 2\delta_5\widetilde{\widetilde{z}} + d_3, \quad (27)$$

$$\phi_1 < \phi_2, \quad (28)$$

$$\tau_1 + \tau_2 > 0. \quad (29)$$

Proof:
The Jacobin matrix at $f_4\left(\widetilde{\widetilde{x}}, \widetilde{\widetilde{y}}, \widetilde{\widetilde{z}}\right)$ is:

$$J_{(\widetilde{\widetilde{x}},\widetilde{\widetilde{y}},\widetilde{\widetilde{z}})} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

where
$c_{11} = \frac{\alpha}{1+f\widetilde{\widetilde{z}}} - 2\delta_1\widetilde{\widetilde{x}} - d_1 - (\delta_2 + \beta_1)\widetilde{\widetilde{y}} - c_1(1-m)\widetilde{\widetilde{z}};$
$c_{12} = -(\delta_2 + \beta_1)\widetilde{\widetilde{x}} < 0;$
$c_{13} = -\left[\frac{\alpha f}{(1+f\widetilde{\widetilde{z}})^2} + c_1(1-m)\right]\widetilde{\widetilde{x}} < 0;$
$c_{21} = (\beta_1 - \delta_4)\widetilde{\widetilde{y}} > 0;$
$c_{22} = (\beta_1 - \delta_4)\widetilde{\widetilde{x}} - 2\delta_3\widetilde{\widetilde{y}} - d_2 - c_2(1-m)\widetilde{\widetilde{z}};$
$c_{23} = -c_2(1-m)\widetilde{\widetilde{y}} < 0;$
$c_{31} = \mu_1 c_1(1-m)\widetilde{\widetilde{z}} > 0; \quad c_{32} = \mu_2 c_2(1-m)\widetilde{\widetilde{z}} > 0;$
$c_{33} = \mu_1 c_1(1-m)\widetilde{\widetilde{x}} + \mu_2 c_2(1-m)\widetilde{\widetilde{y}} - 2\delta_5\widetilde{\widetilde{z}} - d_3.$

The characteristic equation of $J_{(\widetilde{\widetilde{x}},\widetilde{\widetilde{y}},\widetilde{\widetilde{z}})}$ is given by

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \quad (30)$$

Here

$$a_1 = -[c_{11} + c_{22} + c_{33}],$$
$$a_2 = c_{22}c_{33} + c_{11}c_{33} + c_{11}c_{22} - [c_{23}c_{32} + c_{31}c_{13} + c_{21}c_{12}],$$
$$a_3 = -c_{11}c_{22}c_{33} + c_{11}c_{32}c_{23} + c_{12}c_{21}c_{33} - c_{12}c_{31}c_{23} - c_{13}c_{21}c_{32} + c_{13}c_{31}c_{22}.$$

Let

$$\phi_1 = c_{11}c_{32}c_{23} + c_{12}c_{21}c_{33} + c_{13}c_{31}c_{22} - (c_{13}c_{21}c_{32} + c_{11}c_{22}c_{33}) > 0,$$
$$\phi_2 = c_{12}c_{31}c_{23} > 0.$$

From conditions (25)-(27), $a_1 > 0, a_2 > 0$.
Condition (28) gives $a_3 > 0$.
Now,
$a_1a_2 - a_3 = -c_{11}^2[c_{22} + c_{33}] + c_{13}c_{31}[c_{11} + c_{33}] + c_{12}c_{21}[c_{11} + c_{22}] - c_{22}^2[c_{11} + c_{33}] - 2c_{11}c_{22}c_{33} + c_{23}c_{32}[c_{22} + c_{33}] - c_{33}^2[c_{11} + c_{22}] + c_{12}c_{31}c_{23} + c_{13}c_{21}c_{32}$.
Let
$\tau_1 = -c_{11}^2[c_{22}+c_{33}]+c_{13}c_{31}[c_{11}+c_{33}]+c_{12}c_{21}[c_{11}+c_{22}] - c_{22}^2[c_{11} + c_{33}] - 2c_{11}c_{22}c_{33} + c_{23}c_{32}[c_{22} + c_{33}] - c_{33}^2[c_{11} + c_{22}] + c_{12}c_{31}c_{23} > 0$;
$\tau_2 = c_{13}c_{21}c_{32} < 0$.

If $\tau_1 + \tau_2 > 0$, then $a_1a_2 - a_3 > 0$.
Then by Routh-Hurwitz criterion, all the roots of (30) are negative. Henceforth, the theorem follows.

6 Conclusion

A predator prey population model comprising healthy prey, infected prey and predator is developed integrating fear effect and refuge factors of the prey. The disease is transmitted from infected to susceptible prey with linear incidence rate. The predator feeds on both the susceptible and infected prey following Volterra type predation. The positivity and boundedness of the solution demonstrate that the developed system behaves well biologically. Five equilibrium points are located. Conditions for the existence of the equilibrium points are elaborately discussed. Analyzing the stability of each equilibrium point allows understanding of how the system responds to small perturbations around that point. This information is crucial for predicting the long-term behavior of the ecological system and understanding its resilience to external factors. Furthermore, distinct requirements are determined for the system's local stability at each of the equilibrium points.

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