# Average Run Length Computations of Autoregressive and Moving Average Process using the Extended EWMA Procedure

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*Abstract:* - In the past, the control chart served as a statistical tool for detecting process changes. The Exponentially Weighted Moving Average (EWMA) control chart is highly effective for detecting small changes. This research introduces the Extended Exponentially Weighted Moving Average (Extended EWMA) control chart for the Autoregressive and Moving average process with order $p$ = 1 and $q$ = 1 (ARMA(1,1)) The Extended EWMA control chart incorporates two smoothing parameters ( $\lambda_1$ and $\lambda_2$ ) derived from the EWMA control chart. A comparative analysis of the performance of the EWMA control chart. The Average Run Length (ARL) value as determined by the explicit formulas in this research, serves as a metric for evaluating the performance of the Extended EWMA control chart and the EWMA control chart. The Numerical Integral Equation (NIE) method is used to verify the accuracy of the ARL for the explicit formulas of the two control charts which has not been before discovered. The effectiveness of control charts can also be evaluated by analyzing SDRL, ARL, MRL, RMI, AEQL, and PCI values as metrics for various design parameter values. After analyzing the results of the ARL and all five performance meters, it was determined that the Extended EWMA control chart is better than the EWMA control chart at all shift sizes of process changes. Finally, the assessment of the ARMA process is being conducted to evaluate the ARL using a dataset on PM2.5 dust levels in Bangkok, Thailand during January and February of 2024.

*Key-Words:* - Average Run Length, Autoregressive and Moving Average Process, Extended Exponentially Weighted Moving Average control chart, Explicit Formula, Numerical Integral Equation, ARL, NIE method.

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## 1 Introduction

The statistically important tool is the control chart. [1], invented the control chart, which detects the alteration of the control diagram, which is commonly used in the manufacturing industry. The Shewhart control chart is very efficient with small change detection as well. [2], introduced an EWMA control chart for better detection of small changes. The Cumulative Sum (CUSUM) control chart, developed by [3], is extensively utilized in statistical control charting. In 2017, [4], developed and presented a more efficient, Modified EWMA control chart than the EWMA control chart in the detection of minor changes. [5], presented The Extended EWMA control chart as a powerful chart designed to detect minor changes in the process being examined. The effectiveness of control charts can be assessed by utilizing the ARL, [6]. The ARL is divided into two values, [7], The $ARL_0$ is the number of expected observations required before a process is under control, and The $ARL_1$ refers to the amount of observations expected from an uncontrollable process and should be reduced. In 1990, [8], are presented a quantitative analysis comparing the EWMA control chart and the CUSUM control chart. [9], examined the design of the optimal EWMA control chart and compared it to the CUSUM control chart. [10], are presented, and the autocorrelation data will be linked to a statistical model. The ARMA process is a frequently employed model in real-world data analysis. [11], have introduced datasets with exponential white noise to the control chart. [12], shows the performance of the CUSUM control chart for autocorrelated seasonal consistency of trends using the Midpoint Rule method with exponential white noise. [13], presented an approximating of the ARL of changes in the mean of a Seasonal Time Series Model with exponential white noise running on a CUSUM control chart. [14], presented the explicit formulas and NIE of ARL when observations were seasonal autoregressive models with an exogenous variable $SARX(P,r)_L$ with exponential white noise based on the CUSUM control chart. [15], proposed an explicit formula for the ARL using the Fredholm integral equation method in the EWMA control chart on the MAX(q,r) process. [16], are improving the CUSUM control chart for monitoring a change in processes based on a seasonal autoregressive model with one exogenous variable. [17], introduced a Modified EWMA control chart that is derived from its particular examples. [18], developed the explicit formula for the ARL on a Modified EWMA control chart for the AR(1) process. Subsequently, [19], presents analytical explicit formulas of the ARL of Homogenously Weighted Moving Average control chart (HEWMA) based on a MAX process.

However, the derivation of the explicit formula for the ARL on the Extended EWMA control chart for the Autoregressive and Moving Average process with parameters $p = 1$ and $q = 1$ (ARMA(1,1)), when there are two smoothing parameters ($\lambda_1$ and $\lambda_2$) for the ARMA(1,1) process has not been reported previously. The objective of this research is to determine the explicit formula for the ARL on the Extended EWMA control chart for the ARMA(1,1). The explicit formula for the ARL was compared with the NIE method. This research differs from that of other researchers, using five additional statistical performance measurements, Standard of Deviation Run Length (SDRL), Median Run Length (MRL), [20], Related Mean Index (RMI), [21], Average Extra Quadratic Loss (AEQL), [22] and Performance Comparison Index (PCI). The comparison of performance between the Extended EWMA control chart and the EWMA control chart using a dataset on PM2.5 dust levels in Bangkok, Thailand during January and February of 2024.

# 2 Materials and Methods

## 2.1 The Exponential Weighted Moving Average Control Chart

This research describes the properties of the EWMA control chart for ARMA(p,q) processes. The EWMA control chart is defined by a recursive equation, [2].

$$Z_t = (1-\lambda)Z_{t-1} + \lambda W_t, t = 1,2,3,... \tag{1}$$

where $W_t$ is a sequence of an ARMA(p,q) processes with exponential white noise and $0 < \lambda \le 1$, $W_0$ is the initial value of the EWMA statistics, $Z_0 = u$. The control limits of the EWMA control chart consist of the upper control limit (UCL) and the lower control limit (LCL) are:

$$\begin{aligned} UCL &= \mu_0 + \gamma\sigma\sqrt{\frac{\lambda}{2-\lambda}} \\ LCL &= \mu_0 - \gamma\sigma\sqrt{\frac{\lambda}{2-\lambda}} \end{aligned} \tag{2}$$

where $\mu_0$ is process mean of ARMA(p,q) process, $\sigma$ is the process standard deviation parameter, $\gamma$ is a suitable control limit width. Let $b$ be an UCL on $Z_t$. The stopping time $(\tau_b)$ of the Extended EWMA control chart is defined as

$$\tau_b = \inf\{t > 0; Z_t \ge b\}$$

# 3 The Exact Solutions of the ARL on the Extended EWMA Control Chart

## 3.1 The Explicit Formula of the ARL on the Extended EWMA Control Chart for ARMA(p,q) process

The ARMA(p,q) process are defined by the following recursion:

$$\begin{aligned} W_t = {} & \phi_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + ... + \phi_p W_{t-p} \\ & + \vartheta_t - \theta_t \vartheta_{t-1} - \theta_2 \vartheta_{t-2} - ... - \theta_q \vartheta_{t-q} \end{aligned} \tag{3}$$

where $W_t$ is a sequence of the ARMA(p,q) processes with exponential white noise, $\phi_t$ is autoregressive parameter, $W_0 = u$ is the initial value, where $u = [0,b]$ and $b$ is UCL of the Extended EWMA control chart

[6], presented The Extended EWMA control chart. The performance control chart is highly efficient in detecting small changes in the monitored process. The Extended EWMA statistic can be derived as:

$$\omega_t = (1-\lambda_1+\lambda_2)\omega_{t-1} + \lambda_1 W_t - \lambda_2 W_{t-1}, t = 1,2,3,... \tag{4}$$

where $W_t$ is a process with mean, $\lambda_1$ and $\lambda_2$ are exponential smoothing parameters with

$0 \le \lambda_2 < \lambda_1 < 1$ and $\omega_0$ is the initial value of the Extended EWMA statistics, $\omega_0 = u$ and $\vartheta_0 = v$. The UCL and the LCL are:

$$UCL = \mu_0 + \gamma\sigma\sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}$$
$$LCL = \mu_0 - \gamma\sigma\sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}, \tag{5}$$

where $\mu_0$ is the process mean of moving average process, $\sigma$ is the process standard deviation parameter and $\gamma$ is a suitable control limit width, and

$$W_t = \phi_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \ldots + \phi_p W_{t-p} + \vartheta_t - \theta_t\vartheta_{t-1} - \theta_2\vartheta_{t-2} - \ldots - \theta_q\vartheta_{t-q} \tag{6}$$

Hence, the formulation of the Extended EWMA control chart for the ARMA(p,q) process is as.

$$\omega_t = (1-\lambda_1+\lambda_2)\omega_{t-1} + \lambda_1(\phi_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \ldots + \phi_p W_{t-p} + \vartheta_t - \theta_t\vartheta_{t-1} - \theta_2\vartheta_{t-2} - \ldots - \theta_q\vartheta_{t-q}) - \lambda_2 W_{t-1}, t = 1,2,3,\ldots$$

When $t = 1$

$$\omega_1 = (1-\lambda_1+\lambda_2)\omega_0 + \lambda_1(\phi_0 + \phi_1 W_0 + \phi_2 W_{-1} + \ldots + \phi_p W_{1-p} - \theta_1\vartheta_0 - \theta_2\vartheta_{-1} - \ldots - \theta_q\vartheta_{1-q}) - \lambda_2 W_0 + \vartheta_1\lambda_1, t = 1,2,3,\ldots$$

Let $\Delta = \phi_0 + \phi_1 W_0 + \phi_2 W_{-1} + \ldots + \phi_p W_{1-p} - \theta_1\vartheta_0 - \theta_2\vartheta_{-1} - \ldots - \theta_q\vartheta_{1-q}$

So

$$\omega_1 = (1-\lambda_1+\lambda_2)\omega_0 + \lambda_1\Delta - \lambda_2 u + \vartheta_1\lambda_1, t = 1,2,3,\ldots \tag{7}$$

Let's examine the in-control process, where the UCL = $b$ and the LCL = 0.

$$0 < \omega_1 < b$$
$$0 < (1-\lambda_1+\lambda_2)\omega_0 + \lambda_1\Delta - \lambda_2 u + \vartheta_1\lambda_1 < b$$
$$0 < \vartheta_1\lambda_1 < b - (1-\lambda_1+\lambda_2)\omega_0 - \lambda_1\Delta + \lambda_2 u$$
$$0 < \vartheta_1 < \frac{b - (1-\lambda_1+\lambda_2)\omega_0 - \lambda_1\Delta + \lambda_2 u}{\lambda_1}$$
$$0 < \vartheta_1 < \frac{b - (1-\lambda_1+\lambda_2)u - \lambda_1\Delta + \lambda_2 u}{\lambda_1}$$

The function $\rho(u)$ can be obtained by Fredholm integral equation of the second kind as follows;

$$\rho(u) = 1 + \int_0^b \rho(\omega_1) f(\vartheta) d\vartheta \tag{8}$$

$$\rho(u) = 1 + \int_0^b \rho((1-\lambda_1+\lambda_2)\omega_0 + \lambda_1\Delta - \lambda_2 u + \vartheta_1) f(\vartheta) d\vartheta$$

Therefore, the function $\rho(u)$ is obtained as follows:

$$\rho(u) = 1 + \frac{1}{\lambda_1}\int_0^b \rho(k) f\left(\frac{b - (1-\lambda_1+\lambda_2)u - \lambda_1\Delta + \lambda_2 u + \vartheta_1}{\lambda_1}\right) dk$$

Given that $\varepsilon_t \sim Exp(\alpha)$ is determined,

$$f(k) = \frac{1}{\alpha} e^{-\frac{k}{\alpha}}$$

$$\rho(u) = 1 + \frac{e^{\frac{b-(1-\lambda_1+\lambda_2)u-\lambda_1\Delta+\lambda_2 u}{\lambda_1} + \frac{\vartheta_1}{\alpha}}}{\lambda_1\alpha}\int_a^b \rho(k) e^{-\frac{k}{\lambda_{1\alpha}}} dk$$

Setting $\Gamma(u) = e^{\frac{b-(1-\lambda_1+\lambda_2)u-\lambda_1\Delta+\lambda_2 u}{\lambda_1} + \frac{\vartheta_1}{\alpha}}$ and $\Pi = \int_0^b \rho(k) e^{-\frac{k}{\lambda_1\alpha}} dk$

$$\text{Thus } \rho(u) = 1 + \frac{\Gamma(u)}{\lambda_1\alpha}\Pi \tag{9}$$

Consider $\Pi$ and take run $\rho(k)$

$$\Pi = \frac{-\lambda_1\alpha(e^{-\frac{b}{\lambda_1\alpha}} - 1)}{1 + \frac{1}{(\lambda_1-\lambda_2)} e^{\frac{b-(1-\lambda_1+\lambda_2)u-\lambda_1\Delta+\lambda_2 u}{\lambda_1\alpha} + \frac{\vartheta_1}{\alpha}} (e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_{1\alpha}}} - 1)} \tag{10}$$

By substituting the value of an equal to $\alpha_1$ into $\rho(u)$, we can get the explicit formula of the ARL$_1$ on the Extended EWMA control chart as follows:

$$ARL_1 = 1 - \frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_1}}(e^{-\frac{b}{\lambda_1\alpha_1}} - 1)}{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u+\lambda_1\Delta-\lambda_2 u}{\lambda_1\alpha} + \frac{\vartheta_1}{\alpha}} + (e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha_1}} - 1)}$$

$$ARL_1 = 1 - \frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_1}}(e^{-\frac{b}{\lambda_1\alpha_1}} - 1)}{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u+\lambda_1\Delta-\lambda_2 u}{\lambda_1\alpha_1} + \frac{\vartheta_1}{\alpha_1}} + (e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha_1}} - 1)}$$

$$ARL_1 = 1 - \frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_1}}(e^{-\frac{b}{\lambda_1\alpha_1}} - 1)}{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u+\lambda_1(\phi_0+\phi_1 W_0+\phi_2 W_{-1}+\ldots+\phi_p W_{1-p}-\theta_1\vartheta_0-\theta_2\vartheta_{-1}-\ldots-\theta_q\vartheta_{1-q})-\lambda_2 u}{\lambda_1\alpha} + \frac{\vartheta_1}{\alpha}} + (e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha_1}} - 1)} \tag{11}$$

From Equation (11), the ARL finished explicit formula of the Extended EWMA control chart for the ARMA(p,q) process is to be compared to the EWMA control chart.

### 3.2 The Numerical Integral Equation of the ARL on the Extended EWMA Control Chart of ARMA(p,q) process.

Let $\psi(u)$ denote the estimated value of the ARL determined from the $m$ linear equation systems using the composite midpoint quadrature rule.

The evaluation of the ARL approaching NIE on the Extended EWMA control chart is carried out in the following manner:

$$\psi(u) = \int_0^b \rho(\omega_t) f(k) dk \approx \sum_{j=1}^{s} c_j f\left(x_j\right) \tag{12}$$

The system of $s$ linear equations is represented as

$L_{s\times1} = 1_{s\times1} + R_{s\times s} L_{s\times1}$ or $L_{s\times1} = \left(I_s - R_{s\times s}\right)^{-1} 1_{s\times1}$

$L_{s\times1} = \left[L_{NIE}\left(x_1\right), L_{NIE}\left(x_1\right), ..., L_{NIE}\left(x_s\right)\right]^T$,

$I_s = diag\left(1,1,...,1\right)$ and $1_{s\times1} = \left[1,1,...,1\right]^T$.

Let $R_{s\times s}$ be a matrix. The $s$ to $s^{th}$ element matrix $R$ is defined as follows:

$$\left[R_{ij}\right] \approx \frac{1}{\lambda_1} c_j \, f\left(\frac{k_j - (1-\lambda_1+\lambda_2)u + \lambda_1(\phi_0 + \phi_1 W_0 + \phi_2 W_{-1} + ... + \phi_p W_{1-p} - \theta_1 \vartheta_0 - \theta_2 \vartheta_{-1} - ... - \theta_q \vartheta_{1-q}) - \lambda_2 u + \vartheta_1}{\lambda_1}\right)$$

The answer to the NIE can be succinctly expressed as.

$$\psi(u) = 1 + \frac{1}{\lambda_1} \sum_{j=1}^{s} c_j \, f\left(\frac{k_j - (1-\lambda_1+\lambda_2)u + \lambda_1(\phi_0 + \phi_1 W_0 + \phi_2 W_{-1} + ... + \phi_p W_{1-p} - \theta_1 \vartheta_0 - \theta_2 \vartheta_{-1} - ... - \theta_q \vartheta_{1-q}) - \lambda_2 u + \vartheta_1}{\lambda_1}\right) \tag{13}$$

where $k_j$ is a set of the division point on the interval $[0,b]$ as $k_j = \left(j - \frac{1}{2}\right) c_j, j = 1,2,...,s.$ $c_j$ is a weight of composite midpoint formula $c_j = \frac{b}{s}.$

From Equation (13), the NIE method is a comparative criterion to the explicit formulas that the explicit formula is accurate. As a result, both ARL values are similar.

# 4 Existence and Uniqueness of ARL

The answer is obtained from the explicit formula using the ARL of the existence of the NIE, as proven by Banach's fixed-point theorem, [23]. In this study, let $T$ denote an operation on the set of all continuous functions that are defined.

$$T(\rho(u)) = 1 + \frac{1}{\lambda_1} \sum_{j}^{s} c_j \, f\left(\frac{k_j - (1-\lambda_1+\lambda_2)u + \lambda_1(\phi_0 + \phi_1 W_0 + \phi_2 W_{-1} + ... + \phi_p W_{1-p} - \theta_1 \vartheta_0 - \theta_2 \vartheta_{-1} - ... - \theta_q \vartheta_{1-q}) - \lambda_2 u + \vartheta_1}{\lambda_1}\right) dk \tag{14}$$

According to Banach's fixed-point theorem, if an operator $T$ is guaranteed to satisfy the criterion of being a contraction, $T(\rho(u)) = \rho(u)$ has a one solution, as previously stated. For Equation (14) to have a solution that is both present and unique, the Banach fixed-point theorem can be utilized. The Banach fixed-point theorem, also referred to as the contraction mapping theorem, was initially introduced concretely in Banach's.

Typically, it is employed to determine the existence of a solution to an integral problem. Following this, [24], have extensively utilized this tool to address numerous problems related to the presence of solutions in diverse mathematical domains, due to its straightforwardness and practicality. The following information provides the specific details.

**Theorem 1 Banach's Fixed-point Theorem :** Assume that $D: X \to X$ is a contraction mapping with contraction constant $0 \le S < 1$, such that $\left\|D\left(\rho_1\right) - D(\rho_2)\right\| \le S\left\|\rho_1 - \rho_2\right\| \; \forall \rho_1, \rho_2 \in X$, meets this criterion. [25], have established the existence of a single unique a. $\rho(.) \in X$ such that $D(\rho(u)) = \rho(u)$ has a unique fixed point in $X$.

**Proof:** To demonstrate the value of $T$, as determined by the equation $T(\rho(u))$ is a contraction mapping for $\rho_1, \rho_2 \in \Omega[0,b]$. that $\left\|D\left(\rho_1\right) - D(\rho_2)\right\| \le S\left\|\rho_1 - \rho_2\right\|$, $\forall \rho_1, \rho_2 \in \Omega[0,b]$. with $0 \le S < 1$ under the norm $\left\|\rho_\infty\right\| \le \sup_{u\in[0,b]} \left\|\rho(u)\right\|$ From $\rho(u)$ and $D(\rho(u))$.

$$\left\|D\left(\rho_1\right) - D(\rho_2)\right\|_\infty$$

$$= \sup_{u\in[0,b]} \left| \frac{1}{\lambda_1 \alpha} e^{\left(\frac{k_j - (1-\lambda_1+\lambda_2)u + \lambda_1(\phi_0 + \phi_1 W_0 + \phi_2 W_{-1} + ... + \phi_p W_{1-p} - \theta_1 \vartheta_0 - \theta_2 \vartheta_{-1} - ... - \theta_q \vartheta_{1-q}) - \lambda_2 u + \vartheta_1}{\lambda_1}\right)} \int_0^b (\rho_1(k) - \rho_2(k)) e^{-\frac{b}{\lambda_1 \alpha}} dk \right|$$

$$\le \sup_{u\in[0,b]} \left| \frac{1}{\lambda_1 \alpha} e^{\left(\frac{k_j - (1-\lambda_1+\lambda_2)u + \lambda_1(\phi_0 + \phi_1 W_0 + \phi_2 W_{-1} + ... + \phi_p W_{1-p} - \theta_1 \vartheta_0 - \theta_2 \vartheta_{-1} - ... - \theta_q \vartheta_{1-q}) - \lambda_2 u + \vartheta_1}{\lambda_1}\right)} \int_0^b (\rho_1(k) - \rho_2(k)) e^{-\frac{b}{\lambda_1 \alpha}} dk \right|$$

$$= \|\rho_1 - \rho_2\|_\infty \sup_{u\in[0,b]} \left| e^{\left(\frac{k_j-(1-\lambda_1+\lambda_2)u+\lambda_1(\phi_0+\phi_1W_0+\phi_2W_{-1}+...+\phi_pW_{1-p}-\theta_1\vartheta_0-\theta_2\vartheta_{-1}-...-\theta_q\vartheta_{1-q})-\lambda_2u+\vartheta_1}{\lambda_1}\right)} \right|$$

$$\left|1-\left[e^{-\frac{b}{\lambda_1\alpha}}-1\right]\right| \le S\|\rho_1-\rho_2\|_\infty$$

where

$$S = \sup_{u\in[0,b]} \left| e^{\left(\frac{k_j-(1-\lambda_1+\lambda_2)u+\lambda_1(\phi_0+\phi_1W_0+\phi_2W_{-1}+...+\phi_pW_{1-p}-\theta_1\vartheta_0-\theta_2\vartheta_{-1}-...-\theta_q\vartheta_{1-q})-\lambda_2u+\vartheta_1}{\lambda_1}\right)} \right|$$

$$\left|1-\left[e^{-\frac{b}{\lambda_1\alpha}}-1\right]\right|$$

, $0 \le S < 1,$ The uniqueness of the solution is ensured by Banach's fixed-point theorem.

# 5 Numerical Results

In this research, we evaluate the $ARL_0$ and $ARL_1$ by employing explicit formulas and the NIE for an ARMA(p,q) process on the Extended EWMA control chart. In addition, performance indicators such as the SDRL and MRL are used to assess the effectiveness of control charts. The computation for SDRL and MRL for the in-control process is as follows.

$$ARL_0 = \frac{1}{\beta_0}, SDRL_0 = \sqrt{\frac{1-\beta_0}{(\beta_0)^2}}, MRL_0 = \frac{\log(\frac{1}{2})}{\log(1-\beta_0)} \tag{15}$$

where $\beta_0$ represents an error of type I. This research analysis determined that $ARL_0$ = 370. The value of the $ARL_0$ can be computed using Equation (15) as $SDRL_0$ and $MRL_0$ with an approximate value. Conversely, $ARL_1$, $SDRL_1$ and $MRL_1$ are calculated using Equation (16).

$$ARL_1 = \frac{1}{\beta_1}, SDRL_1 = \sqrt{\frac{1-\beta_1}{(\beta_1)^2}}, MRL_1 = \frac{\log(\frac{1}{2})}{\log(1-\beta_1)} \tag{16}$$

where $\beta_1$ represents an error of type II

The minimum values of the $ARL_1$, $SDRL_1$ and $MRL_1$ indicate a higher ability to promptly detect variations in the process mean. To conduct a comparison analysis, we will examine the Extended EWMA control charts and the EWMA control charts for the ARMA(p,q) process.

RMI is employed to assess the efficacy of the Extended EWMA control chart. RMI can be computed.

$$RMI = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{ARL_i(MAX)-ARL_i(MIN)}{ARL_i(MIN)}\right], \tag{17}$$

where $ARL_i(MAX)$ is the $ARL_i$ of row $i$ on the control chart under examination. $ARL_i(MIN)$ is the minimum of the $ARL_1$ for row $i$. A control chart is deemed more effective when it has a lower RMI.

Furthermore, performance measurements can be utilized to evaluate the effectiveness of control charts across a range of modifications. In addition, the AEQL may pertain to costs that have been accrued as a result of an unmanageable situation. This comparison may entail the utilization of various control chart kinds to determine the most efficient strategy for a specific procedure. These include the study model, the research data set, the appropriate parameter value, the control chart that the research is introduced, as well as the application of the actual data to make the chart of this research result as desired.

The AEQL can be determined by using the following formula.

$$AEQL = \frac{1}{\Phi}\sum_{shift_i=shift_{\min}}^{shift_{\max}}(shift_i^{\,2} \times ARL(shift_i)) \tag{18}$$

where *shift* refers to a distinct change in the process. $\Phi$ denotes the aggregate of number of divisons from $shift_{\min}$($\delta_{\min}$) to $shift_{\max}$($\delta_{\max}$). In this research, $\Phi = 10, \delta_{\min} = 0.01$ and $\delta_{\max} = 3.00$. The most effective control chart is the one with the minimum AEQL value.

Additionally, the examination of control chart performance can be carried out by utilizing the performance evaluation criteria of the PCI. The determination of the PCI value entails comparing the AEQL of a certain control chart to the AEQL of the control chart with the minimum value. This helps identify the control chart that has the highest level of efficiency. The PCI can be computed:

$$PCI = \frac{AEQL}{AEQL_{\min}} \tag{19}$$

The ARL was approximated by NIE using the composite midpoint rule on the Extended EWMA control chart for the ARMA(p,q) process with a sample size of 1,000 nodes. When $ARL_0$ = 370, $\phi_0$ = 0.5, $\phi_1$ = 0.1, $\theta_1$ = -0.1 and 0.1, $\theta_2$ = -0.2

and 0.2, $\theta_3$ = 0.3, $\lambda_1$ = 0.05, 0.10, $\lambda_2$ = 0.01, and $\delta$ = 0.01, 0.03, 0.05, 0.10, 0.30, 0.50, 1.00, 2.00 and 3.00, The in-control process is $\alpha_0 = 1$. The results indicate that the ARL of the explicit formulas and the NIE are very similar for the ARMA(1,1), ARMA(1,2), and ARMA(1,3) processes in Table 1, Table 2 and Table 3 in Appendix. The results of the ARL of the Extended EWMA control chart using an explicit formula are shown in Table 4, Table 5 and Table 6 in Appendix. These Tables compare the performance of the Extended EWMA control chart against the EWMA control chart for ARMA(1,1), ARMA(1,2), and ARMA(1,3) processes. Based on the results, the Extended EWMA control chart outperformed the EWMA control chart in terms of the ARL, SDRL, and MRL for $\lambda_1$ values of 0.05 and 0.10. In addition, the results suggest that the Extended EWMA control chart with $\lambda_1$ = 0.10 exhibits the minimum values for RMI, AEQL, and PCI.

Therefore, it can be inferred that the Extended EWMA control chart exhibits greater performance when compared to the EWMA control chart. Moreover, the RMI, AEQL, and PCI derived from each control chart are utilized to assess the effectiveness of the aforementioned control charts. The Extended EWM control chart exhibited superior performance. Based on the minimal values for RMI, AEQL, and PCI, all of them were equal to one.

# 6 Application to Real-world Data

In this study, the explicit formulas of the ARL on the Extended EWMA control chart for the ARMA(1,1) prcess were applied to the dataset on PM2.5 dust levels in Bangkok, Thailand during January and February of 2024 and generated a forecasting process. The ARL was calculated using explicit formulas on the Extended EWMA control chart with $ARL_0$ = 370 for $\lambda_1$ = 0.05, 0.10 and $\lambda_2$ = 0.01, shift ($\delta$) equal to 0.01, 0.03, 0.05, 0.10, 0.30, 0.50, 1.00, 2.00, 3.00 and sample size = 1,000 nodes. The performance of the control chart was evaluated by comparing it to the EWMA control chart using a dataset on PM2.5 dust levels in Bangkok, Thailand during January and February of 2024. The coefficient parameters estimated for the ARMA(1,1) process are determined using maximum likelihood estimation: $\alpha_0$ = 27.3401, $\phi_1$ = 0.947, $\theta_1$ = 0.616. The ARMA(1,1) process can be defined by utilizing the parameter of this forecasting process.

$$\hat{W}_t = 0.947W_{t-1} - 0.616\vartheta_{t-1}$$

Using the explicit formula, we compare the ARL values of the ARMA(1,1) process on the Extended EWMA control chart with the ARL, SDRL, and MRL of the EWMA control charts. This comparison evaluates their efficiency. The results are presented in Table 7 (Appendix) and Figure 1 (Appendix), demonstrating a clear agreement with the findings seen in Table 4, Table 5 and Table 6 in Appendix. Figure 2 (Appendix) displays a comparison of the RMI, AEQL, and PCI derived from each control chart. The purpose is to assess the effectiveness of the control charts.

In this research, the performance of the ARL of the Extended EWMA control chart is assessed and contrasted with that of the EWMA control chart. The findings suggest that the Extended EWMA control charts are superior to the EWMA control chart for the ARMA(1,1) process. Additionally, the Extended EWMA control chart, with $\lambda_1$ = 0.10, better than all three control charts.

# 7 Conclusions

In this study, the formula was successful in finding the ARL value and the accuracy of the Extended EWMA control chart for the ARMA(1,1) compared to the EWMA control chart. the efficacy of control charts was evaluated for the ARL by utilizing the NIE, the explicit formula is subjected to comparison, And use all five measurements as an additional criterion to compare the performance of the two control charts. Both methods demonstrate that the ARL values are similar. The Extended EWMA control chart for the ARMA(1,1) process has superior performance compared to the EWMA control chart. When assessing the comparative efficacy of the ARL under different smoothing factors, it is recommended to utilize a smoothing parameter of $\lambda_1$ = 0.10. The simulation research and the real-world dataset on PM2.5 dust levels in Bangkok, Thailand during January and February of 2024, ultimately, the outcomes were the same. Further research, the extended EWMA control chart can be applied to other aspects, such as health or economics, as well as using an NIE comparison method with the explicit formulas. Several other methods of comparison will generate new control charts.

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**Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)**

- Phunsa Mongkoltawat carried out the writing-original draft preparation and simulation.
- Yupaporn Areepong has organized the conceptualization, writing-review and editing, and validation
- Saowanit Sukparungsee has implemented the methodology and solfware.

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**Conflicts of Interest**

The authors declare no conflict of interest.

# APPENDIX

Table 1. ARL comparison of the Extended EWMA control chart for ARMA(p,q) using explicit formulas against NIE method when $\phi_0 = 0.5$. $\phi_1 = 0.1$, $\theta_1 = 0.1$, $\alpha_0 = 1$ for ARL₀ = 370

| $\delta$ | $\theta$ | $\lambda_2 = 0.01$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\theta_1 = 0.1$ | | $\theta_1 = 0.1,\ \theta_2 = 0.2$ | | $\theta_1 = 0.1,\ \theta_2 = 0.2,\ \theta_3 = 0.3$ | |
| | Process | ARMA(1,1) | | ARMA(1,2) | | ARMA(1,3) | |
| | $\lambda_1$ | 0.05 | 0.10 | 0.05 | 0.10 | 0.05 | 0.10 |
| | *b* | 2.96646 | 3.63105 | 2.96808 | 3.63040 | 2.96850 | 3.63300 |
| 0.00 | Explicit | 370.00200 | 370.55058 | 370.50773 | 370.04983 | 370.54021 | 370.59006 |
| | NIE | 370.00200 | 370.55058 | 370.50773 | 370.04983 | 370.54021 | 370.59006 |
| 0.01 | Explicit | 358.74338 | 357.14330 | 359.22690 | 356.66069 | 359.25546 | 357.17068 |
| | NIE | 358.74338 | 357.14330 | 359.22690 | 356.66069 | 359.25546 | 357.17068 |
| 0.03 | Explicit | 337.80512 | 332.44859 | 338.24730 | 331.99912 | 338.26862 | 332.45412 |
| | NIE | 337.80512 | 332.44859 | 338.24730 | 331.99912 | 338.26862 | 332.45412 |
| 0.05 | Explicit | 318.76091 | 310.26908 | 319.16611 | 309.84910 | 319.18091 | 310.25553 |
| | NIE | 318.76091 | 310.26908 | 319.16611 | 309.84910 | 319.18091 | 310.25553 |
| 0.10 | Explicit | 278.06230 | 263.82544 | 278.39071 | 263.46626 | 278.39183 | 263.77384 |
| | NIE | 278.06230 | 263.82544 | 278.39071 | 263.46626 | 278.39183 | 263.77384 |
| 0.30 | Explicit | 177.21156 | 155.21847 | 177.36603 | 154.99343 | 177.33527 | 155.09284 |
| | NIE | 177.21156 | 155.21847 | 177.36603 | 154.99343 | 177.33527 | 155.09284 |
| 0.50 | Explicit | 125.76868 | 104.21405 | 125.84709 | 104.05419 | 125.80204 | 104.06690 |
| | NIE | 125.76868 | 104.21405 | 125.84709 | 104.05419 | 125.80204 | 104.06690 |
| 1.00 | Explicit | 69.34975 | 52.92752 | 69.36142 | 52.83141 | 69.30529 | 52.78121 |
| | NIE | 69.34975 | 52.92752 | 69.36142 | 52.83141 | 69.30529 | 52.78121 |
| 2.00 | Explicit | 34.94470 | 24.95056 | 34.93253 | 24.89444 | 34.87852 | 24.83357 |
| | NIE | 34.94470 | 24.95056 | 34.93253 | 24.89444 | 34.87852 | 24.83357 |
| 3.00 | Explicit | 3.16420 | 1.17294 | 3.14954 | 1.13249 | 3.10190 | 1.07843 |
| | NIE | 3.16420 | 1.17294 | 3.14954 | 1.13249 | 3.10190 | 1.07843 |

Table 2. ARL comparison of the Extended EWMA control chart for ARMA(p,q) using explicit formulas against NIE method when $\phi_0 = 0.5$. $\phi_1 = 0.1$, $\theta_1 = -0.1$, $\alpha_0 = 1$ for ARL₀ = 370

| $\delta$ | $\theta$ | $\lambda_2 = 0.01$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\theta_1 = -0.1$ | | $\theta_1 = -0.1,\ \theta_2 = 0.2$ | | $\theta_1 = -0.1,\ \theta_2 = 0.2,\ \theta_3 = 0.3$ | |
| | Process | ARMA(1,1) | | ARMA(1,2) | | ARMA(1,3) | |
| | $\lambda_1$ | 0.05 | 0.10 | 0.05 | 0.10 | 0.05 | 0.10 |
| | *b* | 2.96750 | 2.96808 | 2.96900 | 3.63200 | 2.96850 | 3.63300 |
| 0.00 | Explicit | 370.50578 | 370.50773 | 370.91904 | 370.77768 | 370.56128 | 370.67882 |
| | NIE | 370.50578 | 370.50773 | 370.91904 | 370.77768 | 370.56128 | 370.67882 |
| 0.01 | Explicit | 359.23026 | 359.22690 | 359.62363 | 357.35786 | 359.27603 | 357.25643 |
| | NIE | 359.23026 | 359.22690 | 359.62363 | 357.35786 | 359.27603 | 357.25643 |
| 0.03 | Explicit | 338.26024 | 338.24730 | 338.61718 | 332.64039 | 338.28826 | 332.53439 |
| | NIE | 338.26024 | 338.24730 | 338.61718 | 332.64039 | 338.28826 | 332.53439 |
| 0.05 | Explicit | 319.18744 | 319.16611 | 319.51189 | 310.44085 | 319.19975 | 310.33096 |
| | NIE | 319.18744 | 319.16611 | 319.51189 | 310.44085 | 319.19975 | 310.33096 |
| 0.10 | Explicit | 278.42876 | 263.80566 | 278.68608 | 263.95661 | 278.40903 | 263.83937 |
| | NIE | 278.42876 | 263.80566 | 278.68608 | 263.95661 | 278.40903 | 263.83937 |
| 0.30 | Explicit | 177.43595 | 155.23175 | 177.54395 | 155.26439 | 177.34909 | 155.13671 |
| | NIE | 177.21156 | 155.23175 | 177.54395 | 154.26439 | 177.34909 | 155.13671 |
| 0.50 | Explicit | 125.92494 | 104.23758 | 125.97011 | 104.22719 | 125.81467 | 104.10135 |
| | NIE | 125.92494 | 104.23758 | 125.97011 | 104.22719 | 125.81467 | 104.10135 |
| 1.00 | Explicit | 69.43484 | 52.95400 | 69.42895 | 52.91705 | 69.31705 | 52.80609 |
| | NIE | 69.43484 | 52.95400 | 69.42895 | 52.91705 | 69.31705 | 52.80609 |
| 2.00 | Explicit | 34.98837 | 24.97123 | 34.96830 | 24.93601 | 34.88929 | 24.85146 |
| | NIE | 34.98837 | 24.97123 | 34.96830 | 24.93601 | 34.88929 | 24.85146 |
| 3.00 | Explicit | 3.19345 | 1.18909 | 3.17426 | 1.16019 | 3.11154 | 1.09275 |
| | NIE | 3.19345 | 1.18909 | 3.17426 | 1.16019 | 3.11154 | 1.09275 |

Table 3. ARL comparison of the Extended EWMA control chart for ARMA(p,q) using explicit formulas against NIE method when $\phi_0 = 0.5$. $\phi_1 = 0.1$, $\theta_1 = -0.1$, $\theta_2 = -0.2$, $\alpha_0 = 1$ for ARL₀ = 370

| $\delta$ | $\theta$ | $\theta_1$ = -0.1 | | $\theta_1$ = -0.1, $\theta_2$ = -0.2 | | $\theta_1$ = -0.1, $\theta_2$ = -0.2, $\theta_3$ = 0.3 | |
|---|---|---|---|---|---|---|---|
| | | $\lambda_2$ =0.01 | | | | | |
| | Process | ARMA(1,1) | | ARMA(1,2) | | ARMA(1,3) | |
| | $\lambda_1$ | 0.05 | 0.10 | 0.05 | 0.10 | 0.05 | 0.10 |
| | *b* | 2.96750 | 2.96808 | 2.96655 | 3.62855 | 2.96801 | 3.63012 |
| 0.00 | Explicit | 370.50578 | 370.50773 | 370.48056 | 370.10566 | 370.50499 | 370.00294 |
| | NIE | 370.50578 | 370.50773 | 370.48056 | 370.10566 | 370.50499 | 370.00294 |
| 0.01 | Explicit | 359.23026 | 359.22690 | 359.21279 | 356.72779 | 359.22487 | 356.61687 |
| | NIE | 359.23026 | 359.22690 | 359.21279 | 356.72779 | 359.22487 | 357.61687 |
| 0.03 | Explicit | 338.26024 | 338.24730 | 338.25548 | 332.08623 | 338.24655 | 331.96088 |
| | NIE | 338.26024 | 338.24730 | 338.25548 | 332.08623 | 338.24655 | 331.96088 |
| 0.05 | Explicit | 319.18744 | 319.16611 | 319.19400 | 309.95325 | 319.16650 | 309.81575 |
| | NIE | 319.18744 | 319.16611 | 319.19400 | 309.95325 | 319.16650 | 309.81575 |
| 0.10 | Explicit | 278.42876 | 263.80566 | 278.45731 | 263.60281 | 278..39342 | 263.44280 |
| | NIE | 278.42876 | 263.80566 | 278.45731 | 263.60281 | 278.39342 | 263.44280 |
| 0.30 | Explicit | 177.43595 | 155.23175 | 177.50268 | 155.18361 | 177.37365 | 154.99338 |
| | NIE | 177.21156 | 155.23175 | 177.50268 | 155.18361 | 177.37365 | 154.99338 |
| 0.50 | Explicit | 125.92494 | 104.23758 | 125.99841 | 104.24435 | 125.85641 | 104.05877 |
| | NIE | 125.92494 | 104.23758 | 125.99841 | 104.24435 | 125.85641 | 104.05877 |
| 1.00 | Explicit | 69.43484 | 52.95400 | 69.49913 | 52.99346 | 69.37115 | 52.84059 |
| | NIE | 69.43484 | 52.95400 | 69.49913 | 52.99346 | 69.37115 | 52.84059 |
| 2.00 | Explicit | 34.98837 | 24.97123 | 35.03238 | 25.00843 | 34.94060 | 24.90295 |
| | NIE | 34.98837 | 24.97123 | 35.03238 | 25.00843 | 34.94060 | 24.90295 |
| 3.00 | Explicit | 3.19345 | 1.18909 | 2.22606 | 1.21908 | 3.15615 | 1.13953 |
| | NIE | 3.19345 | 1.18909 | 2.22606 | 1.21908 | 3.15615 | 1.13953 |

Table 4. ARL comparison of the Extended EWMA control chart for ARMA(1,1) against EWMA control charts when $\phi_0 = 0.5$. $\phi_1 = 0.1$, $\theta_1 = 0.1$, $\lambda_2 = 0.01$, $\alpha_0 = 1$ for $ARL_0 = 370$

| $\delta$ | Control Chart | Extended EWMA $\lambda_1 = 0.05$ | Extended EWMA $\lambda_1 = 0.10$ | EWMA $\lambda_1 = 0.05$ | EWMA $\lambda_1 = 0.10$ |
|---|---|---|---|---|---|
| | UCL | 2.96646 | 3.63105 | 2.96920 | 3.63200 |
| 0.00 | $ARL_0$ | 370.00200 | 370.55058 | 370.96100 | 370.68423 |
| | $SDRL_0$ | 370.00200 | 370.55058 | 370.96100 | 370.68423 |
| | $MRL_0$ | 370.00200 | 370.55058 | 370.96100 | 370.68423 |
| 0.01 | $ARL_1$ | 358.74338 | 357.14330 | 359.66153 | 359.26077 |
| | $SDRL_1$ | 358.64338 | 357.04330 | 359.56153 | 359.16770 |
| | $MRL_1$ | 247.30530 | 246.20230 | 247.93820 | 247.66200 |
| 0.03 | $ARL_1$ | 337.80512 | 332.44859 | 339.64755 | 334.53658 |
| | $SDRL_1$ | 337.66369 | 332.30716 | 339.50612 | 334.39515 |
| | $MRL_1$ | 232.87120 | 229.17860 | 234.14130 | 230.61800 |
| 0.05 | $ARL_1$ | 318.76091 | 310.26908 | 319.53546 | 312.33100 |
| | $SDRL_1$ | 318.53730 | 310.04547 | 319.31185 | 312.10739 |
| | $MRL_1$ | 219.74280 | 213.88880 | 220.27670 | 215.31020 |
| 0.10 | $ARL_1$ | 278.06230 | 263.82544 | 279.69520 | 265.83410 |
| | $SDRL_1$ | 277.74607 | 263.50921 | 279.37897 | 265.51787 |
| | $MRL_1$ | 191.68660 | 181.87220 | 192.81220 | 183.25690 |
| 0.30 | $ARL_1$ | 177.21156 | 155.21847 | 179.51763 | 157.11150 |
| | $SDRL_1$ | 176.66383 | 154.67074 | 178.96990 | 156.56377 |
| | $MRL_1$ | 122.16350 | 107.00230 | 123.75330 | 108.3072 |
| 0.50 | $ARL_1$ | 125.76868 | 104.21405 | 127.92548 | 106.05878 |
| | $SDRL_1$ | 125.06157 | 103.50694 | 127.21837 | 105.35167 |
| | $MRL_1$ | 86.70059 | 71.84157 | 88.18741 | 73.11326 |
| 1.00 | $ARL_1$ | 69.34975 | 52.92752 | 69.36277 | 54.73000 |
| | $SDRL_1$ | 68.34975 | 51.92752 | 68.36277 | 53.73000 |
| | $MRL_1$ | 47.80732 | 36.48641 | 47.81630 | 37.72897 |
| 2.00 | $ARL_1$ | 34.94470 | 24.95056 | 36.88625 | 26.73442 |
| | $SDRL_1$ | 33.53048 | 23.53634 | 35.47203 | 25.32020 |
| | $MRL_1$ | 24.08967 | 17.20005 | 25.42811 | 18.42979 |
| 3.00 | $ARL_1$ | 3.16420 | 1.17294 | 23.08540 | 15.95210 |
| | $SDRL_1$ | 1.43214 | 0.55911 | 21.35334 | 14.22004 |
| | $MRL_1$ | 1.18129 | 0.50858 | 15.42811 | 10.99683 |
| | RMI | 0.31766 | 0.0000 | 2.21926 | 1.41800 |
| | AEQL | 28.89150 | 20.70582 | 47.67518 | 34.89650 |
| | PCI | 1.39533 | 1.00000 | 2.30250 | 1.68534 |

Table 5. ARL comparison of the Extended EWMA control chart for ARMA(1,2) against EWMA control charts when $\phi_0 = 0.5$. $\phi_1 = 0.1$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\lambda_2 = 0.01$, $\alpha_0 = 1$ for $ARL_0 = 370$

| | Control | Extended EWMA | Extended EWMA | EWMA | EWMA |
|---|---|---|---|---|---|

| $\delta$ | Chart | $\lambda_1 = 0.05$ | $\lambda_1 = 0.10$ | $\lambda_1 = 0.05$ | $\lambda_1 = 0.10$ |
|---|---|---|---|---|---|
| | UCL | 2.96808 | 3.63040 | 2.96880 | 3.63190 |
| 0.00 | $ARL_0$ | 370.50773 | 370.04983 | 370.73988 | 370.46651 |
| | $SDRL_0$ | 370.50773 | 370.04983 | 370.73988 | 370.46651 |
| | $MRL_0$ | 370.50773 | 370.04983 | 370.73988 | 370.46651 |
| 0.01 | $ARL_1$ | 359.22690 | 356.66069 | 359.44785 | 359.05123 |
| | $SDRL_1$ | 359.12690 | 356.56069 | 359.34785 | 358.95123 |
| | $MRL_1$ | 247.63860 | 245.86960 | 247.79090 | 247.51750 |
| 0.03 | $ARL_1$ | 338.24730 | 331.99912 | 339.44759 | 334.34198 |
| | $SDRL_1$ | 338.07409 | 331.82591 | 339.27438 | 334.16877 |
| | $MRL_1$ | 233.17600 | 228.86870 | 234.00350 | 230.48380 |
| 0.05 | $ARL_1$ | 319.16611 | 309.84910 | 319.34781 | 312.14968 |
| | $SDRL_1$ | 318.94250 | 309.62549 | 319.12420 | 311.92607 |
| | $MRL_1$ | 220.02210 | 213.59930 | 220.14740 | 215.18520 |
| 0.10 | $ARL_1$ | 278.39071 | 263.46626 | 279.53339 | 265.68011 |
| | $SDRL_1$ | 278.07448 | 263.15003 | 279.21716 | 265.36388 |
| | $MRL_1$ | 191.91290 | 181.62450 | 192.70070 | 183.15070 |
| 0.30 | $ARL_1$ | 177.36603 | 154.99343 | 179.41662 | 157.01868 |
| | $SDRL_1$ | 176.81830 | 154.44570 | 178.86889 | 156.47095 |
| | $MRL_1$ | 122.27000 | 106.84710 | 123.68360 | 108.24330 |
| 0.50 | $ARL_1$ | 125.84709 | 104.05419 | 127.85347 | 106.99335 |
| | $SDRL_1$ | 125.13998 | 103.34708 | 127.14636 | 106.28624 |
| | $MRL_1$ | 86.75464 | 71.73137 | 88.13777 | 73.75752 |
| 1.00 | $ARL_1$ | 69.36142 | 52.83141 | 69.42107 | 54.69192 |
| | $SDRL_1$ | 68.36142 | 51.83141 | 68.42107 | 53.69192 |
| | $MRL_1$ | 47.81537 | 36.42015 | 47.85649 | 37.70272 |
| 2.00 | $ARL_1$ | 34.93253 | 24.89444 | 36.86304 | 26.71276 |
| | $SDRL_1$ | 33.51831 | 23.48022 | 35.44882 | 25.29854 |
| | $MRL_1$ | 24.08128 | 17.16137 | 25.41211 | 18.41485 |
| 3.00 | $ARL_1$ | 3.14954 | 1.13249 | 23.06902 | 15.93662 |
| | $SDRL_1$ | 1.14174 | 0.59556 | 21.33696 | 14.20456 |
| | $MRL_1$ | 1.14118 | 0.58700 | 15.90299 | 10.98616 |
| | RMI | 0.32930 | 0.00000 | 2.29702 | 1.47237 |
| | AEQL | 28.87843 | 20.63083 | 28.88843 | 34.96265 |
| | PCI | 1.39977 | 1.00000 | 1.40025 | 1.69467 |

Table 6. ARL comparison of the Extended EWMA control chart for ARMA(1,3) against EWMA control charts when $\phi_0 = 0.5$. $\phi_1 = 0.1$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\theta_3 = 0.3$, $\lambda_2 = 0.01$, $\alpha_0 = 1$ for $ARL_0 = 370$

| | Control | Extended EWMA | Extended EWMA | EWMA | EWMA |
|---|---|---|---|---|---|

| $\delta$ | Chart | $\lambda_1 = 0.05$ | $\lambda_1 = 0.10$ | $\lambda_1 = 0.05$ | $\lambda_1 = 0.10$ |
|---|---|---|---|---|---|
| | UCL | 2.96850 | 3.63300 | 2.96950 | 3.63400 |
| 0.00 | $ARL_0$ | 370.54021 | 370.59006 | 370.92196 | 370.90797 |
| | $SDRL_0$ | 370.54021 | 370.59006 | 370.92196 | 370.90797 |
| | $MRL_0$ | 370.54021 | 370.59006 | 370.92196 | 370.90797 |
| 0.01 | $ARL_1$ | 359.25546 | 357.17068 | 359.62196 | 359.47073 |
| | $SDRL_1$ | 359.15546 | 357.07068 | 359.52196 | 359.37073 |
| | $MRL_1$ | 247.65830 | 246.22110 | 247.91100 | 247.80670 |
| 0.03 | $ARL_1$ | 338.26862 | 332.45412 | 339.60703 | 334.72153 |
| | $SDRL_1$ | 338.09541 | 332.28091 | 339.43382 | 334.54832 |
| | $MRL_1$ | 233.19070 | 229.18240 | 234.11340 | 230.74550 |
| 0.05 | $ARL_1$ | 319.18091 | 310.25553 | 319.49410 | 312.49391 |
| | $SDRL_1$ | 318.95730 | 310.03192 | 319.27049 | 312.27070 |
| | $MRL_1$ | 220.03230 | 213.87950 | 220.24820 | 215.42250 |
| 0.10 | $ARL_1$ | 278.39183 | 263.77384 | 279.65219 | 265.95230 |
| | $SDRL_1$ | 278.07560 | 263.45761 | 279.33596 | 265.63607 |
| | $MRL_1$ | 191.91370 | 181.83660 | 192.78260 | 183.33830 |
| 0.30 | $ARL_1$ | 177.33527 | 155.09284 | 179.47178 | 157.13560 |
| | $SDRL_1$ | 176.80754 | 154.54511 | 178.92405 | 156.58787 |
| | $MRL_1$ | 122.24880 | 106.91570 | 123.72170 | 108.32390 |
| 0.50 | $ARL_1$ | 125.80204 | 104.06690 | 127.87959 | 106.04689 |
| | $SDRL_1$ | 125.09493 | 103.35979 | 127.17248 | 105.33978 |
| | $MRL_1$ | 86.72358 | 71.74013 | 88.15578 | 73.10507 |
| 1.00 | $ARL_1$ | 69.30529 | 52.78121 | 69.42012 | 54.69363 |
| | $SDRL_1$ | 68.30529 | 51.78121 | 68.42012 | 53.69363 |
| | $MRL_1$ | 47.77667 | 36.38554 | 47.85583 | 37.70390 |
| 2.00 | $ARL_1$ | 34.87852 | 24.83357 | 36.85123 | 26.69774 |
| | $SDRL_1$ | 33.46430 | 23.41935 | 35.43701 | 25.28352 |
| | $MRL_1$ | 24.04405 | 17.11941 | 25.40397 | 18.40450 |
| 3.00 | $ARL_1$ | 3.10190 | 1.07843 | 23.05620 | 15.92080 |
| | $SDRL_1$ | 1.36984 | 0.65362 | 21.32414 | 14.18874 |
| | $MRL_1$ | 1.13834 | 0.64343 | 15.89415 | 10.97525 |
| | RMI | 0.33929 | 0.00000 | 2.40922 | 1.54835 |
| | AEQL | 28.80694 | 20.55447 | 47.63901 | 34.92037 |
| | PCI | 1.40149 | 1.00000 | 2.31769 | 1.69891 |

Table 7. ARL comparison of the Extended EWMA control chart for ARMA(1,1) using NIE against EWMA control chart when $\alpha_0 = 27.3401$, $\phi_1 = 0.947$, $\theta_1 = 0.616$ for $ARL_0 = 370$

| $\delta$ | Control Chart | Extended EWMA $\lambda_1 = 0.05$ | Extended EWMA $\lambda_1 = 0.10$ | EWMA $\lambda_1 = 0.05$ | EWMA $\lambda_1 = 0.10$ |
|---|---|---|---|---|---|
| | UCL | 81.14050 | 99.36650 | 81.20000 | 99.42950 |

| | | | | | |
|---|---|---|---|---|---|
| 0.00 | $ARL_0$ | 370.09745 | 370.04424 | 370.87888 | 370.78295 |
| | $SDRL_0$ | 370.09745 | 370.04424 | 370.87888 | 370.78295 |
| | $MRL_0$ | 370.09745 | 370.04424 | 370.87888 | 370.78295 |
| 0.01 | $ARL_1$ | 358.83352 | 346.65490 | 359.58227 | 357.35438 |
| | $SDRL_1$ | 358.73352 | 346.55490 | 354.48227 | 357.25438 |
| | $MRL_1$ | 247.36740 | 245.86560 | 247.88360 | 246.34780 |
| 0.03 | $ARL_1$ | 337.88490 | 321.99279 | 338.57353 | 332.62084 |
| | $SDRL_1$ | 337.71169 | 321.81958 | 338.40032 | 332.44763 |
| | $MRL_1$ | 232.92620 | 228.86440 | 233.40090 | 229.29730 |
| 0.05 | $ARL_1$ | 318.83135 | 299.84205 | 319.46609 | 310.40691 |
| | $SDRL_1$ | 318.60774 | 299.61844 | 319.24248 | 310.18330 |
| | $MRL_1$ | 219.79130 | 213.59440 | 220.22890 | 213.98380 |
| 0.10 | $ARL_1$ | 278.11315 | 253.45687 | 288.63544 | 273.89268 |
| | $SDRL_1$ | 277.79692 | 253.14064 | 288.31921 | 273.57645 |
| | $MRL_1$ | 191.72160 | 181.61810 | 198.97530 | 188.81220 |
| 0.30 | $ARL_1$ | 177.21618 | 144.97452 | 179.47908 | 155.13019 |
| | $SDRL_1$ | 176.66845 | 144.42679 | 178.93135 | 154.58246 |
| | $MRL_1$ | 122.16670 | 106.83410 | 123.72670 | 106.94140 |
| 0.50 | $ARL_1$ | 125.75122 | 94.01994 | 127.89571 | 106.05868 |
| | $SDRL_1$ | 125.04411 | 93.31283 | 127.18860 | 105.35157 |
| | $MRL_1$ | 86.68855 | 71.70776 | 88.16689 | 73.11319 |
| 1.00 | $ARL_1$ | 69.30894 | 42.77308 | 79.33972 | 62.70964 |
| | $SDRL_1$ | 68.30894 | 41.77308 | 78.33972 | 61.70964 |
| | $MRL_1$ | 47.77919 | 36.37994 | 54.69406 | 43.22986 |
| 2.00 | $ARL_1$ | 34.89093 | 14.80931 | 36.86497 | 26.70099 |
| | $SDRL_1$ | 33.47671 | 13.39509 | 35.45075 | 25.28677 |
| | $MRL_1$ | 24.05260 | 17.10268 | 25.41344 | 18.40674 |
| 3.00 | $ARL_1$ | 3.10585 | 1.03310 | 23.06385 | 15.91344 |
| | $SDRL_1$ | 1.37379 | 0.69895 | 21.33179 | 14.18138 |
| | $MRL_1$ | 1.14106 | 0.61218 | 15.89942 | 10.97018 |
| | RMI | 0.35411 | 0.00000 | 2.53962 | 1.63711 |
| | AEQL | 28.81308 | 20.50045 | 48.65270 | 35.70610 |
| | PCI | 1.40548 | 1.0000 | 2.37325 | 1.74172 |

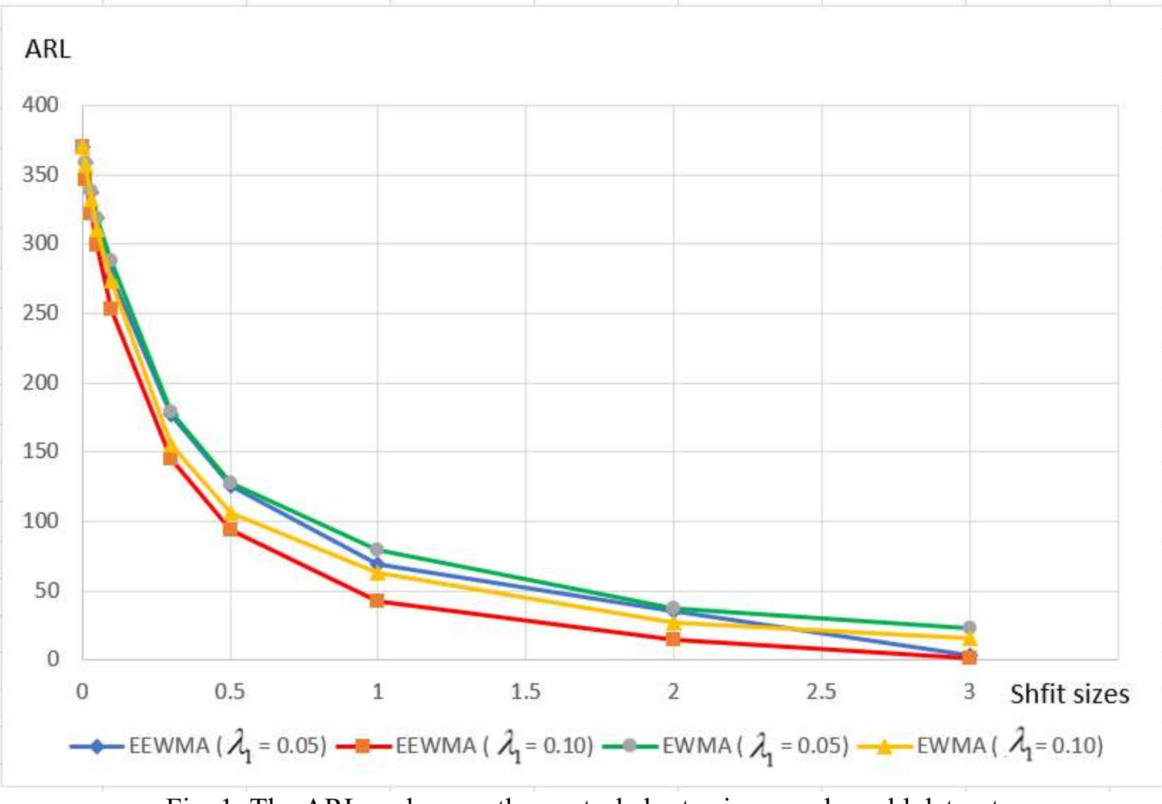


Fig. 1: The $ARL_1$ values on the control chart using a real-world dataset

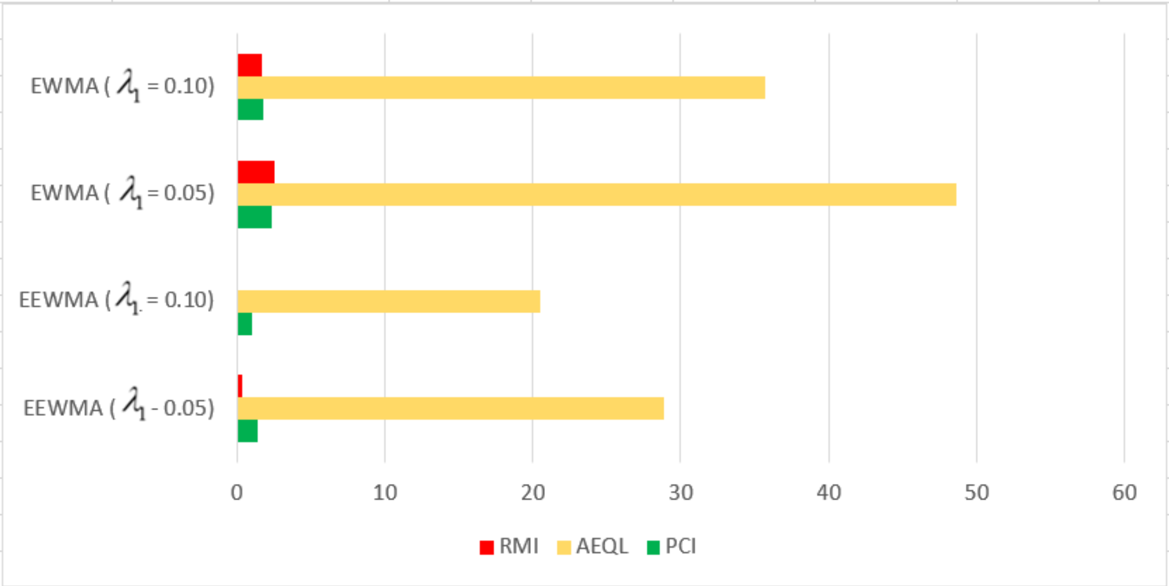


Fig. 2: Comparison the RMI, AEQL, and PCI values with the Extended EWMA control chart and the EWMA control chart for $\lambda_1$ = 0.05, 0.10