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Inverse Problem of Determining the Unknown Coefficients in an Elliptic Equation

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Abstract: - The inverse problem of determining the coefficients of an elliptic equation under different boundary conditions in a given rectangle is considered. These problems lead to the necessity of approximate solution of inverse problems of mathematical physics, which are incorrect in the classical sense. The existence, uniqueness, and stability theorems for the solution of the set inverse problem are proved and a regularizing algorithm for determining the coefficient is constructed.

Key-Words: - Inverse problem, elliptic equation, regularization algorithm, the Green's function, successive approximation method, existence, and uniqueness of the solution.

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1 Introduction

The monographs [1], [2] about inverse problems for differential equations tell the history of this area of mathematical physics, its problems, areas of application, existing solution methods, etc. are widely given. When studying direct problems, the solution of a given differential equation or system of equations is carried out through additional conditions, whereas in inverse problems the equation itself is unknown. Both the definition of the basic equation and its solution require the imposition of additional conditions, rather than problems directly related to them.

Newton's problem of discovering the forces that set the planets in motion according to Kepler's laws was one of the first inverse problems in the dynamics of mechanical systems. It covers the subject of similar problems, including a fairly complete and systematic theory of inverse problems. By developing an approach to the existence, uniqueness and stability of solutions, this work represents a systematic development of the theory of inverse problems for all main types of partial differential equations. Here we discuss modern methods of linear and nonlinear analysis, the theory of differential equations in Banach spaces, applications [1].

The book, [2], offers in-depth coverage of inverse problems for second-order equations and for hyperbolic systems of first-order equations, including the kinematic problem of seismology, the Lamb dynamic problem for equations of the theory of elasticity, and the problem of electrodynamics.

The third edition, [3] is intended for ordinary graduate students of physical sciences who do not have extensive mathematical training. The book is complemented by a companion website that includes MATLAB codes corresponding to examples, illustrated with simple, easy-to-understand problems that highlight the details of specific numerical methods. Updates in the new edition include more discussion of Laplace smoothing, expansion of exercises with basis functions, addition of stochastic descent, improved presentation of Fourier methods and exercises, and much more.

The main classes of inverse problems for equations of mathematical physics and their numerical solution methods are considered in this book which is intended for graduate students and experts in applied mathematics, computational mathematics, and mathematical modelling, [4].

This book, [5], explores methods for specifically solving inverse problems. The inverse problem arises when it is necessary to determine the reasons that caused a particular effect, or when trying to indirectly estimate the parameters of a physical system. The author uses practical examples to illustrate inverse problems in the physical sciences.

Developing an approach to the question of existence, uniqueness and stability of solutions, this work presents a systematic elaboration of the theory of inverse problems for all principal types of partial differential equations. It covers up-to-date methods of linear and nonlinear analysis, the theory of differential equations in Banach spaces, applications of functional analysis, and semigroup theory, [6].

The paper, [7] considers the inverse problem in determining unknown coefficients in a linear elliptic equation. Theorems of existence, uniqueness and stability of the solution of inverse problems for a linear equation of elliptic type are proved. Using the method of sequential measurements, a regularizing algorithm is constructed to determine several coefficients.

A huge number of mathematical models are called Boussinesq-type equations. The classical solution of one nonlinear inverse boundary value problem for the linearized sixth-order Boussinesq equation with an additional integral condition is considered. The first method is based on the application of the Fourier method. The second method is based on the application of the compressive method, which consists in the fact that it is required to determine together with the solution the unknown coefficient depending on the variable t at the unknown function. The problem is considered in the rectangular domain. When solving the original inverse boundary value problem, a transition from the original inverse problem to some auxiliary inverse problem is carried out. With the help of compressed mappings the existence and uniqueness of the solution of the auxiliary problem are proved. Then the transition to the original inverse problem is made again, and as a result the conclusion about the solvability of the original inverse problem is made. The proposed methods of finding solutions to the inverse problem can be used in the study of solvability for various problems of mathematical physics, [8].

The identification of an unknown coefficient in the lower term of elliptic second-order differential equation M u + ku = f with the mixed boundary conditions of the third type is considered. The identification of constant based is based on an integral boundary data. The local existence and uniqueness of strong solution for the inverse problem is proved, [9].

For a mathematical model with external-diffusion kinetics, we consider an inverse problem of determining the inverse isotherm and a kinetic coefficient from two dynamic output curves observed at two points in a single experiment. A gradient-type iterative method utilizing the adjoint problem technique is proposed for this inverse problem, and numerical results are reported, [10].

The purpose of this paper is to prove the uniqueness and existence of solutions of the inverse boundary value problem for the second order elliptic equation.

In the face of higher-order derivatives, the coefficients coincide with the given problem for the rectangular region. In the case under review, similar issues with different border conditions are considered.

2 Problem Formulation

Let $I = \{ \quad ; \quad i_0, e \in I, i_1 = \frac{i_0+1}{2i_0-1},$

$k, q \in \{0,1\}, \qquad w_{qkt} = t + (-1)^t (k+q-2kq),$

$\omega_{qkt} = [1-k(k-1)]\,[t+(-1)^t q],\ t = 0,1.$

Through $b_t, d_t,\ t = 0,1$ denote the constants, which are defined as follows: $k=0, q=0: b_t = w_{00t}, d_t = \omega_{00t},$ $k=0, q=1:$ $b_t = w_{10t}, d_t = \omega_{10t},$ $k=1, q=0:$ $b_t = w_{01t}, d_t = \omega_{01t},$ $q\ k = :$ $b_t = w_{11t}\ d_t = \omega_{11t},\ t = \quad .$

Let us consider the problem with fixed parameters ${}_0i, e, k, q$ and $\{a_{i_0}(x_2), \quad u(x_1, x_2)\}$ the following conditions:

$$-a_1(x_2)u_{x_1x_1} - a_2(x_2)u_{x_2x_2} + c(x_2)u = h(x_1, x_2),$$
$$(x_1, x_2) \in D, \tag{1}$$

$$u(0, x_2) = \phi_1(x_2),$$
$$(e-1)u_{x_1}(l_1, x_2) + (2-e)u(l_1, x_2) = \phi_2(x_2),$$
$$0 \le x_2 \le l_2, \tag{2}$$

$$b_0 u_{x_2}(x_1, 0) + b_1 u(x_1, 0) = \varphi_1(x_1),$$
$$0 \le x_1 \le l_1, \tag{3}$$

$$d_0 u_{x_2}(x_1, l_2) + d_1 u(x_1, l_2) = \varphi_2(x_1),$$
$$0 \le x_1 \le l_1, \tag{4}$$

$$a_{i_0}(x_2) u_{x_1}(0, x_2) = g_{i_0}(x_2),\ 0 \le x_2 \le l_2, \tag{5}$$

whereby

$$b_0\phi_{1x_2}(0)+b_1\phi_1(0)=\varphi_1(0), d_0\phi_{1x_2}(l_2)+d_1\phi_1(l_2)=\varphi_2(0),$$
$$=b_0\phi_{2x_2}(0)+b_1\phi_2(0)=$$
$$=\varphi_1^{(e-1)}(l_1), d_0\phi_{2x_2}(l_2)+d_1\phi_2(l_2)=\varphi_2^{(e-1)}(l_1).$$

Here $D=\{(x_1,x_2)\,|\,0<x_1<l_1, 0<x_2<l_2\}$, $h(x_1,x_2),\phi_i(x_2),\ \varphi_i(x_1), g_{i_0}(x_2), i=1,2$ – given functions $h(x_1,x_2), h_{x_1x_1}(x_1,x_2)\in C^{\alpha}(\overline{D})$, $\phi_1(x_2)\in C^{2+\alpha}[0,l_2]$, $\phi_2(x_2)\in C^{3-e+\alpha}[0,l_2]$, $\varphi_1(x_1)\in C^{1+b_1+\alpha}[0,l_1]$, $\varphi_2(x_1)\in C^{1+d_1+\alpha}[0,l_1]$, $g_{i_0}(x_2)\in C^{\alpha}[0,l_2]$, $0<\alpha<1$.

Definition. The functions $\{a_{i_0}(x_2), u(x_1,x_2)\}$ are called a solution of problem (1)–(5), if $0<a_{i_0}(x_2)\in C[0,l_2], u(x_1,x_2)\in C^2(D)\cap C(\overline{D})$ and satisfy the relation (1) –(5).

It is easy to check th if solutions (1) - (5) exist, then under the assumed assumptions on the smoothness of problem da $a_{i_0}(x_2)\in C^{\alpha}[0,l_2], u(x_1,x_2)\in C^{2+\alpha}(\overline{D})$. Indeed, with the assumptions accepted, it follows from the general theory of elliptic equions th $u(x_1,x_2)\in W_p^2(D)\subset C^{1+\alpha}(\overline{D})$ $p>2$. Therefore, from the additional condition (5) it follows th $a_{i_0}(x_2)\in C^{\alpha}[0,l_2]$. Therefore, $u(x_1,x_2)\in C^{2+\alpha}(\overline{D})$.

Equion (1) can also be written in the following for:

$$-a_{i_0}(x_2)u_{x_{i_0}x_{i_0}}-a_{i_1}(x_2)u_{x_{i_1}x_{i_1}}+c(x_2)u=$$
$$=h(x_1,x_2),(x_1,x_2)\in D.$$

3 Uniqueness and Stability of the Solution

Let us now consider the uniqueness and stability of the solution. Suppose, besides problem (1) - (5), there is a problem where all functions included in $(\overline{1})-(\overline{5})$, are replaced by the corresponding functions with a line. Put:

$$Z(x_1,x_2)=\overline{u}(x_1,x_2)-u(x_1,x_2), \lambda_{i_0}(x_2)=$$
$$\overline{a}_{i_0}(x_2)-a_{i_0}(x_2),$$
$$\delta_1(x_2)=\overline{a}_{i_1}(x_2)-a_{i_1}(x_2),$$
$$\delta_2(x_2)=\overline{c}(x_2)-c(x_2),$$
$$\delta_{i+2}(x_2)=\overline{\phi}_i(x_2)-\phi_i(x_2),$$
$$\delta_{i+4}(x_1)=\overline{\varphi}_i(x_1)-\varphi_i(x_1),\ i=1,2,$$
$$\delta_7(x_1,x_2)=\overline{h}(x_1,x_2)-h(x_1,x_2),$$
$$\delta_8(x_2)=\overline{g}_{i_0}(x_2)-g_{i_0}(x_2).$$

We denote $\widetilde{\delta}_e(x_1,x_2)$, the functions on the boundary each $k,q\in\{0,1\}, e\in I$ coinciding, respectively, with $\delta_{i+2}(x_2), \delta_{i+4}(x_1), i=1,2$ and belonging to $C^{2+\alpha}(\overline{D})$. Denote,

$$k_{ei}=(e-1)(i-1), g_{eij}(l_1)=\left(\frac{l_1}{2}\right)^{(2-j)k_{ei}},$$

$$d_{ej}=\frac{e+j-1}{(j-1)e+1},$$

$$L_{iqk}(x_j)=x_j^{(j-1)(1-k)q}\,\frac{(2-i)l_j-(-1)^{i+1}\left(\frac{1}{2}\right)^{(j-1)(1-k)q}x_j}{l_j}.$$

The function $\widetilde{\delta}_e(x_1,x_2)$ is defined as follows:

$$\widetilde{\delta}_e(x_1,x_2)=\sum_{i,j=1}^{2}l_{ije}(l_1,l_2)\left[P_i(x_j)\right]^{m_{eij}}\delta_{i+2j}(x_{j+(-1)^{j+1}})-$$
$$-n_{ije}(l_1,l_2)\left[P_i(x_1)\right]^{e}\left[P_j(x_2)\right]^{t_j}\delta_{j+4}^{(k_{ei})}\left[(i-1)l_1\right].$$

Here $l_{eij}(l_p), P_i(x_j), m_{eij}, n_{eij}(l_1,l_2), t_j$ are defined as follows:

at $k=0, q=0$:

$$l_{eij}(l_1,l_2)=g_{eij}(l_1), P_i(x_j)=L_{i00}(x_j), m_{eij}=d_{ej}, n_{eij}(l_1,l_2)=g_{ei1}(l_1), t_j=1,$$

at $k=0, q=1$:

$$l_{eij}(l_1,l_2)=g_{eij}(l_1), P_i(x_j)=L_{i10}(x_j), m_{eij}=d_{ej},$$
$$n_{eij}(l_1,l_2)=g_{ei1}(l_1), t_j=1,$$

at $k=1, q=0$:

$$l_{eij}(l_1,l_2)=(-1)^{(j-1)i}l_2^{(2-i)(j-1)}g_{ei1}(l_1), P_i(x_j)=$$
$$=L_{i01}(x_j),$$
$$m_{eij}=2^{(i-1)(j-1)}d_{ej},$$
$$n_{eij}(l_1,l_2)=(-1)^j l_2^{2-j}g_{ei1}(l_1), t_j=2^{j-1},$$

at $k=1, q=1$:

$$l_{eij}(l_1,l_2)=l_2^{(i-1)(j-1)}g_{eij}(l_1), P_i(x_j)=L_{i11}(x_j),$$
$$m_{eij}=2^{(i2-i)(j-1)}d_{ej},$$
$$n_{eij}(l_1,l_2)=l_2^{j-1}g_{ei1}(l_1), t_j=2^{j-1}.$$

Lemma 1. Let the solutions to problem (1) - (5) exist. Then the following estimates are true

$$\left|u(x_1,x_2\right| \le \max\Big[\ \max_D\left|\frac{h(x_1,x_2)}{c(x_1)}\right|,$$
$$\max_{x_2}|\phi_1(x_2)|,(2-e)\max_{x_2}|\phi_2(x_2)|,b_1\max_{x_1}|\varphi_1(x_1)|,$$
$$d_1\max_{x_1}|\varphi_2(x_1)|\Big],$$
$$\left|u_{x_1x_1}(x_1,x_2\right| \le \max\Big[\max_D\left|\frac{h_{x_1x_1}(x_1,x_2)}{c(x_2)}\right|, \tag{6}$$
$$(e-1)\max_{x_2}|\theta_{12}(x_2)|,$$
$$(2-e)\max_i\max_{x_2}|\theta_{i1}(x_2)|,$$
$$b_1\max_{x_1}\left|\varphi_{1\,x_1x_1}(x_1)\right|,d_1\max_{x_1}\left|\varphi_{2\,x_1x_1}(x_1)\right|\Big]$$

Here
$$\theta_{ie}(x_2)=\frac{1}{a_1(x_2)}\left[-a_2(x_2)\phi_{ix_2x_2}(x_2)+c(x_2)\phi_i(x_2)-h^{(k_{ei})}(il_1-l_1,x_2)\right]$$
$i=1,2$

Proof: The first inequality for problem (1) - (3) at each is $e\in I$ obtained from the maximum principle. By differentiating equation (1) twice and using the maximum principle we obtain the second estimate. Analogously at $e=2$ we obtain the evaluation (6). Lemma 1 is proved.

The uniqueness of the solution of the inverse problem (1)-(5) under the assumption of its existence is establishesis.

Theorem 1. Then let $g_{i_0}(x_2)\ne 0, Nl_1l_2<1.$ the solution of the problem (1)-(5) be singular and the following evaluation be true:

$$\left\|\bar{a}_{i_0}(x_2)-a_{i_0}(x_2)\right\|_{C[0,l_2]}+$$
$$+\|\bar{u}-u\|_{C(\bar{D})}\le N_1\Big[\left\|\bar{a}_{i_1}(x_2)-a_{i_1}(x_2)\right\|_{C[0,l_2]}+$$
$$\|\bar{c}(x_2)-c(x_2)\|_{C[0,l_2]}+$$
$$+\left\|\bar{h}(x_1,x_2)-h(x_1,x_2)\right\|_{C(\bar{D})}+\|\bar{\varphi}_1(x_1)-\varphi_1(x_1)\|_{C^{1+b_1}[0,l_1]}+$$
$$+\|\bar{\varphi}_2(x_1)-\varphi_2(x_1)\|_{C^{1+d_1}[0,l_1]}+$$
$$+\left\|\bar{\phi}_1(x_2)-\phi_1(x_2)\right\|_{C^2[0,l_2]}+\left\|\bar{\phi}_2(x_2)-\phi_2(x_2)\right\|_{C^{3-e}[0,l_2]}+$$
$$\left\|\bar{g}_{i_0}(x_2)-g_{i_0}(x_2)\right\|_{C[0,l_2]}\Big] \tag{7}$$

N,N_1 – positive constants that depend on the task data.

Proof. From $(\bar{1})-(\bar{5})$, respectively, subtract (1)–(5) and put $Z_1(x_1,x_2)= =Z(x_1,x_2)-\tilde{\delta}_e(x_1,x_2)$. Then we get
$$-\bar{a}_1(x_2)Z_{1x_1x_1}-\bar{a}_2(x_2)Z_{1x_2x_2}=$$
$$=\delta_{e1}(x_1,x_2)-\bar{c}(x_2)\tilde{\delta}_e(x_1,x_2)-$$
$$-\bar{c}(x_2)Z_1+\sum_{i=1}^{2}\alpha_i(x_1,x_2)\lambda_i(x_2), \tag{8}$$

$$Z_1(0,x_2)=0,$$
$$(2-e)Z_1(l_1,x_2)+(e-1)Z_{1x_1}(l_1,x_2)=0, \tag{9}$$

$$b_0Z_{1x_2}(x_1,0)+b_1Z_1(x_1,l_2)=0, \tag{10}$$

$$d_0Z_{1x_2}(x_1,l_2)+d_1Z_1(x_1,l_2)=0, \tag{11}$$

$$\lambda_{i_0}(x_2)=\delta_{e2}(x_2)+\gamma_1(x_1)Z_{1x_1}(0,x_2), \tag{12}$$

Here
$$\alpha_{i_0}(x_1,x_2)=u_{x_{i_0}x_{i_0}},$$
$$\gamma(x_2)=\bar{a}_{i_0}(x_2)[-u_{x_1}(0,x_2)]^{-1},$$
$$\delta_{e1}(x_1,x_2)=u_{x_{i_1}x_{i_1}}(x_1,x_2)\delta_1(x_2)-\delta_2(x_2)u+$$
$$+\delta_7(x_1,x_2)+\sum_{i=1}^{2}\bar{a}_i(x_2)\tilde{\delta}_{ex_ix_i}(x_1,x_2),$$
$$\delta_{e2}(x_2)=\gamma(x_2)\left\{\left[-\bar{a}_{i_0}(x_2)\right]^{-1}\delta_8(x_2)+\tilde{\delta}_{ex_1}(0,x_2)\right\}.$$

Using the Green's function [9] from (8)-(11), we define the function $Z_1(x_1,x_2)$ through the right side of equality and substitute this expression into the condition (12). Then we obtain:
$$Z(x_1,x_2)=$$
$$=\tilde{\delta}_e(x_1,x_2)+\int_D G(x_1,x_2,\xi_1,\xi_2)\left[\delta_{e1}(\xi_1,\xi_2)-\right.$$
$$\left.-\bar{c}(\xi_2)Z(\xi_1,\xi_2)+\alpha_{i_0}(\xi_1,\xi_2)\lambda_{i_0}(\xi_2)\right]d\xi_1d\xi_2,$$
$$\lambda_{i_0}(x_2)=\delta_{e2}(x_2)+\gamma_1(x_2)\int_D G_{x_1}(0,x_2,\xi_1,\xi_2)$$
$$\left[\delta_{e1}(\xi_1,\xi_2)-\bar{c}(\xi_2)Z(\xi_1,\xi_2)+\right.$$
$$\left.+\alpha_{i_0}(\xi_1,\xi_2)\lambda_{i_0}(\xi_2)\right]d\xi_1d\xi_2. \tag{13}$$

The following estimates are valid for the Green's [9] function:

$$\left|G(x_1,x_2,\xi_1,\xi_2)\right| \le M_1 \ln \frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}},$$
$$\left|G_{x_1}(x_1,x_2,\xi_1,\xi_2)\right| \le M_2\left[(x_1-\xi_1)^2+(x_2-\xi_1)^2\right]^{-1/2},$$
$$M_i > 0, i = 1,2.$$

Now in the system (13) let's put
$$\chi = \max_{x_1,x_2} \left|Z(x_1,x_2)\right| + \max_{x_2}\left|\lambda_{i_0}(x_2)\right|.$$
From system (13) we obtain
$$\chi \le N_2\left[\sum_{i=1}^{2}\left\|\delta_i(x_2)\right\|_{C[0,l_2]} + \left\|\delta_7(x_1,x_2)\right\|_{C(\overline{D})} + \left\|\widetilde{\delta}_e(x_1,x_2)\right\|_{C^2(\overline{D})} + \right.$$
$$\left. + \left\|\delta_8(x_2)\right\|_{C[0,l_2]} \right] + \chi N_3(l_1 l_2)^{1/2}.$$
$N_i, i-2,3$ are some positive numbers. Hence, given the condition of the theorem, we obtain that the evaluation of stability (7) is correct at $(x_1,x_2) \in \overline{D}$. The uniqueness of the solution to the problem follows from evaluation (7) and the theorem is proved.

4 Method of Successive Approximations

The method of successive approximations for solving problem (1)- (5) is applied according to the scheme:
$$-a_{i_0}^{(s)}(x_2)u_{x_{i_0}x_{i_0}}^{(s+1)} - a_{i_1}(x_2)u_{x_{i_1}x_{i_1}}^{(s+1)} +$$
$$+ c(x_2)u^{(s+1)} = h(x_1,x_2),\ (x_1,x_2) \in D, \tag{14}$$

$$u^{(s+1)}(0,x_2) = \phi_1(x_2),$$
$$(2-e)u^{(s+1)}(l_1,x_2) + (e-1)u_{x_1}^{(s+1)}(l_1,x_2) = \phi_2(x_2),$$
$$0 \le x_2 \le l_2, \tag{15}$$

$$b_0 u_{x_2}^{(s+1)}(x_1,0) + b_1 u^{(s+1)}(x_1,0) = \varphi_1(x_1),$$
$$0 \le x_1 \le l_1, \tag{16}$$
$$d_0 u_{x_2}^{(s+1)}(x_1,l_2) + d_1 u^{(s+1)}(x_1,l_2) = \varphi_1(x_1),$$
$$0 \le x_1 \le l_1, \tag{17}$$

$$a_{i_0}^{(s+1)}(x_2)u_{x_1}^{(s+1)}(0,x_2) = g_{i_0}(x_2),$$
$$0 \le x_2 \le l_2, \tag{18}$$

According to the scheme (14) - (18) successive iterations are carried out as follows: first some $a_{i_0}^{(0)}(x_2) > 0$ belongings are chosen $C^\alpha[0,l_2]$ and substituted into equation (14). Then problem (14) - (17) is solved and $u^{(1)}(x_1,x_2)$. The function $u_{x_1}^{(1)}(0,x_2)$, from the conditions (18) is found $a_{i_0}^{(1)}(x_2)$ and this function is used for the next iteration step.

Theorem 2. Let the solution of problem (1)-(5) exist and for all $s = 0,1,...,\ u^{(s)}(x_1,x_2) \in$
$$\in C^2(D), a_{i_0}^{(s)}(x_2) \in C^\alpha\left[0,l_2\right],$$
$$g_1(x_2)u_{x_1}^{(s)}(0,x_2) > 0, Nl_1 l_2 < 1$$
and the derivatives of the function:
$u^{(s)}(x_1,x_2)$ up to second order are uniformly bounded.

Then the functions $\left\{a_{i_0}^{(s)}(x_2),\quad u^{(s)}(x_1,x_2)\right\}$ obtained by the method of successive approximations (14)-(18) at $s \to +\infty$ uniformly converge to the solution of problem (1)-(5) at the rate of geometric progression. N -positive constant, depending on the given tasks.

Proof. Assume
$$Z^{(s)}(x_1,x_2) = u(x_1,x_2) - u^{(s)}(x_1,x_2), \lambda_{ii_0}^{(s)}(x_2) =$$
$$= a_{i_0}(x_2) - a_{i_0}^{(s)}(x_2).$$
From (1) - (5), respectively, subtracting (14) - (18) we get:
$$-a_{i_0}(x_1)Z_{x_{i_0}x_{i_0}}^{(s+1)} - a_{i_1}(x_1)Z_{x_{i_2}x_{i_2}}^{(s+1)} + c(x_1)Z^{(s+1)} =$$
$$= \alpha_{i_0}^{(s)}(x_1,x_2)\lambda_{i_0}^{(s)}(x_2),\ (x_1,x_2) \in D, \tag{19}$$

$$Z^{(s+1)}(0,x_2) = 0, (2-e)Z^{(s+1)}(l_1,x_2) +$$
$$+(e-1)Z_x^{(s+1)}Z_1,x_2) = 0, \tag{20}$$

$$b_0 Z_{x_2}^{(s+1)}(x_1,0) + b_1 Z^{(s+1)}(x_1,0) = 0, \tag{21}$$

$$d_0 Z_{x_2}^{(s+1)}(x_1,l_2) + d_1 Z^{(s+1)}(x_1,l_2) = 0, \tag{22}$$

$$\lambda_{i_0}^{(s+1)}(x_2) = \gamma^{(s)}(x_2)Z_{x_1}^{(s+1)}(0,x_2), \tag{23}$$

Where
$$\alpha_{i_0}^{(s)}(x_1,x_2) = u_{x_{i_0}x_{i_0}}^{(s+1)},$$
$$\gamma^{(s)}(x_2) = a_{i_0}(x_1)[-u_{x_1}^{(s+1)}(0,x_2)]^{-1}.$$

Using the Green's function from (19)- (22) we define $Z^{(s+1)}(x_1,x_2)$ through the right side of equality (19) and substitute this expression in the condition (23).

Then we obtain:

$$\lambda_{i_0}^{(s+1)}(x_2) = \gamma_1^{(s)}(x_2) \int_D G_{x_1}(0, x_2, \xi_1, \xi_2) \left[\alpha_{i_0}^{(s)}(\xi_1, \xi_2)\lambda_{i_0}^{(s)}(\xi_2)\right] d\xi_1 d\xi_2. \quad (24)$$

Put $\chi^{(s)} = \max_{x_2} \left|\lambda_{i_0}^{(s)}(x_2)\right|$. In the former way, as a proof of Theorem 1, it follows from system (24) that $\chi^{(s+1)} \le \chi^{(s)} N_3 (l_1 l_2)^{1/2}$. Thus the theorem is proved.

5 Existence of a Solution

Let us first prove one lemma. We will take $c(x_2) = 1$.

Lemma 2. Let the problem

$$-a_1(x_2)u_{x_1x_1} - a_2(x_2)u_{x_2x_2} + u = h(x_1, x_2), \quad (x_1, x_2) \in D, \quad (25)$$

$$u(0, x_2) = \phi_1(x_2), \quad (2-e)u(l_1, x_2) + (e-1)u_{x_1}(l_1, x_2) = \phi_2(x_2), \quad 0 \le x_2 \le l_2, \quad (26)$$

$$b_0 u_{x_2}(x_1, 0) + b_1 u(x_1, 0) = \varphi_1(x_1), \quad 0 \le x_1 \le l_1, \quad (27)$$

$$d_0 u_{x_2}(x_1, l_2) + d_1 u(x_1, l_2) = \varphi_2(x_1), \quad 0 \le x_1 \le l_1, \quad (28)$$

satisfying conditions:

$b_0\phi_{1x_2}(0) + b_1\phi_1(0) = \varphi_1(0), d_0\phi_{1x_2}(l_2) + d_1\phi_1(l_2) = \varphi_2(0)$,
$b_0\phi_{2x_2}(0) + b_1\phi_2(0) = \varphi_1^{(e-1)}(l_1), d_0\phi_{2x_2}(l_2) + d_1\phi_2(l_2) = \varphi_2^{(e-1)}(l_1)$ at a given $a_1(x_2), a_2(x_2) \ge \mu_0 > 0$ has a solution belonging to $C^2(D) \cap C(\overline{D})$ and
$-Ml_1 \le h(x_1, x_2) \le 0, (-1)^{e-1}\phi_2(x_2) \ge 0,$
$(b_1 - b_0)\varphi_1(0) \ge 0, \varphi_2(0) \ge 0,$
$b_1\varphi_{1x_1}(0) < b_0, d_1\varphi_{2x_1}(0) < d_0, \phi_{1x_2x_2}(x_2) = 0,$
$\left[b_1 m(0) - b_0 m'(0))\right] l_1^{-1} x_1 \le (b_0 - b_1)\left[\varphi_1(0) - \varphi_1(x_1)\right] \le b_1 M x_1,$
$\left[d_0 m'(l_2) + d_0 m(l_2))\right] l_1^{-1} x_1 \le \varphi_2(0) - \varphi_2(x_1) \le d_1 M x_1.$

Then

$$-M - \phi_1(x_2)(2\mu_0)^{-1} l_1 \le u_{x_1}(0, x_2) \le -m(x_2) l_1^{-1}, \quad (29)$$

Where,

$$M = \max\left\{\max_{x_2} l_1^{-1}[\phi_1(x_2) - \phi_2(x_2)],\ b_1 \max_{x_1}\left|\varphi_{1x_1}(x_1)\right|,\ d_1 \max_{x_1}\left|\varphi_{2x_1}(x_1)\right| \right\},$$

$m(x_2) \in C^2[0, l_2], m(x_2) > 0, b_0 m'(0) + b_1 > 0,$
$d_0 m'(l_2) - d_1 < 0, m''(x_2) \ge 0$

Proof. Suppose

$$\upsilon(x_1, x_2) = u(x_1, x_2) + m(x_2)x_1 l_1^{-1} - \phi_1(x_2), V(x_1, x_2) = -u(x_1, x_2) + \phi_1(x_2) - Mx_1 - \phi_1(x_2)(2\mu_0)^{-1} x_1(l_1 - x_1).$$

It is not difficult to check that satisfies the conditions of the problem

$$-a_1(x_2)\upsilon_{x_1x_1} - a_2(x_2)\upsilon_{x_2x_2} + \upsilon = -\phi_1(x_2) + m(x_2)l_1^{-1}x_1 - a_2(x_2)m''(x_2)x_1 l_1^{-1} + h(x_1, x_2),$$

$$\upsilon(0, x_2) = 0,$$

$$(e-1)\upsilon_{x_1}(l_1, x_2) + (2-e)\upsilon(l_1, x_2) = (e-1)m(x_2)l_1^{-1} + (2-e)\left[m(x_2) - \phi_1(x_2)\right] + \phi_2(x_2),$$

$$b_0\upsilon_{x_2}(x_1, 0) + b_1\upsilon(x_1, 0) = \varphi_1(x_1) + b_0\left[-\phi_{1x_2}(0) + m'(0)l_1^{-1}x_1\right] + b_1\left[-\phi_1(0) + m(0)l_1^{-1}x_1\right],$$

$$d_0\upsilon_{x_2}(x_1, l_2) + d_1\upsilon(x_1, l_2) = \varphi_2(x_1) + d_0\left[-\phi_{1x_2}(l_2) + m'(l_2)l_1^{-1}x_1\right] + d_1\left[-\phi_1(l_2) + m(l_2)l_1^{-1}x_1\right].$$

Given the conditions of the lemma, we obtain that the largest positive value of the function $\upsilon(x_1, x_2)$ is reached at $x_1 = 0$.

Then $\upsilon_{x_1}(0, x_2) \le 0$, in other words

$$u_{x_1}(0, x_2) \le -m(x_2) l_1^{-1} \quad (30)$$

Similarly, after substituting $V(x_1, x_2)$ in (25) - (28) and considering the conditions of the lemma, we obtain that the largest positive value of the function $V(x_1,x_2)$ is achieved at $x_1=0$. Therefore $V_{x_1}(0,x_2)\le 0$, or

$$-M-\phi_1(x_2)(2\mu_0)^{-1}l_1\le u_{x_1}(0,x_2) \qquad (31)$$

Combining estimates (30) and (31), we obtain estimate (29). Lemma 2 is proved.

Theorem 3. Empty

$$-Ml_1\le h(x_1,x_2)\le 0,\ (-1)^{e-1}\phi_2(x_2)\ge 0,$$
$$(b_1-b_0)\varphi_1(0)\ge 0,\ \varphi_2(0)\ge 0,$$
$$b_1\varphi_{1x_1}(0)<b_0,d_1\varphi_{2x_1}(0)<d_0,\ \phi_{1x_2x_2}(x_2)=0,$$
$$l_1^{1-e}m(x_2)\le(2-e)\phi_1(x_2)-\phi_2(x_2)\le$$
$$\le(1-e)\left[(2\mu_0)^{-1}l_1\phi_1(x_2)\right]+Ml_1^{2-e},$$
$$\left[b_1\,m(0)-b_0m'(0))\right]l_1^{-1}x_1\le(b_0-b_1)$$
$$\left[\varphi_1(0)-\varphi_1(x_1)\right]\le b_1Mx_1,$$
$$\left[d_0\,m'(l_2)+d_0m(l_2))\right]l_1^{-1}x_1\le\varphi_2(0)-$$
$$-\varphi_2(x_1)\le d_1Mx_1,$$
$$g_1(x_2)<0,\ g_0'\le -g_1(x_2)-\frac{1}{2}\phi_1(x_2)l_1,$$

$m(x_2)-$ is such a non-negative function that it is $g_1(x_2)[m(x_2)]^{-1}$ bounded, g_0' a positive number. Then problem (1) - (5) has at least one solution.

Proof. The proof is done by the method of successive approximations. It follows from the statement of the lemma that:

$$-M[1+\phi_1(x_2)(2g_0')^{-1}l_1]\le u_{x_1}^{(s+1)}(0,x_2)\le -m(x_2)\cdot l_1^{-1},$$
$$0<x_2<l_2,$$

then

$$g_0'M^{-1}\le a_{i_0}^{(s+1)}(x_2)\le\max_{x_2}\{-g_1(x_2)[m(x_2)]^{-1}\}\cdot l_1$$

Thus, for all approximations, the function $a_{i_0}^{(s)}(x_2)$ is strictly positive, continuous, and uniformly bounded. Then it follows from the general theory of elliptic equations that, under the conditions of the theorem, the sequence $\{u^{(s)}(x_1,x_2)\}$ is uniformly bounded by the norm $W_p^2, p>2$. Therefore $\{u^{(s)}(x_1,x_2)\}$, it is compact in $C^1(\overline{D})$. It follows from condition (18) that the sequence $a_{i_0}^{(s+1)}(x_2)$ will be compact in $C[0,l_2]$. Hence, and from (14) - (17) the compactness $\{u^{(s)}(x_1,x_2)\}$ in $C^2(\overline{D})$. In the system (14) - (18) passing to the limit at $s\to+\infty$ we obtain that there exists a pair of functions $\{a_{i_0}(x_2),u(x_1,x_2)\}$ satisfying conditions (1) - (5). The theorem is proved.

At $k=0,q=0$ we get the following expression for the function $\widetilde{\delta}_e(x_1,x_2)$:

$$\widetilde{\delta}_e(x_1,x_2)=(\frac{l_1-x_1}{l_1})^e\delta_3(x_2)+\frac{x_1^e}{2^{e-1}l_1}\delta_4(x_2)+$$
$$\frac{l_2-x_2}{l_2}\Big[\delta_5(x_1)-(\frac{l_1-x_1}{l_1})^e\delta_5(0)-$$
$$-\frac{x_1^e}{2^{e-1}l_1}\delta_5^{(e-1)}(l_1)\Big]+\frac{x_2}{l_2}\ \Big[\delta_6(x_1)-$$
$$-(\frac{l_1-x_1}{l_1})^e\delta_6(0)-\frac{x_1^e}{2^{e-1}l_1}\delta_6^{(e-1)}(l_1)\Big].$$

At $k=0,q=0,e=1$ to solve problem (1) - (5) we can take the following functions:

$$u(x_1,x_2)=-(\frac{x_{22}^2}{2}+\frac{x_2+1}{2}-x_1)x_1+\frac{5}{2},$$
$$a_1(x_2)=\frac{1}{2}(x_2+\frac{5}{2}),a_2(x_2)=x_2+1,c(x_2)=1.$$

In this case $m(x_2)$ is defined as follows:

$$m(x_2)=\frac{x_2^2+x_2+1}{2}l_1.$$

The conditions of Theorem 3 are satisfied for this function.

6 Results

Thus, the inverse problem of finding the coefficients of a linear elliptic equation under various boundary conditions in a given rectangle was studied. To solve the inverse problem, theorems on existence, uniqueness, and stability were proven. Using the method of successive approximations, a regularization algorithm was constructed to determine several coefficients. The inverse problem of finding the coefficients and solving a linear elliptic equation in a given rectangle is studied. A theorem of existence, uniqueness, and stability of the solution to the posed inverse problem is proven.

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