In-depth Study of Eigenvalues in a Boundary Value Problem for a given Partial q-Differential Equation

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Abstract: - In this paper, we are focused on studying a boundary values problem of the second-order differential equation of Euler-type in the classical version of Calculus, given by the following expression:

$$u_{xx} = f\left(\frac{\alpha x+\beta y+a}{\gamma x+\delta y+b}\right)u_{yy},\ \ \alpha, \beta, \gamma, \delta, a, b \in \mathbb{R} \tag{1}$$

and includes the above boundary conditions:

$$u(0, y) = u(N, y) = 0. \tag{2}$$

Firstly, we have proposed the construction of a new function f with the intention of transforming the equation (1) into an Euler-type equation. Since all of these problems are too difficult to solve in Classical Calculus, this study aims to convert them into equations of this type for the ease of study in q-Calculus. Then, we proposed a transformation method for both the equation and the boundary conditions. Thus, the boundary value problem consists of a second-order partial differential equation and boundary conditions dependent on the eigenvalues. By using the procedure of q-difference over a time scale $\mathbb{T}_q$, we obtained a second-order Euler q-difference equation with Dirichlet boundary conditions. Also, we have analyzed the exact number of eigenvalues for all cases that arise from the study of our problem. Here, we have presented three theorems, two of which show the correct number of eigenvalues of two issues with eigenvalues derived from our principal problem. The last one shows a relation between two eigenvalues of a problem for $\mathbb{T}_q^{\mathbb{N}_0}$ and $\mathbb{T}_q^{-\mathbb{N}_0}$. We also have given some examples that prove the above conclusions.

Key-Words: - Time Scale, Quantum Calculus, q-Differential Equation, q −Difference Equation, Eigenvalues, Eigenfunctions.

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1 Introduction

The theory of quantum calculus, or q −calculus, has been attracting the attention of many researchers, and the interest in this subject is still growing for its practical applications, especially in the physical sciences, specifically within the domain of *"Quantum Physics."* It is seen as a connection between mathematics and physics, operating independently of the concept of limits. Jackson was the first to present some applications of q −calculus by introducing q-analogs of derivatives and integrals, q-derivatives and q –integrals. Therefore, the physical meaning of q-deformation can be better understood in terms of the Jackson q-derivative, which corresponds to q −difference equations, than in terms of continuous derivatives and continuous differential equations. The basic rules and exciting definitions of this calculus in comparison with classical Newton-Leibnitz calculus were studied among others in [1], [2], [3]. In particular, some significant results of q-calculus, where the smoothness of a function is no longer a requirement, are presented in [4].

At [5], the idea of generalizing the concept of q-derivative from a real function f with one variable to a two-variable function is given, and a q -directional derivative of a function is constructed.

In ordinary classical calculus, we focus on studying differential equations, whereas in discrete calculus, our concentration is on difference equations. In calculus, we explore the concept of q-derivative or q-difference equations (p-derivative or p -difference equations, h-derivative or h-difference equations), which have applications across various mathematical areas such as number theory, quantum theory, combinatorics, orthogonal polynomials, and essential hypergeometric functions. Recently, there has been significant interest in applying quantum calculus to differential transform methods to obtain analytical approximate solutions for both ordinary and partial differential equations. By using q −calculus, solutions can be generated for certain differential equations. The reduced q −transform techniques were presented in the literature to approach several different linear or nonlinear differential systems, such as ordinary differential equations, functional differential equations, impulsive differential equations, and partial differential equations, among others [6], [7]. Mainly when $q = 1$ the solutions correspond to the classical version of the provided initial value problem solving partial q −differential equations in some Euler-type boundary value problems. In [8] and [9] is studied a boundary value problem which consists into a second-order q −difference equation together with Dirichlet boundary conditions reduced to an eigenvalue problem for a second-order Euler q-difference equation by separation of variables.

On the other hand, the study of q-derivatives on discrete, continuous, and, more generally, on an arbitrary nonempty closed set (i.e., a time scale) is a well-known subject under current solid development. For an introduction to the theory of calculus on time scales, we refer to [10], [11], [12], [13], [14] and [15]. As time scale calculus has evolved, many authors have focused on integrating methodologies from both time-scale and q −calculus. The most famous examples of calculus on time scales are differential calculus ($\mathbb{T} = \mathbb{R}$) difference calculus ($\mathbb{T} = \mathbb{N}$) and quantum calculus ($\mathbb{T}_q = q^{\mathbb{N}_0} = \{q^k, k \in \mathbb{N}_0\}$ where $0 < q < 1$). This paper includes the fundamental definitions and characteristics of delta q −calculus ($\mathbb{T} = \mathbb{R}$) and delta q −calculus on a time scale $\mathbb{T}_q$.

Due to the importance of this quantum calculus on time scales and taking into account that the oscillatory or asymptotic properties of the solutions of q-difference equations are essential to understanding several physics phenomena better, [15], [16], [17] introduced in the literature the concept of determining the eigenvalues and their count for the resulting eigenvalue problem defined on the quantum time scale.

In order to solve a partial differential equation (PDE) of the second order of form $u_{xx} = g(x, y)u_{yy}$. The methods of solutions of this equation depend on properties of g function, as well as the given boundary or initial conditions. This equation, models a lot of real-life problems and describe a lot of PDEs based on the nature of $g(x, y)$, which will have a major effect on the solution technique:

a) The equation reduces to a linear PDE with constant coefficients if $g(x, y)$ is a constant.

b) If $g(x, y)$ is not constant, in most cases the equation can not be solved analiticaly because it can require numerical methods.

Since $f\left(\frac{\alpha x+\beta y+a}{\gamma x+\delta y+b}\right)$is widely used in mathematical modeling, especially in complex biological models.

The organization of this paper is as follows: in section 2, we introduce some basic definitions and preliminary facts used throughout the paper for delta q −calculus and delta q −calculus on a time scale compared to the classical Newton-Leibniz calculus. Section 3 analyzes an equation with partial derivatives of the second order according to various cases and transforms it case by case into an Euler-type q −difference equation. In the end, in Section 4, we determine the eigenvalues and their count for the resulting eigenvalue problem and provide examples to illustrate the effectiveness of the proposed theorems.

2 Preliminaries

This section introduces some of the q −notations used throughout the paper. We use the standard notations found in [1], [2], [3] and [4].

Quantum calculus is a non-limits version of calculus, where derivatives are differences and anti-derivatives are sums, with the derivative of a function $f(t)$ being defined as:

$$f'(t) = \frac{d}{dt}f(t) = \lim_{h\to 0}\frac{f(t+h) - f(t)}{h},$$

related to the existence of the limit.

The h -derivative is defined as:

$$d_h f(t) = \frac{f(t+h) - f(t)}{h}$$

where h is a non zero fixed scalar.

The p -derivative is defined as:

$$D_p f(t) = \frac{f(t^p) - f(t)}{t^p - t}$$

where p is a fixed scalar different from 1.

The delta q -derivative is defined by:

$$\Delta_q f(t) = \frac{f(qt) - f(t)}{(q - 1)t}$$

where q is a fixed scalar different from 1.

Note that these types of derivatives do not use the limit. So, there are different types of quantum calculus. In this section, we recall some notations of

q-calculus with delta q-derivative and some basic facts and results on q −calculus and time scales. Studying q −calculus on a time scale leads to essential facts and results. Some well-known time scales presented in other authors' works are as follows:

$$\mathbb{T}_q^{\mathbb{Z}} = \{q^t, t \in \mathbb{Z}\} \cup \{0\}, \; 0 < q < 1. \tag{3}$$

This time scale is equivalent to:

$$\mathbb{T}_q^{\mathbb{Z}} = \mathbb{T}_q^{\mathbb{N}_0} \cup \mathbb{T}_q^{-\mathbb{N}_0} \cup \{0\}, \; 0 < q < 1, \tag{4}$$

where,

$$\mathbb{T}_q^{\mathbb{N}_0} = \{q^t, t \in \mathbb{N}_0\}, \; 0 < q < 1$$
$$\mathbb{T}_q^{-\mathbb{N}_0} = \{q^{-t}, t \in \mathbb{N}_0\}, \qquad 0 < q < 1$$

or

$$\mathbb{T}_q^{-\mathbb{N}_0} = \{q^t, t \in \mathbb{N}_0\}, \; q > 1. \tag{5}$$

As given in [17], the time scale is defined as a set with zero Minkowski (or box-counting) dimension. Moreover, it is a time scale where $\{a_n\}_{n\in\mathbb{N}}$ is a monotonically decreasing sequence converges to zero and $\mathbb{T}$ is defined as:

$$\mathbb{T} = \{a_n\}_{n\in\mathbb{N}} \cup \{0\}, a_1 = 1.$$

Its particular case time scale is defined by:

$$\mathbb{T}_q = \{q^t, t \in \mathbb{N}_0\} \cup \{0\}, \text{ for } 0 < q < 1 \tag{6}$$

The above collections are defined on (4) and (5) are used in our study's other work. For symmetry, we will focus on a time scale $\mathbb{T}_q$ as follows.

Consider an arbitrary function $f: \mathbb{T}_q \to \mathbb{R}$ with $0 < q < 1$. The Jackson's delta q-difference operator $\Delta_q f$ of the function f, is given by:

$$\begin{cases} \Delta_q f(t) = \dfrac{f(t) - f(qt)}{(1-q)t}, & \text{if } t \neq 0 \\ \Delta_q f(0) = \lim\limits_{t\to 0} \Delta_q f(t), \end{cases} \tag{7}$$

so, this q-derivative can be applied to functions not contained 0 in their domain of definition. If $f(t)$ is differentiable, note that:

$$f'(t) = \frac{d}{dt} f(t) = \lim_{qt\to t} \frac{f(qt) - f(t)}{qt - t} = \lim_{q\to 1} \Delta_q f(t),$$

so, this analog q −derivative is reduced to the ordinary derivative when $q \to 1$. The q-version of the above derivation is the evident and plain ratio in contrast with Leibniz's notation $\frac{d}{dt} f(t)$, since it is known that the latter is a ratio of two "infinitesimals."

Now, we shortly describe the idea of studying Euler-type equations. In particular, Euler-type ordinary differential equations are defined by:

$$t^n f^{(n)}(t) + a_{n-1} t^{n-1} f^{(n-1)}(t) + \cdots + a_1 t f'(t) + a_0 f(t) = g(t),$$

where $a_0, a_1, \ldots, a_{n-1}$ are real numbers. As is known, the linear equation remains linear even after each variable transformation. In the above equation, if transform the independent variable x using the relation $x = \varphi(t)$, where $\varphi(t)$, is an arbitrary function defined and n times differentiable in an interval $]a, b[$ corresponding to the change of x in the interval $]a, b[$ and such that $\varphi'(t) \neq 0$ for every $t \in]a, b[$.

It is known that in these equations, the substitution of the variable $t = e^x$ turns it into a linear equation with constant coefficients, which is easier to solve. What motivates us to use q −calculus is the well-known fact that, as with the ordinary derivative, the action of taking the q −derivative of a function is a linear operator. In other words, for any constants a and b, we have:

$$\Delta_q\big(a f_1(t) + b f_2(t)\big) = a\Delta_q f_1(t) + b\Delta_q f_2(t),$$

and a higher order of delta q −derivatives is as follows:

$$\Delta_q^0 f(t) = f(t), \; \Delta_q^{n+1} f(t) = \Delta_q\left(\Delta_q^n f(t)\right) \quad (n = 0,1,2,3 \ldots).$$

In [8] and [9] is used the separation of variables method for an Euler-type, second-order partial q −differential equation with Dirichlet boundary conditions to arrive at a particular eigenvalue problem. Additionally, by q-calculus for a function $u: \mathbb{T}_q \times \mathbb{T}_q \to \mathbb{R}$, we define the Jackson derivatives of u concerning the first and the second variable, respectively, by:

$$D_{q,x} u(x,y) = \tilde{u}_x(x,y) = \frac{u(x,y) - u(qx,y)}{(1-q)x} \text{ if } x \neq 0$$

and

$$D_{q,y} u(x,y) = \tilde{u}_y(x,y) = \frac{u(x,y) - u(x,qy)}{(1-q)y} \text{ if } y \neq 0.$$

For convenience, in this paper, we will use symbols, $\tilde{u}_x, \tilde{u}_{xx}, \tilde{u}_y, \tilde{u}_{yy}, \tilde{u}_{xy}, \tilde{u}_{yx}$.

$$D_{q,x}^2 = \frac{\tilde{u}_x(x,y) - \tilde{u}_x(qx,y)}{(1-q)x} \text{ if } x \neq 0.$$

When we expand the above equation, we obtain for $x \neq 0$

$$D^2_{q,x} = \frac{\frac{u(x,y)-u(qx,y)}{(1-q)x} - \frac{u(qx,y)-u(q^2x,y)}{q(1-q)x}}{(1-q)x},$$

$$D^2_{q,x} = \tilde{u}_{xx}(x,y) = \frac{u(q^2x,y) - (q+1)u(qx,y) + qu(x,y)}{(1-q)^2qx^2}. \tag{8}$$

Similarly, we can compute the partial derivative $\tilde{u}_{yy}$ for $y \neq 0$

$$D^2_{q,y} = \tilde{u}_{yy}(x,y) = \frac{u(x,q^2y) - (q+1)u(x,qy) + qu(x,y)}{(1-q)^2qy^2}. \tag{9}$$

Note that,

$$u_{xx} = \lim_{q\to 1}\tilde{u}_{xx}, \; u_{yy} = \lim_{q\to 1}\tilde{u}_{yy}.$$

We start with an analysis according to the cases of an equation with partial derivatives of the second order and, case by case, transform it into the following Euler-type:

$$x^2\tilde{u}_{xx} = y^2\tilde{u}_{yy}, \quad u(1,y) = u(q^N,y) = 0,$$

which is a second-order difference equation combined with Dirichlet boundary conditions.

3 Main Results

3.1 Boundary Value Problem

Let $N \in \mathbb{N}_0$ and consider the boundary value problem in ordinary calculus as follows:

$$u_{xx} = f\left(\frac{\alpha x+\beta y+a}{\gamma x+\delta y+b}\right)u_{yy}, \; \alpha,\beta,\gamma,\delta,a,b \in \mathbb{R} \tag{10}$$

with boundary conditions:

$$u(0,y) = u(N,y) = 0. \tag{11}$$

Its associated second-order q-difference equation, together with Dirichlet boundary conditions, is:

$$\tilde{u}_{xx} = f\left(\frac{\alpha x+\beta y+a}{\gamma x+\delta y+b}\right)\tilde{u}_{yy}, \quad u(q^0,y) = u(q^N,y) = 0, \quad \alpha,\beta,\gamma,\delta,a,b \in \mathbb{R}. \tag{12}$$

A generalized solution of the problem (12) is a function $u(x,y)$ that satisfies the equation and Dirichlet boundary conditions.

First, let us solve the problem of constructing the function f such that equation (10) is of the Euler type.

We define f as follows:

$$f(t) = \begin{cases} \left(\frac{\gamma t-\alpha}{\delta t-\beta}\right)^2, & \beta^2+\delta^2 > 0 \\ c^2 & \alpha=\beta=\gamma=\delta=0 \end{cases} \tag{13}$$

and depending on the constants $\alpha,\beta,\gamma,\delta,a,b$, we divide the problem into three cases:

Case 1.

If $a = b = 0$, then function $f\left(\frac{\alpha x+\beta y+a}{\gamma x+\delta y+b}\right)$ is written differently

$$f\left(\frac{\alpha x+\beta y}{\gamma x+\delta y}\right) = \left(\frac{\gamma\left(\frac{\alpha x+\beta y}{\gamma x+\delta y}\right)-\alpha}{\delta\left(\frac{\alpha x+\beta y}{\gamma x+\delta y}\right)-\beta}\right)^2 = \left(\frac{\gamma(\alpha x+\beta y)-\alpha(\gamma x+\delta y)}{\delta(\alpha x+\beta y)-\beta(\gamma x+\delta y)}\right)^2 = \left(\frac{-(\alpha\delta-\beta\gamma)y}{(\alpha\delta-\beta\gamma)x}\right)^2 = \left(\frac{y}{x}\right)^2.$$

The equation (12) is now written in the form:

$$\tilde{u}_{xx} = \left(\frac{y}{x}\right)^2\tilde{u}_{yy}, \quad u(1,y) = u(q^N,y) = 0. \tag{14}$$

Case 2.

If $a^2+b^2 > 0$, then we first consider the case

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0, \; (\alpha\delta-\beta\gamma \neq 0). \tag{15}$$

From this condition, we generate the algebraic system

$$\begin{cases} \alpha x+\beta y+a = 0 \\ \gamma x+\delta y+b = 0 \end{cases}$$

which has a unique solution $(\vartheta,\mu) \neq (0,0)$. We apply the following transformation:

$$x = s+\vartheta, \; y = t+\mu \tag{16}$$

where

$$\begin{cases} \alpha\vartheta+\beta\mu+a = 0 \\ \gamma\vartheta+\delta\mu+b = 0 \end{cases}$$

and obtain in the same way as in ***Case 1***:

$$f\left(\frac{\alpha x+\beta y+a}{\gamma x+\delta y+b}\right) = f\left(\frac{\alpha(s+\vartheta)+\beta(t+\mu)+a}{\gamma(s+\vartheta)+\delta(t+\mu)+b}\right) = f\left(\frac{\alpha s+\beta t}{\gamma s+\delta t}\right) = \left(\frac{t}{s}\right)^2.$$

Let us recall a relation between the derivatives from classic calculus,

$$u_{xx} = u_{ss}, \quad u_{yy} = u_{tt},$$

moreover, we take the boundary value problem in ordinary calculus as follows:

$$u_{ss} = \left(\frac{t}{s}\right)^2 u_{tt},\ u(-\vartheta, t+\mu) = u(N-\vartheta, t+\mu) = 0,\ \vartheta, \mu \in \mathbb{R}. \quad (17)$$

Let us construct its associated equation in q-calculus

$$\tilde{u}_{ss} = \left(\frac{t}{s}\right)^2 \tilde{u}_{tt},\ u\left(q^{-\vartheta}, t+\mu\right) = u\left(q^{N-\vartheta}, t+\mu\right) = 0,\ \vartheta, \mu \in \mathbb{R}. \quad (18)$$

Case 3.
If $a^2 + b^2 > 0$, then we consider the case

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 0. \quad (19)$$

Let us now assume that $\alpha\delta - \beta\gamma = 0$, so there exists $\lambda \in R$, such that $\gamma = \lambda\alpha$, $\delta = \lambda\beta$.
Since,

$$f(t) = \left(\frac{\gamma t - \alpha}{\delta t - \beta}\right)^2 = \left(\frac{\lambda\alpha t - \alpha}{\lambda\beta t - \beta}\right)^2 = \left(\frac{\alpha}{\beta}\right)^2,$$

the problem (10)-(11) is now written in the form:

$$u_{xx} = \left(\frac{\alpha}{\beta}\right)^2 u_{yy}, \quad u(0,y) = u(N,y) = 0, \quad (20)$$

which is a wave-problem, and its solutions are known. This last case, **Case 3**, also includes the subcases when $\alpha = \beta = 0$ or $\gamma = \delta = 0$.
We are assuming that $\gamma = \delta = 0$, while $a^2 + b^2 > 0$, so equation (10) has the form:

$$u_{xx} = f\left(\frac{\alpha x + \beta y + a}{b}\right) u_{yy}.$$

Then for the function f we have:

$$f(t) = \left(\frac{\gamma t - \alpha}{\delta t - \beta}\right)^2 = \left(\frac{\alpha}{\beta}\right)^2,$$

then

$$u_{xx} = \left(\frac{\alpha}{\beta}\right)^2 u_{yy}, \quad u(0,y) = u(N,y) = 0. \quad (21)$$

We are assuming that $\alpha = \beta = 0$, while $a^2 + b^2 > 0$, equation (10) has the form

$$u_{xx} = f\left(\frac{a}{\gamma x + \delta y + b}\right) u_{yy},$$

where,

$$f(t) = \left(\frac{\gamma t - 0}{\delta t - 0}\right)^2 = \left(\frac{\gamma}{\delta}\right)^2,$$

then

$$u_{xx} = \left(\frac{\gamma}{\delta}\right)^2 u_{yy}, \quad u(0,y) = u(N,y) = 0. \quad (22)$$

We are assuming that $\alpha = \beta = \gamma = \delta = 0$, while $a^2 + b^2 > 0$; equation (10) has the form:

$$u_{xx} = c^2 u_{yy},$$

then

$$u_{xx} = c^2 u_{yy}, \quad u(0,y) = u(N,y) = 0. \quad (23)$$

Our work is based on these two problems (14) and (18) respectively

$$\tilde{u}_{xx} = \left(\frac{y}{x}\right)^2 \tilde{u}_{yy}, \quad u(1,y) = u(q^N, y) = 0$$

$$\tilde{u}_{ss} = \left(\frac{t}{s}\right)^2 \tilde{u}_{tt}, \quad u\left(q^{-\vartheta}, t+\mu\right) = u\left(q^{N-\vartheta}, t+\mu\right) = 0, \qquad (\vartheta, \mu) \in \mathbb{R}^2 \backslash (0,0).$$

3.2 Separation of Variables for Equation (14)

In this section, we will look for a generalized function $u(x,y)$ that satisfies the second-order q-difference equation in the problem (14):

$$x^2 \tilde{u}_{xx} = y^2 \tilde{u}_{yy},$$

which is equivalent to a second-order q-recursion relation

$$\frac{x^2\left(u(q^2x,y) - (q+1)u(qx,y) + qu(x,y)\right)}{(1-q)^2 qx^2} = \frac{y^2\left(u(x,q^2y) - (q+1)u(x,qy) + qu(x,y)\right)}{(1-q)^2 qy^2}$$

i.e.,

$$u(q^2x,y) - (q+1)u(qx,y) + qu(x,y) = u(x,q^2y) - (q+1)u(x,qy) + qu(x,y). \quad (24)$$

Using the separation of variables, we derive a specific eigenvalue problem with boundary conditions.

So, let's have $u(x,y) = f(x)g(y)$ so that $u(qx,y) = f(qx)g(y)$ and $u(q^2x,y) = f(q^2x)g(y)$. This is also applied to terms $u(x,qy)$ and $u(x, q^2y)$. We obtain that when we substitute these values into the q-difference equation (24), we obtain that:

$$f(q^2x)g(y) - (q+1)f(qx)g(y) + qf(x)g(y) = f(x)g(q^2y) - (q+1)f(x)g(qy) + qf(x)g(y) \quad (25)$$

Now we divide each side of (25) by $f(x)g(y)$ and then set both sides equal to a constant $-\lambda$ to obtain:

$$\frac{f(q^2x) - (q+1)f(qx) + qf(x)}{f(x)} = \frac{g(q^2y) - (q+1)g(qy) + qg(y)}{g(y)} = -\lambda, \quad (26)$$

and from boundary conditions of the problem (14):

$$f(1)g(y) = f(q^N)g(y) = 0. \tag{27}$$

Hence, from (26) and (27), the eigenvalue problem for the function f is:

$$f(q^2x) - (q+1)f(qx) + (q+\lambda)f(x) = 0,$$
$$f(1) = f(q^N) = 0. \tag{28}$$

We will use a similar substitution technique as the one we use to convert the ordinary Euler-type differential equation into an equation with constant coefficients.

Suppose that $f(x) = \alpha^{\log_q x}$ and $g(y) = \beta^{\log_q y}$. It is obvious that

$$f(qx) = \alpha^{log_q qx} = \alpha^{log_q q + log_q x} = \alpha\alpha^{log_q x}$$
$$= \alpha f(x)$$

and

$$f(q^2x) = f(q(q(x))) = \alpha f(qx) = \alpha^2 f(x).$$

Now, we will make the following substitutions into the Euler-type equation in (28), and we will get

$$\alpha^2 f(x) - (q+1)\alpha f(x) + (q+\lambda)f(x) = 0.$$

with characteristic equation

$$\alpha^2 - (q+1)\alpha + (q+\lambda) = 0. \tag{29}$$

We solve (29) as below:

$$\alpha = \frac{q+1 \pm \sqrt{(q+1)^2 - 4(q+\lambda)}}{2}$$
$$= \frac{q+1}{2} \pm \sqrt{\left(\frac{q-1}{2}\right)^2 - \lambda}.$$

Hence, we let:

$$\alpha_1 = \frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 - \lambda} \quad \text{and} \quad \alpha_2 = \frac{q+1}{2} - \sqrt{\left(\frac{q-1}{2}\right)^2 - \lambda}.$$

From the sign of $\Delta = \left(\frac{q-1}{2}\right)^2 - \lambda$ we have the following three cases:

Case I. $\Delta > 0$;
Case II. $\Delta = 0$;
Case III. $\Delta < 0$.

The eigenvalues and general solutions of the Euler-type equation in problem (28) for each of these cases are as follows:

Case I. $\lambda < \left(\frac{q-1}{2}\right)^2 \quad \alpha_1 \neq \alpha_2,\ \alpha_1, \alpha_2 \in \mathbb{R}$

$f(x) = c_1 f_1(x) + c_2 f_2(x)$ where $f_1(x)$ and $f_2(x)$ are linearly independent, so

$$f(x) = c_1 {\alpha_1}^{log_q x} + c_2 {\alpha_2}^{log_q x}. \tag{30}$$

Our next step is to find the eigenvalues of (29). We apply the first Dirichlet condition $f(1) = 0$ to (30), and we obtain that:

$$f(1) = c_1 {\alpha_1}^{log_q 1} + c_2 {\alpha_2}^{log_q 1} = c_1 + c_2 = 0,$$

so that

$$c = c_1 = -c_2.$$

Now we will use the relationship between c_1 and c_2 and apply it to the general solution (30), and then we will use the other Dirichlet condition $f(q^N) = 0$

$$f(q^N) = c\left({\alpha_1}^{log_q q^N} - {\alpha_2}^{log_q q^N}\right)$$
$$= c({\alpha_1}^N - {\alpha_2}^N) = 0.$$

Since for $c = 0$ we obtain the trivial solution, we shall discuss the following conditions for $N \in \mathbb{N}$

$${\alpha_1}^N = {\alpha_2}^N.$$

This can occur for $\alpha_1 = \alpha_2$ or for even N, $\alpha_1 = -\alpha_2$.
For $\alpha_1 = \alpha_2$ we have that

$$\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 - \lambda} = \frac{q+1}{2} - \sqrt{\left(\frac{q-1}{2}\right)^2 - \lambda}$$

i.e.,

$$\sqrt{\left(\frac{q-1}{2}\right)^2 - \lambda} = 0$$

so that:

$$\lambda = \left(\frac{q-1}{2}\right)^2,$$

which is not a valid value for λ in ***Case I***.

Next, for even N, $\alpha_1 = -\alpha_2$ we have that

$$\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 - \lambda} = -\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 - \lambda}.$$

which leads us to $q = -1$, contradicting the fact, $0 < q < 1$. Thus, there are no eigenvalues in this case.

Case II. $\lambda = \left(\frac{q-1}{2}\right)^2 \quad \alpha_1 = \alpha_2,\ \alpha_1, \alpha_2 \in \mathbb{R}$

$f(x) = c_1\, log_q x\, f_1(x) + c_2 f_1(x)$ where $f_1(x)$ and $log_q x\, f_1(x)$ are linearly independent.

$$f(x) = (c_1\, log_q x + c_2)\left(\frac{q+1}{2}\right)^{log_q x}. \tag{31}$$

Now we look at the first Dirichlet condition in (31) to find

$$f(1) = (c_1\, log_q 1 + c_2)\left(\frac{q+1}{2}\right)^{log_q 1} = 0,$$

so $c_2 = 0$.

Let $c = c_1$, and then apply the second Dirichlet condition, which gives us the following equality

$$0 = f(q^N) = c\, log_q\, q^N \left(\frac{q+1}{2}\right)^{log_q q^N} = cN\left(\frac{q+1}{2}\right)^N.$$

For this to be true either $c = 0, N = 0$ or $q = -1$, from which we do not obtain any eigenvalues. Hence, there are no eigenvalues.

Case III. $\lambda > \left(\frac{q-1}{2}\right)^2$, $\alpha_{1,2} = |\alpha|(\cos\theta \pm i sin\theta)$, $\alpha_1, \alpha_2 \in \mathbb{C}$

So, we have

$$f(x) = c_1 {\alpha_1}^{log_q x} + c_2 {\alpha_2}^{log_q x} = c_1\big(|\alpha|(\cos\theta + i sin\theta)\big)^{log_q x} + c_2\big(|\alpha|(\cos\theta - i sin\theta)\big)^{log_q x}$$

$$f(x) = |\alpha|^{log_q x}\left[c_1(\cos\theta + i sin\theta)^{log_q x} + c_2(\cos\theta - i sin\theta)^{log_q x}\right],$$

$$f(x) = |\alpha|^{log_q x}\left[c_1 \cos(\theta\, log_q x) + i c_1 sin(\theta\, log_q x) + c_2 \cos(\theta\, log_q x) - i c_2 sin(\theta\, log_q x)\right]$$

$$f(x) = |\alpha|^{log_q x}\left[(c_1 + c_2)\cos(\theta\, log_q x) + i(c_1 - c_2) sin(\theta\, log_q x)\right].$$

From substitutions $c_1 + c_2 = c_3$, $i(c_1 - c_2) = c_4$, this leads to the subsequent result

$$f(x) = |\alpha|^{log_q x}\left[c_3 cos(\theta\, log_q x) + c_4 sin(\theta\, log_q x)\right], \tag{32}$$

where $|\alpha| = \sqrt{(Re\alpha)^2 + (Im\alpha)^2}$ and $\theta = arccos\left(\frac{Re\alpha}{|\alpha|}\right)$, $c_3, c_4 \in \mathbb{R}$.

At the end, we will look at the case where $\lambda > \left(\frac{q-1}{2}\right)^2$, we will use the equation (32) with the first Dirichlet condition to find

$$f(1) = |\alpha|^{log_q 1}\left[c_3 cos(\theta\, log_q 1) + c_4 sin(\theta\, log_q 1)\right] = c_3 = 0.$$

We let $c = c_4$, and apply the other Dirichlet condition to obtain:

$$0 = f(q^N) = |\alpha|^{log_q q^N}\left(c sin(\theta\, log_q q^N)\right) = c|\alpha|^N sin(\theta N). \tag{33}$$

Note that

$$|\alpha| = \sqrt{\left(\frac{q+1}{2}\right)^2 - \left(\frac{q-1}{2}\right)^2 + \lambda} = \sqrt{q+\lambda} \text{ and}$$

$$\theta = arccos\left(\frac{Re\alpha}{|\alpha|}\right) = arccos\left(\frac{q+1}{2\sqrt{q+\lambda}}\right).$$

Therefore, we obtain from (33) that:

$$0 = f(q^N) = c\left(\sqrt{q+\lambda}\right)^N sin(\theta N). \tag{34}$$

If we look at the conditions at (34), $c = 0$, $\lambda = -q$ it may be suitable because $-q > \left(\frac{q-1}{2}\right)^2$ is a contradiction. So, we consider only that $sin(\theta N) = 0$. This leads us to $\theta_k N = k\pi$, which gives us the values $\lambda_k, k \in \mathbb{N}_0$ where

$$arccos\left(\frac{q+1}{2\sqrt{q+\lambda_k}}\right) = \frac{k\pi}{N}. \tag{35}$$

Since the cosine function is an even function, we do not lose anything if we continue to take $N \in \mathbb{N}$. Solving (35) for λ_k we have that:

$$\lambda_k = \left(\frac{q+1}{2cos\left(\frac{k\pi}{N}\right)}\right)^2 - q, \tag{36}$$

for $k = 0, \dots, (N-2)/2$ if N is an even integer and $k = 0, \dots, (N-1)/2$ if N is an odd one. Hence, we obtain the following main result.

Theorem 3.1

Let $N \in \mathbb{N}$. The problem (14) has exactly $\left[\frac{N+1}{2}\right]$ where $[\cdot]$ denotes the greatest integer function, eigenvalues, and they can be calculated from the formula (36). The corresponding eigenfunctions are given by

$$f_k(x) = \left(\sqrt{q+\lambda_k}\right)^{log_q x} sin\left(\frac{k\pi}{N} log_q x\right). \tag{37}$$

Example 3.1

Let $N = 6$. By (36), for $k = 0$ and $k = 1$, the eigenvalue and corresponding eigenfunctions are:

$$\lambda_0 = \lambda_6 = \left(\frac{q-1}{2}\right)^2 \text{ and } f_0(x) = 0,$$

$$\lambda_1 = \frac{q^2-q+1}{3} \text{ and } f_1(x) = \left(\frac{q+1}{\sqrt{3}}\right)^{log_q x} sin\left(\frac{\pi log_q x}{6}\right).$$

For $k = 2$, the eigenvalue and its corresponding eigenfunction are:

$$\lambda_2 = q^2 + q + 1 \text{ and } f_2(x) = (q+1)^{log_q x} sin\left(\frac{\pi log_q x}{3}\right)$$

$$\lambda_2 = \lambda_4 = \left(\frac{q+1}{-1}\right)^2 - q \text{ and } f_4(x) = [-(q+1)]^{log_q x} sin\left(\frac{2\pi log_q x}{3}\right),$$

$$\lambda_1 = \lambda_5 = \left(\frac{q+1}{-\sqrt{3}}\right)^2 - q \text{ and } f_5(x) = \left[-\left(\frac{q+1}{\sqrt{3}}\right)\right]^{log_q x} sin\left(\frac{5\pi log_q x}{6}\right).$$

Next, for $k = 3$ from (36), we will have that $q = -1$ which does not lead us to an eigenvalue. Similarly, we have that for $k = 4$ and $k = 5$ we conclude that $q < -1$. We will obtain the same result as the above argument when the values of k grow. Hence, there are only two eigenvalues as given above. In particular, if $q = \frac{1}{2}$, the eigenvalues are $\frac{1}{16}, \frac{1}{4}$ and $\frac{7}{4}$.

Theorem 3.2
Let $\mathbb{T}$ be the set $\{q^n\}_{n\epsilon\mathbb{N}_0} \cup \{0\}$, where $0 < q < 1$, and let $\{\lambda_m\}_{m\epsilon\mathbb{N}_0}$ be the sequence of eigenvalues of problem (14). Let $\mathbb{T}^*$ be the set $\{q^{-n}\}_{n\epsilon\mathbb{N}_0}$, where $0 < q < 1$, and let $\{\mu_m\}_{m\epsilon\mathbb{N}_0}$ be the sequence of eigenvalues of problem (14). Then, $q^2\mu_m = \lambda_m$, $m\epsilon\mathbb{N}_0$.

Proof:
For $q = 2$, the eigenvalues are $\frac{1}{4}, 1$ and 7, (see Bohner 2007). For $q = \frac{1}{2}$, the eigenvalues are $\frac{1}{16}, \frac{1}{4}$ and $\frac{7}{4}$.

$$\lambda_0 = \left(\frac{q-1}{2}\right)^2, \quad \lambda_1 = \frac{q^2 - q + 1}{3}, \quad \lambda_2 = q^2 + q + 1, \ldots,$$

$$\lambda_m = \left(\frac{q+1}{2cos\left(\frac{m\pi}{N}\right)}\right)^2 - q$$

$$\mu_0 = \frac{\left(\frac{1}{q} - 1\right)^2}{4} = \frac{1 - 2q + q^2}{4q^2} = \frac{\lambda_0}{q^2}$$

$$\mu_1 = \frac{\frac{1}{q^2} - \frac{1}{q} + 1}{3} = \frac{q^2 - q + 1}{3q^2} = \frac{\lambda_1}{q^2}$$

$$\mu_2 = \frac{1}{q^2} + \frac{1}{q} + 1 = \frac{q^2 + q + 1}{q^2} = \frac{\lambda_2}{q^2}.$$

$$\vdots$$

$$\mu_m = \left(\frac{\frac{1}{q} + 1}{2cos\left(\frac{m\pi}{N}\right)}\right)^2 - \frac{1}{q} = \frac{\left(\frac{q+1}{2cos\left(\frac{m\pi}{N}\right)}\right)^2 - q}{q^2} = \frac{\lambda_m}{q^2}.$$

3.3 Separation of Variables for equation (18)

$$\tilde{u}_{ss} = \left(\frac{t}{s}\right)^2 \tilde{u}_{tt}, \quad u\left(q^{-\vartheta}, t + \mu\right) = u\left(q^{N-\vartheta}, t + \mu\right) = 0, \qquad (\vartheta, \mu) \in \mathbb{R}^2 \backslash (0,0).$$

We use the separation of variables to arrive at a specific eigenvalue problem. So, let $(s,t) = f(s)g(t)$ and proceed as above, and by sign of $\left[\left(\frac{q-1}{2}\right)^2 - \lambda\right]$, we have the following three cases:

Case I.

$$\lambda < \left(\frac{q-1}{2}\right)^2 \text{ and } \quad f(s) = c_1{\alpha_1}^{log_q s} + c_2{\alpha_2}^{log_q s}, \tag{38}$$

Case II.

$$\lambda = \left(\frac{q-1}{2}\right)^2 \text{and } f(s) = (c_1 + c_2\, log_q\, s)\left(\frac{q+1}{2}\right)^{log_q s} \tag{39}$$

Case III.

$$\lambda > \left(\frac{q-1}{2}\right)^2 \text{and } f(s) = |\alpha|^{log_q s}\left(c_3 cos\left(\theta\, log_q\, s\right) + c_4 sin\left(\theta\, log_q\, s\right)\right), \tag{40}$$

where $\theta = arccos\left(\frac{Re\alpha}{|\alpha|}\right)$ dhe $c_3, c_4 \in \mathbb{R}$.
The eigenvalues and general solutions of the Euler-type equation in problem (18) with Dirichlet conditions:

$$f\left(q^{-\vartheta}\right) = f\left(q^{N-\vartheta}\right) = 0$$

for each case are as follows:
Case I. $\lambda < \left(\frac{q-1}{2}\right)^2 \quad \alpha_1 \neq \alpha_2,\ \alpha_1, \alpha_2 \in \mathbb{R}$.
We apply the first Dirichlet condition $f\left(q^{-\vartheta}\right) = 0$ to (38) and obtain that:
$f\left(q^{-\vartheta}\right) = c_1{\alpha_1}^{log_q q^{-\vartheta}} + c_2{\alpha_2}^{log_q q^{-\vartheta}} = c_1{\alpha_1}^{-\vartheta} + c_2{\alpha_2}^{-\vartheta} = 0$ so, we have that $c_1 = -c_2\left(\frac{\alpha_1}{\alpha_2}\right)^{\vartheta}$.

Now we use the relationship between c_1 and c_2 and apply it to the general solution (30), and after that, we use the other Dirichlet condition $f(q^{N-\vartheta}) = 0$.

$$f\left(q^{N-\vartheta}\right) = c_1{\alpha_1}^{log_q q^{N-\vartheta}} + c_2{\alpha_2}^{log_q q^{N-\vartheta}} = 0,$$
$$c_1{\alpha_1}^{N-\vartheta} + c_2{\alpha_2}^{N-\vartheta} = 0$$
$$-c_2{\alpha_2}^{-\vartheta}{\alpha_1}^N + c_2{\alpha_2}^{-\vartheta}{\alpha_2}^N = 0$$
$$c_2{\alpha_2}^{-\vartheta}({\alpha_2}^N - {\alpha_1}^N) = 0$$

Since for $c_2 = 0, {\alpha_2}^{-\vartheta} = 0$ we have obtained a trivial solution, we shall discuss the following condition for $N \in \mathbb{N}$

$${\alpha_1}^N = {\alpha_2}^N.$$

This can occur for $\alpha_1 = \alpha_2$ or $\alpha_1 = -\alpha_2$ for even N.
For $\alpha_1 = \alpha_2$ we have that:

$$\frac{q+1}{2}+\sqrt{\left(\frac{q-1}{2}\right)^2-\lambda}=\frac{q+1}{2}-\sqrt{\left(\frac{q-1}{2}\right)^2-\lambda}$$

i.e.,

$$\sqrt{\left(\frac{q-1}{2}\right)^2-\lambda}=0 \text{ so that } \lambda=\left(\frac{q-1}{2}\right)^2,$$

which is not a valid value for λ in ***Case I***.

Next, for $\alpha_1=-\alpha_2$ we have that:

$$\frac{q+1}{2}+\sqrt{\left(\frac{q-1}{2}\right)^2-\lambda}=-\frac{q+1}{2}+\sqrt{\left(\frac{q-1}{2}\right)^2-\lambda}.$$

where we obtain that, $q=-1$, contradicting the fact that $0<q<1$. Thus, there are no eigenvalues in this case.

Case II. $\lambda=\left(\frac{q-1}{2}\right)^2$ $\alpha_1=\alpha_2,\ \alpha_1,\alpha_2\in\mathbb{R}$

Now we look at the first Dirichlet condition in (39) to find:

$$f(q^{-\vartheta})=(c_1\log_q q^{-\vartheta}+c_2)\left(\frac{q+1}{2}\right)^{\log_q q^{-\vartheta}}=0$$

$$(-c_1\vartheta+c_2)\left(\frac{q+1}{2}\right)^{-\vartheta}=0,$$

which leads us to $q=-1$, and again contradict the fact that $0<q<1$. Let's have $c_2=c_1\vartheta$ and then apply the second Dirichlet condition, which gives the following equality:

$$\begin{aligned}0=f(q^{N-\vartheta})&=c_1(\log_q q^{N-\vartheta}\\&\quad+\vartheta)\left(\frac{q+1}{2}\right)^{\log_q q^{N-\vartheta}}\\&=c_1N\left(\frac{q+1}{2}\right)^{N-\vartheta}.\end{aligned}$$

For this to be true, either $c_1=0, N=0$ or $q=-1$, from which we do not obtain any eigenvalues. Hence, also, in this case there are no eigenvalues.

Case III. $\lambda>\left(\frac{q-1}{2}\right)^2$, $\alpha_{1,2}=|\alpha|(\cos\theta\pm i\sin\theta)$, $\alpha_1,\alpha_2\in\mathbb{C}$

Now, we will use the equation (40) with the first Dirichlet condition to find:

$$\begin{aligned}f(q^{-\vartheta})=|\alpha|^{\log_q q^{-\vartheta}}&\left(c_3\cos(\theta\log_q q^{-\vartheta})\right.\\&\left.+c_4\sin(\theta\log_q q^{-\vartheta})\right)=0\end{aligned}$$

$$f(q^{-\vartheta})=|\alpha|^{-\vartheta}\left(c_3\cos(-\vartheta\theta)+c_4\sin(-\vartheta\theta)\right)=0$$

Let $c_3\cos(\vartheta\theta)=c_4\sin(\vartheta\theta)$ and apply the other Dirichlet condition to obtain:

$$f(q^{N-\vartheta})=|\alpha|^{\log_q q^{N-\vartheta}}\left(c_3\cos(\theta\log_q q^{N-\vartheta})+c_4\sin(\theta\log_q q^{N-\vartheta})\right)=0 \tag{41}$$

$$|\alpha|^{N-\vartheta}\left(c_3\cos(\theta(N-\vartheta))+c_4\sin(\theta(N-\vartheta))\right)=0$$

$$\begin{aligned}|\alpha|^{N-\vartheta}\big[c_3[&\cos(\theta N)\cos(\theta\vartheta)+\sin(\theta N)\sin(\theta\vartheta)]\\&+c_4[\sin(\theta N)\cos(\theta\vartheta)\\&-\cos(\theta N)\sin(\theta\vartheta)]\big]=0\end{aligned}$$

$$\begin{aligned}|\alpha|^{N-\vartheta}\big[[&c_4\cos(\theta N)\sin(\theta\vartheta)+c_3\sin(\theta N)\sin(\theta\vartheta)]\\&+[c_4\sin(\theta N)\cos(\theta\vartheta)\\&-c_4\cos(\theta N)\sin(\theta\vartheta)]\big]=0\end{aligned}$$

$$\sin(\theta N)[c_3\sin(\theta\vartheta)+c_4\cos(\theta\vartheta)]=0$$

From the last equality, we have that $c_3\sin(\theta\vartheta)+c_4\cos(\theta\vartheta)=0$, or $\sin(\theta N)=0$. If the first equality is true, we have that:

$$\frac{c_3}{c_4}c_3\cos(\vartheta\theta)+c_4\cos(\theta\vartheta)=0$$

$$\cos(\theta\vartheta)\left(\frac{{c_3}^2+{c_4}^2}{c_4}\right)=0.$$

Then, $|\alpha|=0,\ c_3=\ c_4=0$ or $\cos(\theta\vartheta)=0$.

So, we will consider only the other possible solution, which is $\sin(\theta N)=0$. This leads us to $\theta_k N=k\pi$, which gives us the values $\lambda_k, k\in N_0$, where

$$\arccos\left(\frac{q+1}{2\sqrt{q+\lambda_k}}\right)=\frac{k\pi}{N}. \tag{42}$$

Since the cosine function is an even function, we do not lose anything if we continue to take $N\in\mathbb{N}_0$. Solving for λ_k, we find that

$$\lambda_k=\left(\frac{q+1}{2\cos\left(\frac{k\pi}{N}\right)}\right)^2-q \tag{43}$$

for $k=0,\ldots,(N-2)/2$ if N is an even integer and $k=0,\ldots,(N-1)/2$ if N is an odd integer. Hence, we determine the following main result.

Theorem 3.3

Let $N\in\mathbb{N}$. The problem (18) has exactly $\lfloor(N-1)/2\rfloor$ where $\lfloor\cdot\rfloor$ denotes the greatest integer function, eigenvalues, and they can be calculated from the formula (43). The corresponding eigen functions are given by

$$f(s)=\left(\sqrt{q+\lambda_k}\right)^{\log_q s}\sin\left(\frac{k\pi}{N}\log_q s\right) \tag{44}$$

where,

$$\lambda_k=\left(\frac{q+1}{2\cos\left(\frac{k\pi}{N}\right)}\right)^2-q.$$

4 Conclusion

In this article, we have introducesd some basic properties of delta q-calculus and delta q-calculus on a time scale $\mathbb{T}_q$ compared to the classical Newton-Leibniz calculus. We have analyzed a non classical q-difference equation, by using a transformation for the function which is involved in it. To continue, we have determined and combined some explicit formulas for eigenvalues λ_k and their count for the resulting eigenvalue problem and we have provided examples to illustrate the effectiveness of the proposed theorems. The obtained result will help us inour further study to find a lower and upper bound for the n-th eigenvalue to analyze the asymptotic behavior of eigenvalues on a Time Scale $\mathbb{T}_q$. These results will be combined with some well-known properties of oscillation theory to study the countability of these eigenvalues.

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