Nano Topology via Idealization

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Abstract: - Examining some weak forms of open and closed sets in nano ideal topological spaces serves as the most substantial objective of this work . Along with their connections to specific other kinds of nano open sets, several new attributes and numerous essential features of these nano sets are researched. In light of the aforementioned novel ideas. In this work we extend nano topological model to a nano ideal topological space. Also, we show that the concept of nano local function in terms of nano ideal topological space is exanimated. Our main objective is to establish new results about some nano open sets and the relationships between them. Many concepts available in the classical theorem maybe discussed using the theory of nano ideal topological space.
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1 Introduction

Some early applications of ideal topological structure can be found in various branches of mathematical modellings[1]. Many perspectives have been used to investigate approximate topological space [2]. Since 1930, [3] has explored ideals in topological spaces.The study written by Vaidyanathaswamy[4] in 1945 contributed to establishing the topic's significance.An ideal or dual filter on X is a nonempty set of finitely many subsets of X with hereditary conditions on additivity. Specifically a nonempty family $I \subseteq P(X)$(where $P(x)$ is the set of all subsets of X is referred to an ideal if and only if (i) $A \in I$ gives $P(A) \subseteq I$ and(ii)$A, B \in I$ gives $A \bigcup B \in I$. Given a topological space (X , τ) with an ideal I on X, a set operator()*: $P(X) \rightarrow P(X)$, is call a local function [4] of A with respect to τ and I is defined as follows: for $A \subseteq X$, $A^*(I, \tau) = \{ x \in X{:}G \bigcap A \notin I$, for every $G \in \tau(x) \}$ where , $\tau(x)= \{G \in \tau{:}\ x \in G \}$,A kuratowski closure operator $C1^*()$ for a topology $\tau* \ (I, \tau)$ is called *-topology, finer than τ is defined by $C1^*(A)= A \cup A^*(I, \tau)$.When there is no possibility of misunderstanding , we will simply write A^* for $A^*(I, \tau)$and $\tau*$ for $\tau*(I,\tau)$. The space (X, τ, I) is referred to as an ideal topological space if I is an ideal on X.

In [5] the idea of a nano-topology was first proposed, which they characterized in terms of approximations and the boundary area of a subset of the universe using an equivalence relation. They also introduced the concepts of nano closed sets, nano-interior, and nano-closure.In [6] the idea of topological nano-spaces was introduced and its properties were investigated. The links between some weak forms of nano open sets in nano topological spaces and some weak forms of nano open sets in nano ideal topological spaces are examined in this research. We additionally draw attention to some findings in ([6][7]) that are not correct.

2 Preliminaries

We recall the following terms, which are vital in the sequel, before commencing our task.

Definition 2.1. [8] Let R be an equivalence relation on U known as the indiscernibility relation, and let U be a nanempty finite set of objects, Then different equivalence classes for U are created. It is argued that elements in some equivalence classes are indiscernible from one another. The approximation space is referred to as the pair (U, R).

Definition 2.2. [8] Check Figure 1. Let$X \subseteq U$ and (U, R) be an approximation space.

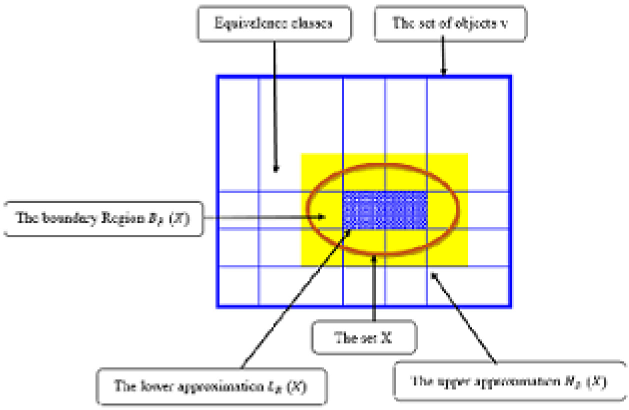

Figure 1: A rough set in a rough approximation space.

(i) The lower approximation of X with respect to$R$ is the set of all objects, which can be for certain categorized as X with regard to R and it is indicated by L_R*(X)*, that is L_R *(X)* = $\cup_{x\in\cup}$ { *R(χ): R(x)* $\subseteq X$ } where *R(x)* denotes the equivalence class specified by $x \in \cup$.
(ii) The set of all objects that mightbe categorized as X with respect to R is the upper approximation of X with respect to R, and it is indicated by H_R*(X)*, that is H_R*(X)* $\cup_{x\in\cup}$ { *R(x): R(x)* $\bigcap X \neq \phi$ }.
(iii) The collection of all objects that connot be classified as either X or -X with respect to R is known as the boundary region of X with respect to R and is indicated by B_R *(X)*, where B_R *(X)* = H_R *(X)* - L_R *(X)*.

Pawlak's definition states that X is a rough set if H_R *(X)* $\neq L_R$ *(X)*.

Proposition 2.3. [8] If *(U, R)* is an approximation space and *X, y* $\subseteq$ *U*, which possess the qualities of pawlak's rough sets.
(i) $L_R(X) \subseteq X \subseteq H_R(X)$ (Contraction and Extension).
(ii) $L_R(\phi) = H_R(\phi) = \phi$ (Normality) and L_R $U = H_R$ $U = U$ (Co-normality).
(iii) $H_R(X \cup Y) = H_R(X) \cup H_R(Y)$ (Addition).
(iv) $H_R(X \cap Y) \subseteq H_R(X) \cap H_R(Y)$.
(v) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$.
(vi) $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$ (Multiplication).
(vii) $L_R(X) \subseteq L_R(Y)$ *and* $H_R(X) \subseteq H_R(Y)$ *whenever* $X \subseteq Y$ (Monotone).
(viii) $H_R(X^c) = [H_R(X)]^c$ *and* $L_R(X^c) = [H_R(X)]^c$ *where* X^c denotes the complement of X in U (Duality).
(ix) $H_R(H_R(X)) = L_R(H_R(X)) = H_R(X)$ (Idempotency).
(x) $L_R(L_R(X)) = H_R(L_R(X)) = L_R(X)$ (Idempotency).

Definition 2.4. [5] Let U be the universe, R be an equivalence relation on U, then for $X \subseteq U$, $\tau_R(X)$ = { *U, ϕ,* $L_R(X)$*,* $H_R(X)$*,* $B_R(X)$ } is referred to the nano topology on U which complies the following axioms:
(i) U and $\phi \in \tau_R(X)$.
(ii) The combination of any subcollection's parts of $\tau_R(X)$ is in $\tau_R(X)$;
(iii) Any finite subcollection's intersection of components of $\tau_R(X)$ is in $\tau_R(X)$.
In other words, the pair $(U, R(X))$ is referred to as a nano topological space, and $R(X)$ is a topology on U that is known as the nano topology on U with regard to X. In U, the components of $R(X)$ are known as nano open sets, and a nano open set's complement is known as a nano closed set. The components of $[R(X)]c$ are referred to as $R's$ dual nano topology (X).

Definition 2.5.[5] If $\tau_R(X)$ is nano topology on U with respect to X, then the family β = { U , $L_R(X)$, $B_R(X)$ } is the basic for $\tau_R(X)$.

Remark 2.6. With respect to X, let (*U,*$\tau_R(X)$) be *a* nano topological space, and let *X,...U*, and R each represent an equivalence relation on U. The family of equivalence classes of U by R is thus denoted by *U/R*.

Definition 2.7. [5] If (*U,*$\tau_R(X)$) is *a* nano topological space regarding X, where $X \subseteq U$ and if $A \subseteq U$, then:
(i) The union of all the set A is nano open subsets is the definitionof the set's nano interior, which is represented by the symbol *nInt(A)*. This means that the greatestnano open subset of A is *nInt(A)*.
(ii) The intersection of all nano closed sets containing A is known as the nano closure of the set A, and it is represented by the symbol *nCI(A)*. This means that the smallest nano closed set that contains A is *nCI(A)*.

Definition 2.8. ([5], [9], [10])Let (*U,*$\tau_R(X)$)be *a* nano topological space and $A \subseteq U$. Then A is said to be :
*(i)*Nano regular open if A = *nInt*(*nCI(A)*),
(ii) Nano α-open if $A \subseteq$ *nInt*(*nCI*(*nInt(A)*)),
(iii) Nano semi-open if $A \subseteq$ *nCI*(*nInt(A)*),
(iv) Nano preopen if $A \subseteq$ *nInt*(*nCI(A)*),
(v) Nano γ-open (or nano *b*-open) if $A \subseteq$ *nCI*(*nInt(A)*) $\cup$ *nInt*(*nCI(A)*)
(vi) Nano β-open if $A \subseteq$ *nCI*(*nInt(*nInt(A)))

Nano regular closed (resp. nano-closed,nano semiclosed, nano preclosed, nano-close, nano-close)sets are the complement of a nano regular open (resp. nano-open, nano semi-open, nano preopen, nano-open, nano-open) set. NSO stands for the family of all nano semi-open sets in a nano topological space (*U,*$\tau_R(X)$) *(U, X)*.

3 Nano ideal topological spaces

The nano lacal function in a nano ideal topological space is examined in this section.

Definition 3.1. [6] Let (*U,*$\tau_R(X)$*, I*) be *a* nano ideal topological space. *A* set operatar $()^*_n$:*P(U)*→ *P(U)* is called the nano local function. And for *a* subset $A \subseteq U$, A^*_n*(I,* $\tau_R(X)$)= { $x \in U$: $G_\chi \bigcap A \notin I$, *for every* $G_x \in \tau_R(X)$ } is called the nano local function of *A* with respect to *I* and $\tau_R(X)$, we shall merely write A^*_n for A^*_n*(I,* $\tau_R(X)$).

Example 3.2. Let (*U*,$\tau_R(X)$)be *a* nano topological space with an ideal *I* on*U* and for every $A \subseteq U$:
(i) If *I* = { ϕ }, then A^*_n = *nCI(A)*,
*(ii)*If *I*= *P(U)*, then A^*_n= ϕ.

The following theorem contains many basic and useful facts concerning the nano local function.

Theorem 3.3. [7] Let (*U*,$\tau_R(X)$) be *a* nano topological space with an ideal *I*, *J* on *U* and *A, B* be subsets of *U*. Then the subsequent statements are true:
(i) If $A \subseteq B$, then $A^*_n \subseteq B^*_n$,
(ii) If $I \subseteq J$, then $A^*_n(J) \subseteq A^*_n(I)$,
(iii) A^*_n = *nCI*(A^*_n) $\subseteq$ *nCI(A)*, this means that A^*_n is*a* nano closed subset of *nCI(A)*,
(iv) $(A^*_n)^*_n \subseteq A^*_n$,
(v) $(A \cup B)^*_n = A^*_n \cup B^*_n$,
(vi) $(A \cap B)^*_n \subseteq A^*_n \cap B^*_n$,
(vii) $A^*_n - B^*_n = (A-B)^*_n$ - $B^*_n \subseteq (A-B)^*_n$,
(viii) If $V \in \tau_R(X)$, *then*V $\cap A^*_n$ = V $\cap$ $(V \cap A)^*_n \subseteq (V \cap A)^*_n$,
(ix) If $E \subseteq I$, *then* $(A \cup E)^*_n = A^*_n = (A-E)^*_n$.

***Proof*.**
(i) Let $A \subseteq B$ and x $\notin A^*_n$, Then $G \in \tau_R(X)$ contains x such that $G \cap B \in I$. Since $A \subseteq B$, $G \cap A \in I$ and hence $x \notin A^*_n$, Thus $A^*_n \subseteq B^*_n$.
(ii) Let x $\in A^*_n(J)$, then for every $G \in \tau_R(X)$, $G \cap A \notin J$, this implies that $G \cap A \notin I$, so $x \in A^*_n(I)$. Hence $A^*_n(J) \subseteq A^*_n(I)$.
(iii) In general $A^*_n \subseteq$ *nCI*(A^*_n). Let $x \in$ *nCI*(A^*_n). Then $A^*_n \cap G \neq \phi$ for every $G_x \in \tau_R(X)$. Therefore, there exist some $y \in A^*_n \cap G$ and $G \in \tau_R(X)$ containing x. Since $y \in A^*_n$, $A \cap G \notin I$ and hence *nCI*(A^*_n) $\subseteq A^*_n$, $A^*_n = nCI(A^*_n)$.
(iv) Let x $\in (A^*_n)^*_n$,then for every $G \in \tau_R(X)$ containing x , $G \cap A^*_n \notin I$ and hence $G \cap A^*_n \neq \phi$. Let $y \in G \cap A^*_n$. Then $G \in \tau_R(X)$ containing y and $y \in A^*_n$. Hence $G \cap A^*_n \notin I$ and x $\in A^*_n$. This implies that $(A^*_n)^*_n \subseteq A^*_n$.
(v) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then from (i), $A^*_n \subseteq (A \cup B)^*_n$ and $B^*_n \subseteq (A \cup B)^*_n$.Hence $A^*_n \cup B^*_n \subseteq (A \cup B)^*_n$. Conversely, let x $\in (A \cup B)^*_n$. Then for every nano open set G of x such that $(G \cap A) \cup (G \cap B) = G \cap (A \cup B) \notin I$. Therefore, $G \cap A \notin I$ or $G \cap B \notin I$. This implies that $x \in A^*_n$ or x $\in B^*_n$. That is x $\in A^*_n \cup B^*_n$. Therefore, we have $(A \cup B)^*_n \subseteq A^*_n \cup B^*_n$. Thus we get $(A \cup B)^*_n \subseteq A_n km^* \cup B^*_n$.
(vi) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by (i), $(A \cap B)^*_n \subseteq A^*_n$ and $(A \cap B)^*_n \subseteq B^*_n$. Hence $(A \cap B)^*_n \subseteq A^*_n \cap B^*_n$
(vii) Since $A \cup B = (A-B) \cup B$, by (i), $(A \cup B)^*_n = [(A-B) \cup B]^*_n$, by(v), $A^*_n \cup B^*_n = (A-B)^*_n \cup B^*_n$ and hence $A^*_n \subseteq (A-B)^*_n \cup B^*_n$, therefore $A^*_n - B^*_n \subseteq (A-B)^*_n$.
(viii) If $V \in \tau_R(X)$ and x $\in V \cap A^*_n$. Then x $\in V$ and x $\in A^*_n$. Let G be any nano open set containing x. Then $G \cap V \in \tau_R(X)$ and $G \cap (V \cap A) = (G \cap V) \cap A \notin I$. This shows that x $\in (V \cap A)^*_n$ and hence $V \cap A^*_n \subseteq (V \cap A)^*_n$. Hence $V \cap A^*_n \subseteq V \cap (V \cap A)^*_n$. By (i), $(V \cap A)^*_n \subseteq A^*_n$ and $V \cap A^*_n \supseteq V \cap (V \cap A)^*_n$. Therefore $V \cap A^*_n = V \cap (V \cap A)^*_n \subseteq (V \cap A)^*_n$.
*(ix)*Since $A - E \subseteq A$, by (i), $(A-E)^*_n \subseteq A^*_n$. Also, by (v), $(A \cup E)^*_n = A^*_n \cup E^*_n = A^*_n \cup \phi = A^*_n$. Hence $(A-E)^*_n \subseteq A^*_n = (A \cup E)^*_n$, since $E^*_n = \phi$.

The instance that follows demonstrates that the converse implications of the preceding theorem's (i), (ii) and (iii) do not generally hold.

Example 3.4. Let $U = \{a,b,c,d\}$ be the universe.
(i) If $X = \{a,b\}; U/R = \{\{a\},\{c\},\{b,d\}\}$ are the family of equivalence classes of U by the equivalence relation R. One can deduce that $L_R(X) = \{a\}, H_R(X) = \{a,b,d\}$ and $B_R(X) = \{b,d\}$, then $\tau_R(X) = \{U,\phi,\{a\},\{b,d\},\{a,b,d\}\}$. Let $I = \{\phi,\{a\}\}$. For $A = \{A,C\}$ and $B = \{a,d\}$, we have $A^*_n = \{c\}, B^*_n = \{b,c,d\}$, that is $A^*_n \subseteq B^*_n$, but $A \nsubseteq B$. Also, let $I = \{\phi,\{a\}\}$ and $J = \{\phi,\{b\}\}$.For A, it is obvious that, for $A = \{a,c,d\}.A^*_n(I) = \{b,c,d\}, A^*_n(J) = \{a,b,c,d\} = U$, that is $A^*_n(I) \subseteq A^*_n(J)$ while $I \nsubseteq J$.
(ii) Let $X = \{a,d\}; U/R = \{\{b\},\{d\},\{a,c\}\}$. One can draw the conclusion that $L_R(X) = \{d\}, H_R(X) = \{a,c,d\}$ and $B_R(X) = \{a,c\}$, then $\tau_R(X) = \{U,\phi,\{d\},\{a,c\},\{a,c,d\}\}$. Let $I = \{\phi,\{\{d\}\}$. For $A = \{b,d\}$, we have $nCl(A) = nCl(\{b,d\}) = \{b,d\}, A^*_n = \{b,d\}^*_n = \{b\}$ and $nCl(A^*_n) = nCl(\{b\}) = \{b\}$. Therefore, $nCl(A) \nsubseteq A^*_n = nCl(A^*_n)$.

Theorem 3.5. Let $(U,\tau_R(X),I)$ be a nano ideal topological space and A, B be subsets of U. Then:
(i) A^*_n is a nano closed.
(ii) $\phi^*_n = \phi$,
(iii) $(U-E)^*_n = U^*_n$ if $E \in I$,
(iv) $[U-(A-E)]^*_n = [(U-A) \cup E]^*_n$, if $E \in I$.

Proof.
(i) It is clear.
(ii) Obvious, since ϕ always belongs to I.
(iii) Let x $\in (U-E)^*_n$, then for every nano open neighbourhood G containing $G_x \cap (U-E) \notin I$, implies, $(G_x \cap U) - (G_x \cap E) \notin I$, implies, $G_x \cap U \notin I$, so $x \in U^*_n$. Thus $(U-E)^*_n \subseteq U^*_n$. Also, let $x \in U^*_n$, implies, $G_x \cap U \notin I$, for every nano open neighbourhood G containing x, implies, $G_x \cap (U-E) \notin I$, so $x \in (U-E)^*_n$ and thus concludes the proof.
(iv) Follow by using Theorem 3.3(ix).

Definition 3.6. Let $(U,\tau_R(X))$ be a nano topological space with an ideal I on U, the set operator nCl^* is called a nano *-closure and is defined as : $nCl^*(A) = A \cup A^*_n$, for $A \subseteq X$.

Theorem 3.7. [11] The set operator nCl^* meets the requirement listed below:
(i) $A \subseteq nCl^*(A)$,
(ii) $nCl^*(\phi) = \phi$ and $nCl^*(U) = U$,
(iii) If $A \subseteq B$, then $nCl^*(A) \subseteq nCl^*(B)$,
(iv) $nCl^*(A) \cup nCl^*(B) = nCl^*(A \cup B)$,
(v) $nCl^*(nCl^*(A)) = nCl^*(A)$.

Definition 3.8. ([5], [7]) Let $(U, \tau_R(X), I)$ be a nano ideal topological space, then $A \subseteq U$ is said to be :
(i) Nano regular *I-open* (nano *RI-open*) if $A = nInt(nCl^*(A))$,
(ii) Nano regular *I-closed*(nano *RI-closed*) if its complement is nano *RI-open*
(iii) Nano semi *I-open* if $A = nCl(nInt^*(A))$,
(iv) Nano semi *I-closed* if its complement is nano semi *I-open* ,
(v) Nano *αI-open* if $A \subseteq nInt(nCl^*(nInt(A)))$,
(vi) Nano *αI-closed* if its complement is nano *αI-open* ,
(vii) Nano *pre-I-open* if $A \subseteq nInt(nCl^*(A))$,
(viii) Nano *pre-I-closed* if its complement is nano *pre-I-open* ,
(ix) Nano *βI-open* if $A \subseteq nCl(nInt(nCl^*(A)))$,
(x) Nano *βI-closed* if its complement is nano *βI-open* .

Definition 3.9. [6] An ideal I in a nano ideal topological space $(U, \tau_R(X), I)$ is called $\tau_R(X)$- *codense* ideal if $\tau_R(X) \cap I = \{\phi\}$.

Definition 3.10. [6] A subset A in a nano ideal topological space $(U, \tau_R(x), I)$ is said to be :
(i) Nano *$\star$-dense-in-itself* if $A \subseteq A^*_n$,
(ii) Nano *$\star$-closed* if $A^*_n \subseteq A$,
(iii) Nano *$\star$-perfect* if $A = A^*_n$,
(iv) Nano *I-dense* if $A^*_n = U$.

The connections between the aforementioned nano sets are depicted in the diagram below.

Nano *$\star$-dense-in-itself* $\Leftarrow$ Nano *$\star$-perfect* $\Rightarrow$ Nano *$\star$-closed*

The examples that follow demonstrate that the diagram's converse implications cannot be satisfied.

Example 3.11. Let $U = \{a, b, c, d\}$ be the universe, $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ be the family of equivalence classes of U by the equivalence relation R and $X = \{a, b\}$. Then one can deduce that $L_R(X) = \{a\}, H_R(X) = \{a, b, d\}$, then the nano topology $\tau_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$. For I $=\{\phi, \{a\}\}$, we have :
(i) If $A = \{a, c\}$, then parimala et al. [6] observed that $A^*_n = \{c\}$. Here A is a nano $\star$ *closed*, but not nano *$\star$-perfect* .
(ii) If $B = \{c, d\}$, then we have $B^*_n = \{b, c, d\}$. So B is a nano *$\star$-dense-in-itself* but not nano *$\star$-perfect*.

4 More on nano *I-open* sets

Definition 4.1. [11] A subset A in a nano ideal topological space $(U, \tau_R(x), I)$ is said to be nano *I-open* if $A \subseteq nINT(A^*_n)$. A subset $F \subseteq (U, \tau_R(X), I)$ is said to nano *I-closed* if its complement $U \setminus F$ is nano *I-open* .

We connote by $NIO(U, \tau_R(X), I) = \{A \subseteq U : A \subseteq nInt(A^*_n)\}$ or simply write $NIO(U, \tau_R(X))$ or $NIO(U)$ for $NIO(U, \tau_R(X), I)$ when there is no possibility of mistake with the ideal.

The largest nano *I-open* set contained in A is known as the nano *I-interior* of A, denoted by ***nI-Int(A)***.

Remark 4.2. The following example demonstrates how nano *I-openness* and nano openness are conceptually distinct.

Example 4.3. Let $U = \{a, b, c, d\}$ be the universe, $X = \{a, b\} \subset U$ and $U/R = \{\{a\}, \{c\}, \{b, d\}\}$. This can be extrapolated as: $L_R(X) = \{a\}, H_R(X) = \{a, b, d\}$. Then $\tau_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$. For $I = \{\phi, \{a\}\}$.
(i) Set $A = \{a, b, d\}$, we have $A^*_n = \{b, c, d\}$ and $nInt(A^*_n) = \{b, d\}$, that is $A \nsubseteq nInt(A^*_n)$. As can be seen, A is nano open but nano *I-open*.
(ii) Set $B = \{d\}$, we have $B^*_n = \{b, c, d\}$ and $nInt(B^*_n) = \{b, c, d\}$, that is $B \subseteq nInt(B^*_n)$. A is nano *I-open* but not nano open, as evidenced by this.

Remark 4.4. The following example demonstrates that every nano *I-open* set is a nano preopen set and that, generally speaking, the opposite is not true.

Example 4.5. Let $U = \{a, b, c, d\}$ be the universe, and let $X = \{a, b\} \subset U$ and $U/R = \{\{a\}, \{c\}, \{b, d\}\}$. This can be presumed as: $L_R(X) = \{a\}, H_R(X) = \{a, b, d\}$. Then $\tau_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$. For $I = \{\phi, \{a\}\}$ and $A = \{a, b, c\}$, we have $A^*_n = \{b, c, d\}$ and $nInt(A^*_n) = \{b, d\}$, that is $A \nsubseteq nInt(A^*_n)$, but $nCl(A) = U$ and $nInt(nCl(A)) = U$. A is a nano preopen, but not nano *I-open*, as demonstratedby this.

Proposition 4.6. Nano *I-open* also refers to the arbitrary union of nano *I-open* sets.

Proof. Let $(U, \tau_R(X), I)$ be a nano ideal topological space and $W_i \in NIO(U, \tau_R(X), I)$ for $i \in \bigtriangledown$; this implies that for every $i \in \bigtriangledown, W, \in nInt((W_i)^*_n)$ and so $\cup_i\{W_i : i \in \bigtriangledown\} \subseteq \cup_i(nInt((W_i)^*_n) \subseteq nIn(\cup_i((W_i)^*_n) \subseteq nIn(\cup_i((W_i))^*_n$ for every $i \in \bigtriangledown$. Hence $\cup_i\{W_i : i \in \bigtriangledown\} \in NIO(U, \tau_R(X), I)$

Theorem 4.7. [11] Let $(U, \tau_R(X), I)$ be a nano ideal topological space and $A, B \subseteq U$.Then:

(i) If $A \in NIO(U, \tau_R(X), I)$ and $B \in \tau_R(X)$, then $A \cap B \in NIO(A)$,
(ii) If $A \in NIO(U, \tau_R(X), I)$ and $B \in NSO(U, \tau_R(X))$, then $A \cap B \in NSO(A)$,
(iii) If $A \in NIO(U, \tau_R(X), I)$ and $B \in \tau_R(X)$, then $A \cap B \subseteq nInt(B \cap (B \cap A)^*_n)$,

Proposition 4.8. For a nano ideal topological space $(U, \tau_R(X), I)$ and $A \subseteq U$, we have:
(i) If $I = \phi$, then $A^*_n = nCl(A)$.and hence each of nano *I-open* set and nano preopen sets coincide.
(ii) If $I = P(U)$, then $A^*_n = \phi$. and hence A is a nano *I-open* if and only if $A = \phi$.

Proposition 4.9. For any nano *I-open* set A of a nano ideal topological space $(U, \tau_R(X), I)$, we have $A^*_n = nInt(A^*_n)^*_n$.

Proposition 4.10. If $A \subseteq (U, \tau_R(X), I)$ is a nano *I-closed*, then $A \supseteq (nInt(A))^*_n$.

Proof. It results from Theorem 3.3 and the concept of namo *I-closed* sets (iii).

Remark 4.11. The idea of nano *I-closeness* creates a significant departure from the notion of nano topology in general.

Proposition 4.12. If $A \subseteq (U, \tau_R(X), I)$, we have $((nInt(A))^*_n)^c \neq nInt((A^c)^*_n)$ in general (Example 4.13), where A^c denotes the complement of A.

Example 4.13. Let $U = \{a, b, c, d\}$be the universe, $X = \{a, c\} \subset U$, let $U/R = \{\{a\}, \{d\}, \{b, c\}\}$. The following may be inferred as: $L_R(X) = \{a\}, H_R(X) = \{a, b, c\}$ and $B_R(X) = \{b, c\}$. Then $\tau_R(X) = \{U, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$. For $I = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Set $A = \{a, d\}$, then it is clear that $nInt(A) = \{a\}$ and $(\{a\})^*_n = \{a, d\}$, so $((nInt(A))^*_n)^c = \{b, c\}$. But $nInt(A^c) = nInt(\{b, c\}) = \{b, c\}$, so $(nInt(A^c))^*_n) = (\{b, c\})^*_n = \{b, c, d\}$. This also complies with proposition 4.12.

Proposition 4.14. If $A \subseteq (U, \tau_R(X), I)$ is a nano *I-open* and nano *semi- closed*, then $A = nInt(A^*_n)$.

Proof. Theorem 3.3 (iii) dictates this.

Proposition 4.15. Every nano *I-open* set is *nano pre-I-open*.

Proof. Let $(U, \tau_R(X), I)$ be a nano ideal topological space and let $A \subseteq U$ be nano.

Proposition 4.16. We have the following for a subset A of a nano ideal topological space $(U; \tau_R(X); I)$:
(i) If A is nano $\star$*-closed* and $A \in NIO(U)$, then $nLnt(A) = nInt(A^*_n)$.
(ii) A is nano $\star$*-closed* if and only if A is nano open and nano *I-closed* .
(iii) If A is nano $\star$*-perfect* , then $A = nInt(A^*_n(I_n))$, for every $A \in NIO(U, \tau_R(X))$.
(iv) If A is nano regular closed and nano *I-open*, then $A^*_n(I_n) = nInt(A^*_n(I_n))$, where I_n is the ideal of nano nowhere dense sets ($I_n = \{A \subseteq U : nlnt(nCl(A))\} = \phi$).

Proof. (i), (ii), and (iii) are obvious.
(vi) Is implied by the description of nano *I-open* and the assumption that A is nano regular closed if and only if $A = A^*_n(I_n)$.

Proposition 4.17. The implications of various weak nano open set types as stated above in the nano ideal topological space $(U; \tau_R(X); I)$ are depicted in Figure 2.

Nonetheless, Examples 2.1 and 2.2[9] and the subsequent example show that the converses of the preceding diagram's assumptions are not generally true.

Example 4.18. Let $U = \{a, b, c, d\}$ be the universe and $U/R = \{\{b\}, \{d\}, \{a, c\}\}$ and $X = \{a, d\} \subset U$. Hence, it entails that: $\tau_R(X) = \{U, \phi, \{d\}, \{a, c\}, \{a, c, d\}\}$.
(a) If $I = \{\phi, \{d\}\}$. Then:
(i) $A = \{a, d\}$ is nano *pre-I-open* but not nano α*I-open*.
(ii) $B = \{a, b, c\}$ is nano *semi-I-open* but not nano α*I-open*.
(iii) Also $B = \{a, b, c\}$ is nano β*I-open* but not nano *pre-I-open*.
(iv) $C = \{b, d\}$ is nano *semi-open* but not nano *semi-I-open*.

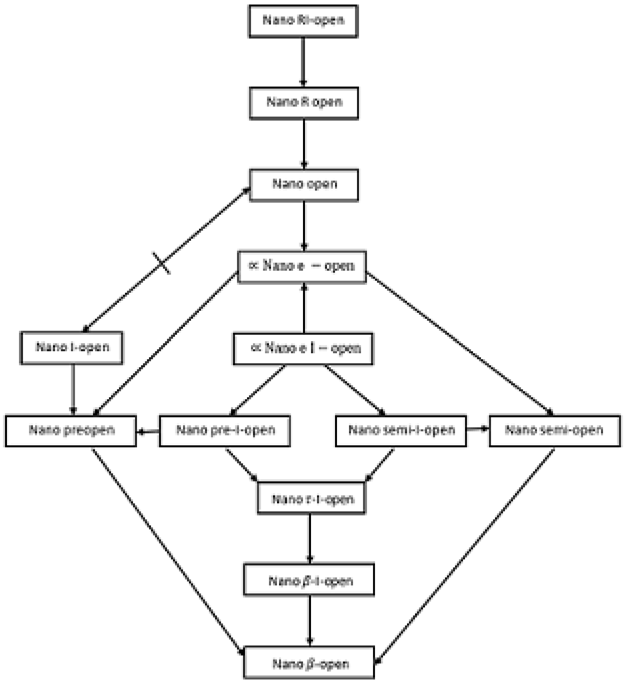

Figure 2: Weak nano open sets

(b) If $I = \{\phi, \{a\}\}$. Then $A = \{a, d\}$ is nano preopen but not nano *pre-I-open*.
(c) If $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then $A = \{a, b, d\}$ is nano β*-open* but not nano β*I-open*.

5 Conclusions

We anticipate that this research is merely the start of a new framework. Many people will be motivated to contribute to the development of nano ideal topology in the area of mathematical nanostructures.

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Conflict of Interest

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