New Solutions of Benney-Luke Equation Using The (G'/G,1/G) Method

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Abstract: The Benney-Luke equation has contributed to studying the propagation of the water wave surfaces. This paper illustrates the (G'/G,1/G)-method to obtain the solutions of the Benney-Luke equation and an extension of the Benney-Luke equation. The new types of solutions are also constructed to gather the performance and visualization in three dimensions for observing the behaviors. The solutions are found in the expressions of hyperbolic functions giving the general performance by selecting arbitrary constants.

Key-Words: (G'/G,1/G)-method; exact solution; Benney-Luke equation, extended Benney-Luke equation.

Received: November 7, 2023. Revised: March 6, 2024. Accepted: March 19, 2024. Published: April 24, 2024.

1 Introduction

Finding exact solutions has contributed to developing the studies in mechanics and dynamics, [1], [2]. The important role of the (G'/G,1/G)-method has been performed in finding solutions of nonlinear partial differential equations (NPDEs). Using the (G'/G,1/G)-method, various solutions of the fifth-order nonlinear equation have been found and illustrated in three dimensions, [3]. Based on the traveling wave solution, the NPDEs will be turned into ordinary differential equations and applied two variable (G'/G,1/G)-expansion method using the mathematical application such as Maple, Mathematica to find the suitable solutions corresponding to the case construction, [4]. The solutions gained using the two-term (G/G,1/G)-expansion method are closed-form solutions demonstrating hyperbolic function, trigonometric, and rational function solutions, [5]. The (G'/G,1/G)-expansion method is also useful in solving the nonlinear medium equal width (MEW) wave equation that represents one-dimensional wave propagation related to the dispersion process, [6].
The procedure of the (G'/G, 1/G)-method transforms from NPDEs to ordinary differential equations (ODE) based on the traveling wave and construct the solutions with parameters, [7]. The (G'/G, 1/G)-method is extended from the well-known (G'/G)-method and has demonstrated more effectively and more general than the (G'/G)-method, [8]. The (G'/G, 1/G)-method is considered direct, concise, and elementary for attaining the solutions of nonlinear evolution equations (NLEEs), [9]. Compared to the sine-Gordon expansion method (SGEM) obtaining kink-type solutions, bell-shaped solitary wave solutions, and anti-bell-shaped type soliton solutions, the (G'/G, 1/G)-method supports the other types of the solutions, [10]. Besides that, the (G'/G, 1/G)-method is perfectly complemented for the special solutions of the (3+1DJM) equation that performs a stationary wave in physics, [11].
The Benney-Luke equation is one of the NPDEs representing the two-way propagation of water tension. By applying the (G'/G)-method, hyperbolic, and trigonometric solutions have been found replied on the setting of shifting variables, [12]. The solutions of the Benney-Luke equation have been found in the expression of hyperbolic and trigonometric solutions by choosing appropriate parameters, [13], [14]. Another presentation banked on the variation direct method (VDM), various solutions such as dark solitary type solution, dark-like solitary type solution, kinky-dark solitary type solution, and periodic

wave type solutions have been built, [15]. Moreover, the (1/G')-expansion method has supported the new tools for getting solutions of nonlinear evolution equations (NLEEs), and shown the necessary way to gather other hyperbolic solutions, [16]. One more successful performance, the modified simple equation method has straightforwardly contributed to finding exact traveling wave solutions and the solitary wave solutions of the Benney-Luke equation, [17].
In the present study, the Benney-Luke equation, [13], given with $a=1, b=1, n=1$ as follows:

$$u_{tt}-u_{xx}+u_{xxxx}-u_{xxtt}+u_t u_{xx}+2u_x u_{xt}=0, \quad (1.1)$$

and an extended Benney-Luke equation given as the following will be considered to seek the exact analytic solutions:

$$u_{tt}-u_{xx}+u_{xxxx}-u_{xxtt}+u_t u_{xx}+2u_x u_{xt}+u_t u_{xx}=0. \quad (1.2)$$

The Benney-Luke equations have played a significant function in studying the two-way propagation of water wave surfaces, and the (G'/G,1/G)-method will be applied to find the solutions of equation (1.1), and equation (1.2).

2 Methodology

Given an NPDE formed

$$F(u,u_t,u_x,u_{xx},u_{tt})=0. \quad (2.1)$$

The (G'/G,1/G)-method will be constructed as the following steps:
Step 1. Setting the traveling wave variable

$$u(x,t)=U(\xi),\ \xi=x-Vt. \quad (2.2)$$

Substituting (2.2) into (2.1) to get an ODE formed

$$P(u,u',u'',...)=0, \quad (2.3)$$

where V is a constant, $u'=\frac{du}{d\xi}$.
Step 2. Given solutions of (2.3) can be expressed in form:

$$u(\xi)=\sum_{i=0}^{N}a_i f^i+\sum_{i=1}^{N}b_i f^{i-1}g, \quad (2.4)$$

where a_i and b_i be the constants will be achieved later.
Step 3. Find the value of N in (2.4) by balancing the greatest order of derivative term and nonlinear term in (2.3).
Step 4. Substitute (2.4) into (2.3) along with case 1 (shown below) to find the value $a_i, b_i, V, \mu, A_1, A_2$, and λ to gather the exact analytic solutions expressed in the types of hyperbolic solutions, trigonometric solutions, and rational solutions.

- Now, considering the second-order linear ordinary differential equation (LODE):

$$G''(\xi)+\lambda G(\xi)=\mu. \quad (2.5)$$

Using the transform $f=\frac{G'}{G}, g=\frac{1}{G}$ to get the system of equations

$$f'=-f^2+\mu g-\lambda, g'=-fg.$$

Case 1. With $\lambda<0$, the specific solution of LODE (2.5) is formed of hyperbolic functions:

$$G(\xi)=A_1\sinh\sqrt{-\lambda}\xi+A_2\cosh\sqrt{-\lambda}\xi+\frac{\mu}{\lambda}.$$

and we gather

$$g^2=\frac{-\lambda}{\lambda^2\sigma+\mu^2}\left(f^2-2\mu g+\lambda\right).$$

where A_1, A_2 are the given constants and $\sigma=A_1^2-A_2^2$.
Case 2. With $\lambda>0$, the specific solution of LODE (2.5) is formed of trigonometry functions as

$$G(\xi)=A_1\sin\sqrt{\lambda}\xi+A_2\cos\sqrt{\lambda}\xi+\frac{\mu}{\lambda}.$$

and we have

$$g^2=\frac{\lambda}{\lambda^2\sigma-\mu^2}\left(f^2-2\mu g+\lambda\right).$$

where A_1, A_2 are the given constants and $\sigma=A_1^2+A_2^2$.
Case 3. $\lambda=0$ then the general solution of LODE (2.5) is formed of rational functions:

$$G(\xi)=\frac{\mu}{2}\xi^2+A_1\xi+A_2.$$

and we have

$$g^2=\frac{\lambda}{A_1^2-2\mu A_2}\left(f^2-2\mu g\right).$$

where A_1, A_2 are the given constants.
Step 5. Using the same way by replacing (2.4) by (2.3) coming along with cases 1, 2, 3 (shown above) to gain the exact solutions of hyperbolic type solution, trigonometry type solutions, and rational type solutions.

3 Appication

3.1 (G/G,1/G)-expansion method applied for Benney-Luke equation

Apply two variable $(G'/G,1/G)$-expansion method to find the traveling wave solutions of nonlinear

$$u_{tt}-u_{xx}+u_{xxxx}-u_{xxtt}+u_t u_{xx}+2u_x u_{xt}=0. \quad (3.1)$$

Step 1. Setting the traveling wave variable

$$u(x,t) = U(\xi), \text{ where } \xi = x - Vt.$$

We construct the following terms

$$\begin{aligned} u_{tt} &= V^2 U_{\xi\xi}; u_{xx} = U_{\xi\xi}; \\ u_{xxxx} &= U_{\xi\xi\xi\xi}; u_{xxtt} = V^2 U_{\xi\xi\xi\xi}; \\ u_t u_{xx} &= -VU_\xi U_{\xi\xi}; 2u_x u_{xt} = -2VU_\xi U_{\xi\xi}. \end{aligned} \quad (3.2)$$

Substituting (3.2) into (3.1) to get the ODE formed

$$V^2 U_{\xi\xi} - U_{\xi\xi} + U_{\xi\xi\xi\xi} - V^2 U_{\xi\xi\xi\xi} - VU_\xi U_{\xi\xi} - 2VU_\xi U_{\xi\xi} = 0. \quad (3.3)$$

Simplify on both sides Eq. (3.3), we have

$$\left(V^2 - 1\right) U_{\xi\xi} + \left(1 - V^2\right) U_{\xi\xi\xi\xi} - 3VU_\xi U_{\xi\xi} = 0.$$

Integrate both sides of the equation and putting constant equals zero, we have

$$\left(V^2 - 1\right) U_\xi + \left(1 - V^2\right) U_{\xi\xi\xi} - \frac{3}{2} V \left(U_\xi\right)^2 = 0, \quad (3.4)$$

where V is a constant, $U_\xi = \frac{dU}{d\xi}$.
Step 2. Given solutions of (3.4) will be illustrated as follows:

$$U(\xi) = \sum_{i=0}^{N} a_i f^i + \sum_{i=1}^{N} b_i f^{i-1} g.$$

where a_i and b_i be the constants will be achieved later.
Step 3. Balancing both terms $U_{\xi\xi\xi}$, and $\left(U_\xi\right)^2$:

$$N + 3 = 2(N+1),$$

then $N = 1$.
Step 4. So the solution is formed of

$$U(\xi) = a_0 + a_1 f(\xi) + b_1 g(\xi), \quad (3.5)$$

where a_0, a_1, and b_1 are constants. Using the transform $f = \frac{G'}{G}, g = \frac{1}{G}$ to get the system of equations as follows

$$f' = -f^2 + \mu g - \lambda, g' = -fg. \quad (3.6)$$

Case 1. With $\lambda < 0$, the specific solution of LODE (3.4) is attained

$$G(\xi) = A_1 \sinh\sqrt{-\lambda}\xi + A_2 \cosh\sqrt{-\lambda}\xi + \frac{\mu}{\lambda}. \quad (3.7)$$

and we have

$$g^2 = \frac{-\lambda}{\lambda^2 \sigma + \mu^2}\left(f^2 - 2\mu g + \lambda\right). \quad (3.8)$$

where A_1, A_2 are the given constants and $\sigma = A_1^2 - A_2^2$. For $\lambda < 0$, substituting Eq. (3.5) into Eq. (3.4) and by using Eq. (3.6) and Eq. (3.7)-(3.8) yields a set of algebraic equations for $a_0, a_1, b_1, \mu, \lambda$, and V. Solving the constructed simultaneous equation, the solutions of algebraic values will be collected. We substitute the values to Eq. (1.1) and select the exact solutions. These systems are expressed in terms of

$$1, f(\xi)g(\xi), f(\xi)^2, f(\xi), g(\xi), g(\xi)f(\xi)^3, g(\xi)f(\xi)^2, g(\xi)f(\xi)^4, f(\xi)^3. \quad (3.9)$$

corresponding to the following:

$$\begin{aligned} 1 : &-2\lambda\left(\lambda^2\left(A_1^2 - A_2^2\right) + \mu^2\right)^3 a_1 \\ &\left(-2(A_1 - A_2)\left(V^2 - \frac{3}{4}Va_1 - 1\right)\right. \\ &\times (A_1 + A_2)\lambda^3 + \left(\left(V^2 - 1\right)A_1^2\right. \\ &+ \left(-V^2 + 1\right)A_2^2\Big)\lambda^2 + \mu^2\left(V^2 - 1\right)\lambda \\ &\left.+ \mu^2\left(V^2 - 1\right)\right). \end{aligned}$$

$$\begin{aligned} g(\xi)f(\xi)^3 : &12b_1\left(V^2 - \frac{1}{2}Va_1 - 1\right) \\ &\left(\lambda^2\left(A_1^2 - A_2^2\right) + \mu^2\right)^4. \end{aligned}$$

$$\begin{aligned} f(\xi)g(\xi) : &-14\left(-5(A_1 - A_2)\left(V^2 - \frac{3}{5}Va_1 - 1\right)\right. \\ &\times \frac{(A_1 + A_2)\lambda^3}{7} \\ &+ \frac{(A_1 - A_2)(A_1 + A_2)(V-1)(V+1)\lambda^2}{7} \\ &\left.+ \mu^2\left(V^2 - \frac{3}{7}Va_1 - 1\right)\lambda + \frac{V^2\mu^2}{7} - \frac{\mu^2}{7}\right) \\ &\times b_1\left(\lambda^2\left(A_1^2 A_2^2\right) + \mu^2\right)^3. \end{aligned}$$

$$\begin{aligned} f(\xi)^2 : &10\left(8(A_1 - A_2)a_1\left(V^2 - \frac{3}{8}Va_1 - 1\right)\right. \\ &\frac{(A_1 + A_2)\lambda^3}{5} \\ &+ \left(-\frac{(A_1 - A_2)(A_1 + A_2)(V-1)(V+1)a_1}{5}\right. \\ &\left.+ \frac{3Vb_1^2}{10}\right)\lambda^2 + a_1\left(V^2 - \frac{3}{10}Va_1 - 1\right)\mu^2\lambda \\ &\left.- \frac{\mu^2 a_1 (V-1)(V+1)}{5}\right) \\ &\times \left(\lambda^2\left(A_1^2 - A_2^2\right) + \mu^2\right)^3. \end{aligned}$$

$$f(\xi):12b_1\lambda^2\left(V^2-\frac{1}{2}Va_1-1\right)\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\mu.$$

$$g(\xi):2\Bigg(-5\left(A_1-A_2\right)\left(V^2-\frac{3}{5}Va_1-1\right)\times\left(A_1+A_2\right)\lambda^3+\Bigg(\left(V^2-1\right)A_1^2+\left(-V^2+1\right)A_2^2\Bigg)\lambda^2+\mu^2\left(V^2-1\right)\lambda+\mu^2\left(V^2-1\right)\Bigg)\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3a_1\mu.$$

$$g(\xi)f(\xi)^2:-24\Bigg(\left(A_1-A_2\right)a_1\left(V^2-\frac{1}{4}Va_1-1\right)\times\left(A_1+A_2\right)\lambda^2+\frac{Vb_1^2\lambda}{4}+a_1\left(V^2-\frac{1}{4}Va_1-1\right)\mu^2\Bigg)\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\mu.$$

$$f(\xi)^4:12\Bigg(\left(A_1-A_2\right)a_1\left(V^2-\frac{1}{4}Va_1-1\right)\left(A_1+A_2\right)\lambda^2+\frac{Vb_1^2\lambda}{4}+a_1\left(V^2-\frac{1}{4}Va_1-1\right)\mu^2\Bigg)\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3.$$

$$f(\xi)^3:12b_1\lambda\left(V^2-\frac{1}{2}Va_1-1\right)\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\mu.$$

Case 2. $\lambda > 0$ then the general solution of LODE (3.4) is formed

$$G(\xi)=A_1\sin\sqrt{\lambda}\xi+A_2\cosh\sqrt{\lambda}\xi+\frac{\mu}{\lambda}.$$

and we attain

$$g^2=\frac{\lambda}{\lambda^2\sigma-\mu^2}\left(f^2-2\mu g+\lambda\right),$$

where A_1, A_2 are the given constants and $\sigma=A_1^2+A_2^2$.
There is no solution existing in this case.
Case 3. $\lambda = 0$ then the specific solution of LODE (3.4) is formed

$$G(\xi)=\frac{\mu}{2}\xi^2+A_1\xi+A_2.$$

and we have

$$g^2=\frac{\lambda}{A_1^2-2\mu A_2}\left(f^2-2\mu g\right),$$

where A_1, A_2 are the given constants.
There is no solution existing in this case.
Step 5: will be moved to the next section.

3.2 (G/G,1/G)-expansion method applied for extended Benney-Luke equation

Apply two variable $(G'/G, 1/G)$-expansion method to find the traveling wave solutions of the extended Benney-Luke equation as follows:

$$u_{tt}-u_{xx}+u_{xxxx}-u_{xxtt}+u_tu_{xx}+2u_xu_{xt}+u_tu_{xx}=0. \quad (3.10)$$

Step 1. Setting the shifting wave variable

$$u(x,t)=U(\xi),\text{ where }\xi=x-Vt.$$

where V is a given nonzero value.
We establish the following terms

$$\begin{aligned}u_{tt}&=V^2U_{\xi\xi};u_{xx}=U_{\xi\xi};\\u_{xxxx}&=U_{\xi\xi\xi\xi};u_{xxtt}=V^2U_{\xi\xi\xi\xi};\\u_tu_{xx}&=-VU_\xi U_{\xi\xi};2u_xu_{xt}=-2VU_\xi U_{\xi\xi}.\end{aligned} \quad (3.11)$$

Substituting (3.11) into (3.10) to get the ODE formed

$$V^2U_{\xi\xi}-U_{\xi\xi}+U_{\xi\xi\xi\xi}-V^2U_{\xi\xi\xi\xi}-2VU_\xi U_{\xi\xi}-2VU_\xi U_{\xi\xi}=0. \quad (3.12)$$

Simplify on both sides equation (3.12), we have

$$\left(V^2-1\right)U_{\xi\xi}+\left(1-V^2\right)U_{\xi\xi\xi\xi}-4VU_\xi U_{\xi\xi}=0.$$

Integrate both sides of the equation and putting a constant equal to zero, we have

$$\left(V^2-1\right)U_\xi+\left(1-V^2\right)U_{\xi\xi\xi}-2V\left(U_\xi\right)^2=0,\quad (3.13)$$

where V is a constant, $U_\xi=\frac{dU}{d\xi}$.
Step 2. Given solutions of (3.13) can be expressed in form:

$$U(\xi)=\sum_{i=0}^{N}a_if^i+\sum_{i=1}^{N}b_if^{i-1}g.$$

where a_i and b_i be the constants will be achieved later.
Step 3. Balancing both sides equation (3.13), two terms $U_{\xi\xi\xi}$, and $\left(U_\xi\right)^2$:

$$N+3=2(N+1),$$

then $N=1$.
Step 4. So the solution is built in the form

$$U(\xi)=a_0+a_1 f(\xi)+b_1 g(\xi). \tag{3.14}$$

where a_0, a_1, and b_1 are constants.
Using the transform $f=\frac{G'}{G}, g=\frac{1}{G}$ to get the system of equations

$$f'=-f^2+\mu g-\lambda, g'=-fg. \tag{3.15}$$

Case 1. $\lambda<0$ then the general solution of LODE (2.5) is formed of

$$G(\xi)=A_1 \sinh\sqrt{-\lambda}\xi+A_2\cosh\sqrt{-\lambda}\xi+\frac{\mu}{\lambda}. \tag{3.16}$$

and we have

$$g^2=\frac{-\lambda}{\lambda^2\sigma+\mu^2}\left(f^2-2\mu g+\lambda\right). \tag{3.17}$$

where A_1, A_2 are the given constants and $\sigma=A_1^2-A_2^2$. For $\lambda<0$, substituting Eq. (3.14) into Eq. (3.13) and by using Eq. (3.15) and Eq. (3.16)-(3.17) leads to a set of algebraic equations for $a_0, a_1, b_1, \mu, \lambda$, and V. Solving the constructed simultaneous equation, we select the exact solutions satisfied Eq. (1.2) These systems of

$$1, f(\xi)g(\xi), f(\xi)^2, f(\xi), g(\xi), f(\xi)^4, f(\xi)^3, g(\xi)f(\xi)^2, g(\xi)f(\xi)^3.$$

corresponding to the following systems

$$1: -2\lambda a_1\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\Big(-2\left(A_1-A_2\right)\left(A_1+A_2\right)\left(V^2-Va_1-1\right)\lambda^3+\Big(\left(V^2-1\right)A_1^2+\left(-V^2+1\right)A_2^2\Big)\lambda^2+\mu^2\left(V^2-1\right)\lambda+\mu^2\left(V^2-1\right)\Big).$$

$$f(\xi)g(\xi): 12\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^4\left(V^2-\frac{2}{3}Va_1-1\right)b_1.$$

$$f(\xi)^2: -24\Big(\left(A_1+A_2\right)\left(A_1-A_2\right)\left(V^2-\frac{1}{3}Va_1-1\right)a_1\lambda^2+\frac{V\lambda b_1^2}{3}+\mu^2\left(V^2-\frac{1}{3}Va_1-1\right)a_1\Big)\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\mu.$$

$$f(\xi): -14\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\Big(-\frac{5\left(V^2-\frac{4}{5}Va_1-1\right)\left(A_1+A_2\right)\left(A_1-A_2\right)\lambda^3}{7}+\frac{\left(A_1-A_2\right)\left(A_1+A_2\right)\left(V-1\right)\left(V+1\right)\lambda^2}{7}+\mu^2\left(V^2-\frac{4}{7}Va_1-1\right)\lambda+\frac{V^2\mu^2}{7}-\frac{\mu^2}{7}\Big)b_1.$$

$$g(\xi): 10\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\times\Big(8\left(A_1+A_2\right)\left(V^2-\frac{1}{2}Va_1-1\right)\times\frac{\left(A_1-A_2\right)a_1\lambda^3}{5}+\Big(-\left(A_1-A_2\right)\left(A_1+A_2\right)\frac{\left(V-1\right)\left(V+1\right)a_1}{5}+\frac{2Vb_1^2}{5}\Big)\lambda^2+\left(V^2-\frac{2}{5}Va_1-1\right)\mu^2 a_1\lambda-\frac{\mu^2 a_1\left(V-1\right)\left(V+1\right)}{5}\Big).$$

$$f(\xi)^4: 12\lambda\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\left(V^2-\frac{2}{3}Va_1-1\right)\mu b_1.$$

$$f(\xi)^3: 12\Big(\left(A_1+A_2\right)\left(A_1-A_2\right)\left(V^2-\frac{1}{3}Va_1-1\right)a_1\lambda^2+\frac{V\lambda b_1^2}{3}+\mu^2\left(V^2-\frac{1}{3}Va_1-1\right)a_1\Big)\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3.$$

$$g(\xi)f(\xi)^2: 12\Big(\left(A_1-A_2\right)\left(V^2-\frac{1}{3}Va_1-1\right)a_1\left(A_1+A_2\right)\lambda^2+\frac{V\lambda b_1^2}{3}+\mu^2\left(V^2-\frac{1}{3}Va_1-1\right)a_1\Big)\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3.$$

$$g(\xi)f(\xi)^3: 12\left(V^2-\frac{2}{3}Va_1-1\right)\mu\left(\lambda^2\left(A_1^2-A_2^2\right)+\mu^2\right)^3\lambda b_1.$$

Case 2. $\lambda>0$, there is no solution existing in this case.
Case 3. $\lambda=0$, there is no solution existing in this case.
Step 5: will be transferred to the next section.

4 Solution Benney-Luke equation Using (G'/G,1/G)

4.1 Solution of the Benney-Luke Eq. (1.1)

By solving the system of simultaneous equations, the solutions are obtained as follows:

Case 1:

Using the condition of the first case we have the constant

$$\lambda=-\frac{1}{4},\mu=0,a_1=\frac{4(V^2-1)}{V},b_1=0,$$

we calculate the term of the variant equation established as the following

$$U(\xi)=a_0+a_1f(\xi),$$

where a_0 is arbitrary. Using

$$G(\xi)=A_1\sinh(\frac{\xi}{2})+A_2\cosh(\frac{\xi}{2}).$$

and $f(\xi)=\frac{G'}{G}$. We have the hyperbolic solution

$$u_{11}=a_0-\frac{2(V^2-1)(A_1\cosh\left(\frac{Vt}{2}-\frac{x}{2}\right)-A_2\sinh\left(\frac{Vt}{2}-\frac{x}{2}\right))}{V(A_1\sinh\left(\frac{Vt}{2}-\frac{x}{2}\right)-A_2\cosh\left(\frac{Vt}{2}-\frac{x}{2}\right))},$$

where V,A_1,A_2 are arbitrary constants. The solution u_{11}, kink type solution, is depicted in Figure 2 (Appendix).

Case 2:

Using the condition of the first case we have the constant

$$\lambda=-1,\{\mu=0,\mu=\mu\},a_1=\frac{2(V^2-1)}{V},$$
$$\{b_1=\frac{\pm2\sqrt{A_1^2-A_2^2}\left(V^2-1\right)}{V},$$
$$b_1=\pm\frac{\sqrt{4A_1^2-4A_2^2+4\mu^2}(V^2-1)}{V}\}.$$

We calculate the term of the variant equation established as the following

$$U(\xi)=a_0+a_1f(\xi)+b_1g(\xi),$$

where a_0 is arbitrary. Using

$$G(\xi)=A_1\sinh\xi+A_2\cosh\xi+\mu.$$

and

$$f(\xi)=\frac{G'}{G},g(\xi)=\frac{1}{G}.$$

We have the hyperbolic solution

$$u_{12}=a_0+a_1\frac{A_1\cosh(Vt-x)-A_2\sinh(Vt-x)}{-A_1\sinh(Vt-x)+A_2\cosh(Vt-x)+\mu}-b_1\frac{1}{-A_1\sinh(Vt-x)+A_2\cosh(Vt-x)+\mu},$$

where

$$a_1=\frac{2\left(V^2-1\right)}{V},b_1=\frac{\pm2\sqrt{A_1^2-A_2^2}\left(V^2-1\right)}{V},$$
$$\{\mu=0,\mu=\mu\},b_1=\frac{\pm\sqrt{4A_1^2-4A_2^2+4\mu^2}(V^2-1)}{V}.$$

and V,A_1,A_2 are arbitrary constants. The solution u_{12}, traveling wave solution, is performed in Figure 3, and Figure 1 (Appendix).

4.2 Solution of the extended Benney-Luke equation (1.2)

By solving the system of simultaneous equations, the solutions are obtained as the following:

Case 1:

Using the condition of the first case we have the constant

$$\lambda=-\frac{1}{4},\mu=0,a_1=\frac{3(V^2-1)}{V},b_1=0.$$

We calculate the term of the variant equation established as the following

$$U(\xi)=a_0+a_1f(\xi),$$

a_0 is arbitrary. Using

$$G(\xi)=A_1\sinh(\frac{\xi}{2})+A_2\cosh(\frac{\xi}{2}),$$

and $f(\xi)=\frac{G'}{G}$, we have the exact solution

$$u_{21}=a_0-\frac{3(V^2-1)(A_1\cosh\left(\frac{Vt}{2}-\frac{x}{2}\right)-A_2\sinh\left(\frac{Vt}{2}-\frac{x}{2}\right))}{2V(A_1\sinh\left(\frac{Vt}{2}-\frac{x}{2}\right)-A_2\cosh\left(\frac{Vt}{2}-\frac{x}{2}\right))},$$

where V,A_1,A_2 are arbitrary constants. The solution u_{21}, kink type solution, is delineated in Figure 4 (Appendix).

Case 2:

Using the condition of the first case we have the constant

$$\lambda=-1,\mu=0,a_1=\frac{3(V^2-1)}{2V},$$
$$b_1=\frac{\pm\sqrt{9A_1^2-9A_2^2+9\mu^2}\left(V^2-1\right)}{2V}.$$

We calculate the term of the variant equation established as the following

$$U(\xi) = a_0 + a_1 f(\xi) + b_1 g(\xi),$$

a_0 is arbitrary. Using

$$G(\xi) = A_1 \sinh \xi + A_2 \cosh \xi - \mu,$$

and

$$f(\xi) = \frac{G'}{G}, g(\xi) = \frac{1}{G}.$$

We have the exact solution

$$u_{22} = a_0 + a_1 \frac{A_1 \cosh(Vt - x) + A_2 \sinh(Vt - x)}{A_1 \sinh(Vt - x) + A_2 \cosh(Vt - x)} - c_1 \frac{1}{-A_1 \sinh(Vt - x) + A_2 \cosh(Vt - x)},$$

where

$$a_1 = \frac{3(V^2 - 1)}{2V}, c_1 = \frac{\pm 3\sqrt{A_1^2 - A_2^2 + \mu^2}(V^2 - 1)}{2V}.$$

where V, A_1, A_2 are arbitrary constants. The solution u_{22}, Traveling wave solution, is illustrated in Figure 5 (Appendix).
All the analytic solutions attained corresponding to Eq. (1.1), and Eq. (1.2) have been exactly substituted for checking satisfaction.

5 Conclusion

By utilizing the prevalent (G'/G,1/G)-method, the hyperbolic function solutions of the Benney-Luke equation, and the extended Benney-Luke equation have been found and illustrated in both two and three dimensions. The (G'/G,1/G)-method has supported the variety of solutions of the Benney-Luke equation that are meaningful in studying the propagation of the water wave surface. Compared to previous studies, the present method has attained a new form of kink-type solutions, and traveling wave solutions that play a significant role in studying the water surface tensions.

Acknowledgment:
The authors acknowledge the anonymous reviewer's comments, which improve the manuscript's quality, and kindly provide advice and valuable discussions.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)
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Sekson Sirisubtawee: Conceptualization, formal analysis, methodology, resources, supervision, validation, visualization, and writing review and editing.
Phayung Meesad: Conceptualization, formal analysis, methodology, resources, supervision, validation, visualization, and writing review and editing.
Nattawut Khansai: Employed for extended Benney-Luke equation consisting of conceptualization, investigation, methodology, project administration, and writing review and editing.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself No funding was received for conducting this study.

Conflicts of Interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

APPENDIX

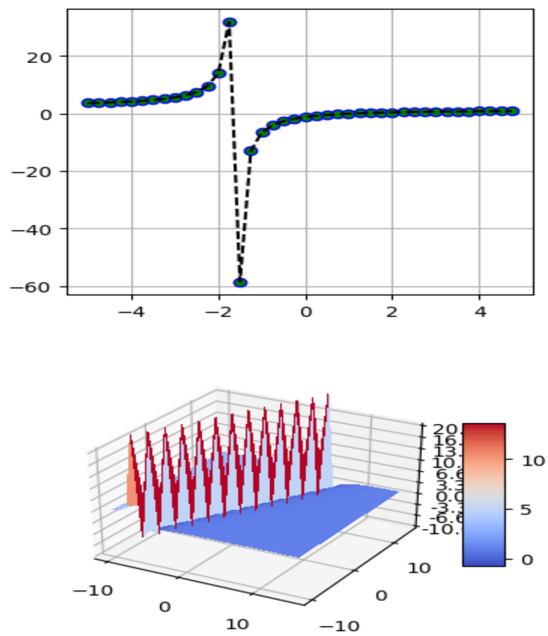

Figure 1: Traveling wave solution, u_{12}, when $V = 0.5; a_1 = 0.5;; A_1 = 0.5; A_2 = 0.5; a_0 = 0.5; b_1 = 0.5, \mu = 0.2$.

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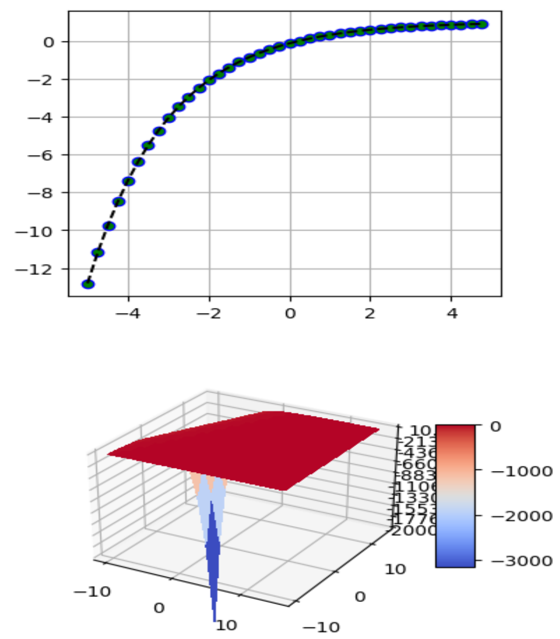

Figure 2: Kink type solution, u_{12}, performance when $V = 0.5; a_1 = 0.5; a_0 = 0.5; A_1 = 0.5; A_2 = 0.5; b_1 = 0.5.$

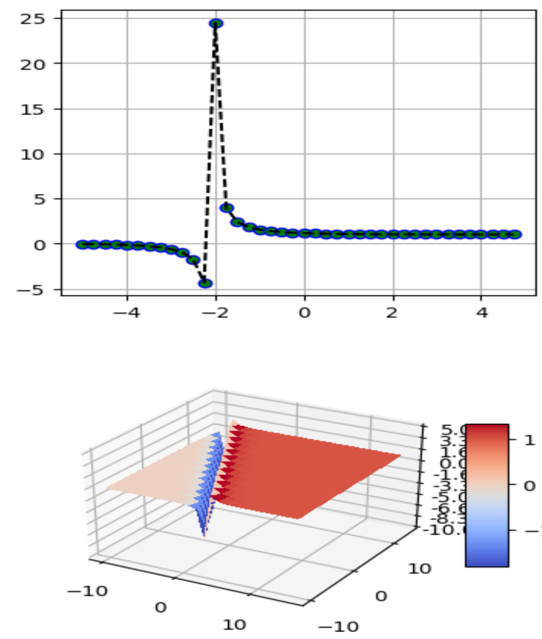

Figure 3: Traveling wave solution performance, u_{12}, when $V = -0.5; a_1 = 0.5; a_0 = 0.5; A_1 = 0.5; A_2 = 0.5; b_1 = 0.5.$

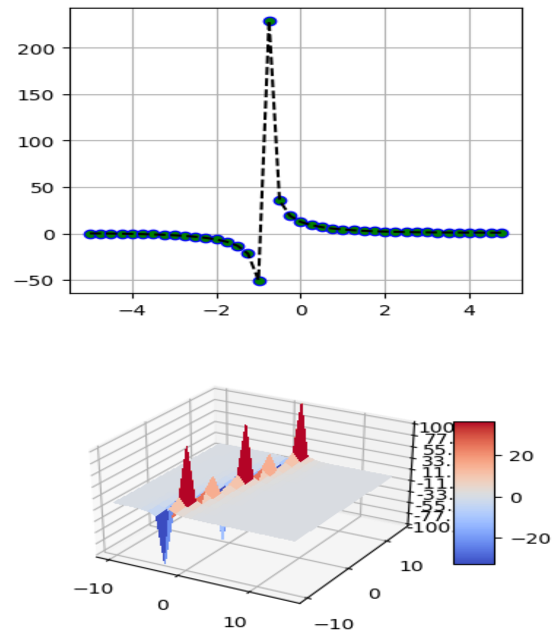

Figure 4: Kink type solution, u_{21}, when $V = \frac{1}{3}\sqrt{10} - 1/3; a_1 = \frac{3(V^2-1)}{2V}; a_0 = 0.5; A_1 = 0.7; A_2 = 0.5.$
.

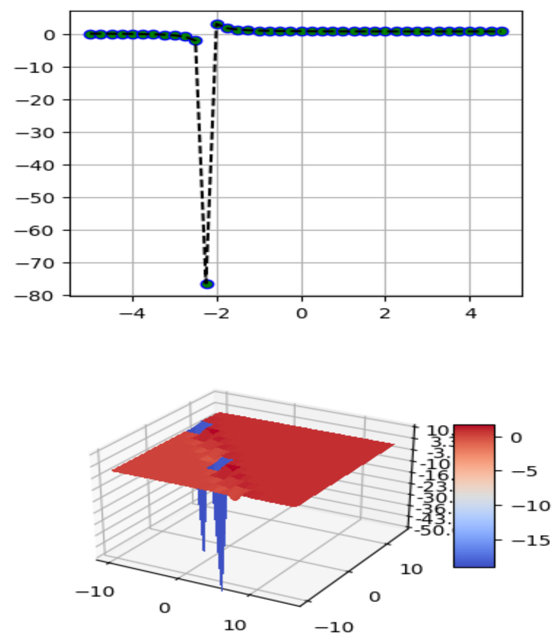

Figure 5: Traveling wave solution, u_{22}, when $V = 0.5; a_1 = 0.2; a_0 = 0.5; A_1 = 0.7; A_2 = 0.5; b_1 = 0.5; b1 = 0.5; \mu = 0.5;.$