Adjoint Separating Systems

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Abstract: Combinatorial group testing is a method that could be used to efficiently test many individuals for diseases like COVID-19 by pooling and testing their samples. This paper develops ideas concerning separating systems as an initial theoretical framework for studying combinatorial group testing methods. A system of subsets of a finite set S is called separating if it enables to separate individual elements of S, i.e., for any two different aspects of S there is a set in the system containing just one of them. In this paper, we present an easy and flexible method to construct "small" separating systems on "large" sets from "large" separating systems on "small" sets. The point is that small systems are practical for saving time and money, while large ones are much easier to construct.

Key-Words: system of subsets, separating systems, combinatorial group testing, pooling, adjoint poolscapes, construction of poolscapes, non-adaptive testing

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1 Introduction

Separating systems were introduced in [1] and, a bit later, they turned out to be utilizable in non-adaptive combinatorial group testing (see, e.g., [2], [3] and [4]). Combinatorial group testing is a method that could be used for efficiently testing many individuals for diseases like COVID-19 by pooling and testing their samples ([5], [6], [7], [8], [9]). The idea of group testing, based initially on health care needs, has proven to be applicable in many other fields, such as computer science (e.g., [10], [11]) and engineering ([12]).

This note presents an easy and flexible method to construct "small" separating systems on "large" sets from "large" separating systems on "small" sets. The point is that small systems are practical for saving time and money, while large ones are much easier to construct.

2 Preliminaries

Throughout the paper, let S be a finite set, $s = |S| \geq 2$. We denote by $\mathcal{P}(S)$ the set of all subsets of S and by $\mathcal{P}^+(S)$ that of non-empty ones; we have $|\mathcal{P}(S)| = 2^s \geq 4$ and $|\mathcal{P}^+(S)| = 2^s - 1 \geq 3$. In the sequel, a *system* (on S) will mean any non-empty subset of $\mathcal{P}^+(S)$. The number of all systems is $2^{2^s-1} - 1 \geq 7$ (≥ 127 for $s \geq 3$). Of course, if $S \subseteq T$, every system on S is also a system on T.

Following [1], we will say that a system $\mathcal{A}$ ($\subseteq \mathcal{P}^+(S)$) is *separating* (on S) if for all $a, b \in S$, $a \neq b$, there is at least one set $A \in \mathcal{A}$ such that $|A \cap \{a, b\}| = 1$.

In the subsequent text, separating systems will be called *poolscapes*, while *pools* will be the sets in them. The following assertion is obvious.

Lemma 2.1. *Let $\mathcal{A}$ be a poolscape (on S). Then:*
(i) *$\mathcal{A} \setminus \{S\}$ is a poolscape.*
(ii) *If $\mathcal{B} \subseteq \mathcal{P}^+(S)$ is a system such that $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{B}$ is a poolscape as well.* □

Let $\mathcal{A}$ be a system. Define a mapping $\mathcal{R}(\mathcal{A}) : S \to \mathcal{P}(\mathcal{A})$ by $\mathcal{R}(\mathcal{A})(\mathrm{A}) = \{ A \in \mathcal{A} \,|\, a \in A \}$ for every $a \in S$. The following assertion is straightforward.

Lemma 2.2. *Let $\mathcal{A}$ be a system on S and $a \in S$. Then:*
(i) *$\mathcal{R}(\mathcal{A})(\mathrm{A}) = \emptyset$ if and only if $a \in S \setminus \bigcup \mathcal{A}$.*
(ii) *$\mathcal{R}(\mathcal{A})(\mathrm{A}) = \mathcal{A}$ if and only if $a \in \bigcap \mathcal{A}$.*
(iii) *$\mathcal{A}$ is a poolscape if and only if the mapping $\mathcal{R}(\mathcal{A})$ is injective.* □

Taking into account 2.2(iii), we realize easily that poolscapes are divided into four basic classes:

(A) $|\bigcap\mathcal{A}| = 1 = |S \setminus \bigcup\mathcal{A}|$;

(B) $\bigcap\mathcal{A} = \emptyset$, $|S \setminus \bigcup\mathcal{A}| = 1$;

(C) $|\bigcap\mathcal{A}| = 1$, $S = \bigcup\mathcal{A}$;

(D) $\bigcap\mathcal{A} = \emptyset$, $S = \bigcup\mathcal{A}$.

Clearly, a system $\mathcal{A}$ is a poolscape of class (C) or (D) if and only if for all $a, b \in S$ there is at least one pool A with $|A \cap \{a, b\}| = 1$.

Till the end of the section, let $\mathcal{A}$ be a poolscape (on S) and r be the smallest positive integer satisfying $s \le 2^r$. Using 2.2(iii), we easily come by the following statements.

Lemma 2.3. $1 \le r \le |\mathcal{A}| \le 2^s - 1$. □

Lemma 2.4. *If* $s = 2^r$ *and* $\mathcal{A}$ *is not of class* (A) *then* $r + 1 \le |\mathcal{A}|$. □

Lemma 2.5. *Assume* $S \notin \mathcal{A}$ *and put* $\overline{\mathcal{A}} = \{\, S \setminus A \mid A \in \mathcal{A} \,\}$. *Then:*
(i) $\mathcal{R}(\overline{\mathcal{A}})(\mathrm{A}) = \{\, S \setminus A \mid A \in \mathcal{A}, a \notin A \,\}$ *for every* $a \in S$.
(ii) $\overline{\mathcal{A}}$ *is a poolscape,* $S \notin \overline{\mathcal{A}}$, $|\overline{\mathcal{A}}| = |\mathcal{A}|$ *and* $\overline{\overline{\mathcal{A}}} = \mathcal{A}$.
(iii) $\overline{\mathcal{A}}$ *is of class* (A) *(*(B),(C),(D)*, resp.) if and only if* $\mathcal{A}$ *is of class* (A) *(*(C),(B),(D)*, resp.).* □

Lemma 2.6. *If* $\bigcup\mathcal{A} = S$ *(i.e.,* $\mathcal{A}$ *is of class* (C) *or* (D)*) and if* $S \subseteq S'$, $|S'| = |S| + 1$, *then* $\mathcal{A}$ *is a poolscape on* S' *as well (and* $\mathcal{A}$ *is of class* (B) *on* S'*).* □

Example 2.7. Let $S = \{1, 2\}$. The systems $\mathcal{A}_1 = \{\{1\}\}$, $\mathcal{A}_2 = \{\{2\}\}$, $\mathcal{A}_3 = \{\{1\},\{2\}\}$, $\mathcal{A}_4 = \{\{1\},\{1,2\}\}$, $\mathcal{A}_5 = \{\{2\},\{1,2\}\}$ and $\mathcal{A}_6 = \{\{1\},\{2\},\{1,2\}\}$ are all poolscapes on S. None of them is of class (B).

3 Large systems are poolscapes

Proposition 3.1. *Let* $\mathcal{A}$ *be a system (on* S*) that is not a poolscape. Then:*
(i) $1 \le |\mathcal{A}| \le 2^{s-1} - 1$.
(ii) $|\mathcal{A}| = 2^{s-1} - 1$ *if and only if there is a two-element subset* T *of the set* S *such that* $\mathcal{A} = \{\, A \mid T \subseteq A \subseteq S \,\} \cup \{\, B \mid \emptyset \ne B \subseteq S \setminus T \,\}$.

Proof. Since $\mathcal{A}$ is not a poolscape, there is a two-element subset $T \subseteq S$ such that $|A \cap T| \ne 1$ for each $A \in \mathcal{A}$. Put $\mathcal{A}_1 = \{\, A \in \mathcal{A} \mid T \subseteq A \,\}$, $\mathcal{A}_2 = \{\, A \in \mathcal{A} \mid A \cap T = \emptyset, A \cup T \in \mathcal{A} \,\}$, $\mathcal{A}_3 = \{\, A \in \mathcal{A} \mid A \cap T = \emptyset, A \cup T \notin \mathcal{A} \,\}$. Clearly, the system $\mathcal{A}$ is the disjoint union of the sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$. Furthermore, $\mathcal{A}_2 \cup \mathcal{A}_3 \subseteq \mathcal{P}^+(S \setminus T)$, $|\mathcal{A}_2| + |\mathcal{A}_3| = |\mathcal{A}_2 \cup \mathcal{A}_3| \le 2^{s-2} - 1$, $\mathcal{A}_1 \subseteq \{\, B \cup T \mid B \in \mathcal{P}(S \setminus T) \,\}$, $|\mathcal{A}_1| \le 2^{s-2}$ and $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \le 2^{s-2} + 2^{s-2} - 1 = 2^{s-1} - 1$. Finally, if $|\mathcal{A}| = 2^{s-1} - 1$ then $\mathcal{A}_1 = \{\, B \cup T \mid B \in \mathcal{P}(S \setminus T) \,\}$, $|\mathcal{A}_2 \cup \mathcal{A}_3| = 2^{s-2} - 1$ and $\mathcal{A}_2 \cup \mathcal{A}_3 = \mathcal{P}^+(S \setminus T)$. The rest is clear. □

Theorem 3.2. *Let* $\mathcal{A}$ *be a system (on* S*) such that* $|\mathcal{A}| \ge 2^{s-1}$. *Then:*
(i) $\mathcal{A}$ *is a poolscape of class* (C) *or* (D).
(ii) *If* $|\mathcal{A}| > 2^{s-1}$ *then* $\mathcal{A}$ *is of class* (D).
(iii) $\mathcal{A}$ *is of class* (C) *if and only if there is an element* $a \in S$ *such that* $\mathcal{A} = \{\, A \mid a \in A \subseteq S \,\}$ *(then* $|\mathcal{A}| = 2^{s-1}$ *and* a *is uniquely determined).*

Proof. It follows directly from 3.1(i) that $\mathcal{A}$ is a poolscape. If $a \in S \setminus \bigcup\mathcal{A}$ then $\mathcal{A} \subseteq \mathcal{P}^+(S \setminus \{a\})$, and therefore $|\mathcal{A}| < 2^{s-1}$, a contradiction. Thus $\bigcup\mathcal{A} = S$ and $\mathcal{A}$ is of one from the classes (C),(D). If $a \in \bigcap\mathcal{A}$ then $|\mathcal{B}| = |\mathcal{A}|$, where $\mathcal{B} = \{\, A \setminus \{a\} \mid A \in \mathcal{A} \,\}$. Since $\mathcal{B} \subseteq \mathcal{P}^{(}S \setminus \{a\})$ and $|\mathcal{B}| \ge 2^{s-1}$, we get $\mathcal{B} = \mathcal{P}^{(}S \setminus \{a\})$. The rest is clear. □

Remark 3.3. The number of the systems (poolscapes) $\mathcal{A}$ (on S) with $|\mathcal{A}| \ge 2^{s-1}$ is precisely $2^{2^s-2} \ge 4$ (≥ 64 for $s \ge 3$). Indeed, that number equals to the sum $x = \sum_{k=2^{s-1}}^{2^s-1} \binom{2^s-1}{k}$. However, $x = \sum_{k=0}^{2^{s-1}-1} \binom{2^s-1}{k}$ and $2x = \sum_{k=0}^{2^s-1} \binom{2^s-1}{k} = 2^{2^s-1}$.

4 Forming poolscapes from poolscapes

Construction 4.1. Let S, T be finite sets, $|S| = s \ge 2$, $|T| = t \ge 2$. Let $\alpha : S \to \mathcal{P}^{(}T)$ be a mapping. For every $b \in T$, put $A_b = \{\, a \in S \mid b \in \alpha(a) \,\}$ $(\subseteq S)$ and define a mapping $\beta : T \to \mathcal{P}^{(}S)$ by $\beta(b) = A_b$ for every $b \in T$. Further, for every $a \in S$, put $B_a = \{\, b \in T \mid a \in \beta(b) \,\}$. Finally, put $\mathcal{A} = \beta(T)$ and $\mathcal{B} = \alpha(S)$.

The following assertion is a mere observation.

Lemma 4.2. (i) $\mathcal{A} \subseteq \mathcal{P}(S)$ *and* $1 \le |\mathcal{A}| \le t$.
(ii) $\alpha(a) = \{\, b \in T \mid a \in \beta(b) \,\}$ *and* $\beta(b) = \{\, a \in S \mid b \in \alpha(a) \,\}$ *for every* $a \in S$, $b \in T$.
(iii) $\mathcal{A}$ *is a system on* S *(i.e.,* $\emptyset \ne \mathcal{A} \subseteq \mathcal{P}^+(S)$*) if and only if* $T = \bigcup_{a \in S} \alpha(a)$.
(iv) *If* $b_1, b_2 \in T$, $b_1 \ne b_2$, *then* $A_{b_1} \ne A_{b_2}$ *if and only if* $|\alpha(a) \cap \{b_1, b_2\}| = 1$ *for at least one* $a \in S$.
(v) $|\mathcal{A}| = t$ *if and only if* $\mathcal{B} \setminus \{\emptyset\}$ *is a poolscape (on* T*).*
(vi) $\bigcup\mathcal{A} = \{\, a \in S \mid \alpha(a) \ne \emptyset \,\}$.
(vii) $\bigcup\mathcal{A} = S$ *if and only if* $\alpha(S) \subseteq \mathcal{P}^+(T)$.
(viii) $\bigcap\mathcal{A} = \{\, a \in S \mid \alpha(a) = T \,\}$.
(ix) $\bigcap\mathcal{A} = \emptyset$ *if and only if* $\alpha(S) \subseteq \mathcal{P}(T) \setminus \{T\}$.
(x) *If* $a_1, a_2 \in S$, $a_1 \ne a_2$, *then* $|A_b \cap \{a_1, a_2\}| = 1$ *for at least one* $b \in T$ *if and only if* $\alpha(a_1) \ne \alpha(a_2)$.
(xi) $\mathcal{A} \setminus \{\emptyset\}$ *is a poolscape (on* S*) if and only if* α *is an injective mapping.*
(xii) $A_b = S$ *if and only if* $b \in \bigcap\mathcal{B}$.
(xiii) *The following conditions are equivalent:*

1. $|\mathcal{B}| = s$.

2. α is injective.

3. $\mathcal{A} \setminus \{\emptyset\}$ is a poolscape (on S).

(xiv) *The following conditions are equivalent:*

1. $|\mathcal{A}| = t$.

2. β is injective.

3. $\mathcal{B} \setminus \{\emptyset\}$ is a poolscape (on T).

□

Theorem 4.3. *Let S, T be finite sets, $|S| = s \geq 2$, $|T| = t \geq 2$. Let $\mathcal{B}$ be a poolscape defined on the set T such that $|\mathcal{B}| = s$ and $\bigcup \mathcal{B} = T$ (i.e., $\mathcal{B}$ is of one of the classes* (C) *and* (D)*). Let $\alpha : S \to \mathcal{B}$ be a bijection. Put $A_b = \{ a \in S \,|\, b \in \alpha(a) \}$ for every $b \in T$. Then:*
(i) *$\mathcal{A} = \{ A_b \,|\, b \in T \}$ is a poolscape defined on the set S and $|\mathcal{A}| = t$.*
(ii) *$\bigcup \mathcal{A} = S$ (i.e., $\mathcal{A}$ is of one of the classes* (C) *and* (D)*).*
(iii) *$\mathcal{A}$ is of class* (C) *(*(D)*, resp.) if and only if $T \in \mathcal{B}$ ($T \notin \mathcal{B}$, resp.).*
(iv) *If $T \in \mathcal{B}$ then $\bigcap \mathcal{A} = \{\alpha^{-1}(T)\}$.*
(v) *$S \in \mathcal{A}$ ($S \notin \mathcal{A}$, resp.) if and only if $\bigcap \mathcal{B} \neq \emptyset$ ($\bigcap \mathcal{B} = \emptyset$). That is, if and only if $\mathcal{B}$ is of class* (C) *(*(D)*, resp.).*
(vi) *If $\bigcap \mathcal{B} = \{b\}$ then $A_b = S$.*
(vii) *The mapping $\beta : T \to \mathcal{A}$, where $\beta(b) = A_b$ for each $b \in T$, is a bijection.*
(viii) *$\alpha(a) = \{ b \in T \,|\, a \in \beta(b) \}$ and $\beta(b) = \{ a \in S \,|\, b \in \alpha(a) \}$ for every $a \in S$, $b \in T$.*

Proof. (i) Use 4.2(iii),(iv),(xii),(xiv).
(ii) Use 4.2(viii).
(iii) and (iv). See 4.2(x),(ix).
(v) and (vi). Use 4.2(xii).
(vii) Use 4.2(xiv).
(viii) See 4.2(ii). □

5 Adjoint poolscapes

Considering 4.3, we feel fully eligible to formulate such a definition:

Definition 5.1. Let S and T be finite sets and $\mathcal{A}$ and $\mathcal{B}$ be poolscapes defined on S and T, resp., such that $|\mathcal{B}| = |S| = s \geq 2$ and $|\mathcal{A}| = |T| = t \geq 2$. The poolscapes $\mathcal{A}$ and $\mathcal{B}$ are said to be *adjoint* (via α, β) provided that there exist bijections $\alpha : S \to \mathcal{B}$ and $\beta : T \to \mathcal{A}$ such that the following two conditions are true:

(α) $\alpha(a) = \{ b \in T \,|\, a \in \beta(b) \}$ for every $a \in S$;

(β) $\beta(b) = \{ a \in S \,|\, b \in \alpha(a) \}$ for every $b \in T$.

(Notice that these conditions are equivalent.)

Observation 5.2. Let $\mathcal{A}, \mathcal{B}$ be adjoint poolscapes on the sets S, T. Here, we collect a handful of easy (not facile, nonetheless) observations:
(i) $\bigcup \mathcal{A} = S$ and $\bigcup \mathcal{B} = T$.
(ii) $\bigcap \mathcal{A} = \emptyset$ if and only if $T \notin \mathcal{B}$.
(iii) $\bigcap \mathcal{B} = \emptyset$ if and only if $S \notin \mathcal{A}$.
(iv) $\mathcal{A}$ ($\mathcal{B}$, resp.) is of class (C) if and only if $T \in \mathcal{B}$ ($S \in \mathcal{A}$, resp.). Otherwise, $\mathcal{A}$ ($\mathcal{B}$, resp.) is of class (D).
(v) $s < 2^t$ and $t < 2^s$.
(vi) Assume that $S \notin \mathcal{A}$ and $T \notin \mathcal{B}$. Then the poolscapes $\mathcal{A}, \mathcal{B}$ are of class (D) and the same is true for poolscapes $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ (see 2.5). Besides, $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ are adjoint via bijections $\overline{\alpha}, \overline{\beta}$, where $\overline{\alpha}(a) = T \setminus \alpha(a)$ for every $a \in S$ and $\overline{\beta}(b) = S \setminus \beta(b)$ for every $b \in T$.

Indeed, (i) follows from 4.2(iii) (and its dual), and (ii) is an immediate consequence of 4.2(Viii). Further, α is a bijection, and hence $s = |S| = \mathcal{B}| \leq |\mathcal{P}^+(T)| = 2^t - 1$. Similarly, $t \leq 2^s - 1$ and (v) follows. The rest is clear.

6 Construction of poolscapes of minimum size

Throughout this section, let S be a set with $|S| = s \geq 2$. Clearly (see 2.8), if $s = 2$ and $S = \{x, y\}$ then poolscapes of minimum size on S are $\mathcal{A}_1 = \{\{x\}\}$, $\mathcal{A}_2 = \{\{y\}\}$ (these poolscapes are of class (A)).

Construction 6.1. Let $s \geq 3$, s not a power of 2, and let t be the smallest positive integer satisfying $s \leq 2^t$. Clearly, $2 \leq t$ and $3 \leq 2^{t-2} + 1 \leq s \leq 2^t - 1$. Furthermore, put $T = \{1, 2, \ldots, t\}$ and choose any system $\mathcal{B} \subseteq \mathcal{P}^+(T)$ such that $|\mathcal{B}| = s$. (Exactly $\binom{2^t-1}{s}$ such systems are at our disposal.) By 3.2, $\mathcal{B}$ is a poolscape of class (D) on the set T.

Now, choose a bijection $\alpha : S \to \mathcal{B}$ (there are s! (≥ 6) possibilities for choosing α) and denote by $\mathcal{A}$ the poolscape from 4.3 (the poolscapes $\mathcal{A}, \mathcal{B}$ are adjoint). Then $\mathcal{A} = \{ A_i \,|\, 1 \leq i \leq t \}$, where $A_i = \{ a \in S \,|\, i \in \alpha(a) \}$ for every $i = 1, 2, \ldots, t$, the poolscape $\mathcal{A}$ is of class (C) ((D), resp.) iff $T \in \mathcal{B}$ ($T \notin \mathcal{B}$, resp.), the mapping $\beta : T \to \mathcal{A}$, where $\beta(i) = A_i$ for every $i = 1, 2, \ldots, t$, is a bijection, $\alpha(a) = \{ i \,|\, 1 \leq i \leq t,\, a \in A_i \}$ and $\mathcal{B} = \{\alpha(a) \,|\, a \in S\}$.

It follows from 2.3 that $|\mathcal{A}| = t \leq |\mathcal{A}_1|$ for any poolscape $\mathcal{A}_1$ defined on the set S.

If $t = 2$ then $s = 3$, $\mathcal{B} = \{\{1\}, \{2\}, \{1, 2\}\} = \mathcal{P}^+(T)$ and $\mathcal{A}$ is of class (C). If, moreover, $S = \{x, y, z\}$, $\alpha(x) = \{1\}$, $\alpha(y) = \{2\}$ and $\alpha(z) = \{1, 2\}$ then $A_1 = \{x, z\}$ and $A_2 = \{y, z\}$.

Now, assume that $t \geq 3$ and ($5 \leq$) $2^{t-1} + 1 \leq s \leq 2^t - 2$. Then we always can choose $\mathcal{B}$ in such a way that either $T \in \mathcal{B}$ or $T \notin \mathcal{B}$ (so that $\mathcal{A}$ will be of class (C) or (D), resp.).

Finally, if $t \geq 3$ and $(7 \leq)\ s = 2^t - 1$ then $\mathcal{B} = \mathcal{P}^+(T)$ and $\mathcal{A}$ is of class (C).

Construction 6.2. Let $s = 2^t$, $t \geq 2$. Take $a \in S$ and put $S_1 = S \setminus \{a\}$. We have $|S_1| = 2^t - 1$ and. Denote by $\mathcal{A}$ a poolscape on the set S_1 constructed by Construction 6.1. Then $|\mathcal{A}| = t$ and $\mathcal{A}$ is of class (C). Lemma 2.6 implies that $\mathcal{A}$ is a poolscape on S (and $\mathcal{A}$ is of class (B) this time). Finally, 2.3 implies that $|\mathcal{A}| = t \leq |\mathcal{A}_1|$ for each poolscape $\mathcal{A}_1$ defined on S.

Construction 6.3. Let $t \geq 1$, $s = 2^t$ and $T = \{1, \dots, t, t+1\}$. Put $\mathcal{B} = \{\, B \subseteq T \mid 1 \in B \,\}$. Then $\mathcal{B}$ is a poolscape of class (C) on T and $|\mathcal{B}| = 2^t = s$ (cf. 3.2(iii)). Now, choose a bijection $\alpha : S \to \mathcal{B}$ and use Construction 4.1 to construct $\mathcal{A}_0$ (thus $\mathcal{A}_0$ and $\mathcal{B}$ are adjoint poolscapes by 4.3). Then $\mathcal{A} = \{\, A_i \mid 1 \leq i \leq t+1 \,\}$, where $A_i = \{\, a \in S \mid i \in \alpha(a) \,\}$, the poolscape $\mathcal{A}_0$ is of class (C) by 5.2(iv) and $|\mathcal{A}_0| = t + 1$. Apparently, $A_1 = S$, and hence $\mathcal{A} = \mathcal{A}_0 \setminus \{A_1\} = \mathcal{A}_0 \setminus \{S\}$ is a poolscape by 2.1(i). Of course, $|\mathcal{A}| = t \leq |A_1|$ for any poolscape $\mathcal{A}_1$ defined on S and $\mathcal{A}$ is of class (A); we have $\bigcap \mathcal{A} = \{\alpha^{-1}(T)\}$ and $\bigcup \mathcal{A} = S \setminus \{\alpha^{-1}(1)\}$.

7 Conclusion

This paper develops ideas concerning separating systems that form the initial theoretical framework for studying combinatorial group testing methods. It gathers basic knowledge for working with separating systems and introduces basic constructs to create separating systems. Using combinatorial methods and set theory to develop separating systems is a suitable theoretical framework that could contribute to creating new testing methods and as an essential theoretical starting point for proofs in separating systems optimization.

The notion of adjoint poolscapes yields a correspondence between classes defined on sets S and T so that a large poolscape on a small set S corresponds to a small poolscape on a large set T and vice versa. This correspondence can be used in the design of group tests, which have applications in statistics, computer science, medicine, and many other fields. In future research, similar correspondence could be investigated in more general settings.

References:

[1] A. Rényi, On random generating elements of a finite Boolean algebra, *Acta Sci. Math. Szeged*, Vol. 22 , 1961, pp. 75–81.

[2] C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs*, Chapman and Hall, New York 2006.

[3] D.-Z. Du and F. K. Hwang, *Pooling Designs and Nonadaptive Group Testing*, Series on Appl. Math. 18, World Scientific, 2006.

[4] G. Wiener, E. Hoszu, J. Tapolcai, On separating systems with bounded set size, *Discrete applied mathematics*, Vol. 276, SI, 2020, 172–176.

[5] A. Jančařík, Combinatorial group testing algorithms improved for d = 3. *WSEAS Transactions on Information Science and Applications*, Vol. 20, 2023, pp. 453–455.

[6] T. Bardini Idalino, L. Moura, Structure-aware combinatorial group testing: a new method for pandemic screening. In *International Workshop on Combinatorial Algorithms*. Cham: Springer International Publishing, 2022. p. 143–156.

[7] F. Huang, P. Guo, Y. Wang, Optimal group testing strategy for the mass screening of SARS-CoV-2. *Omega*, Vol. 112, 2022, no. 102689.

[8] V. H. da Silva, C. P. Goes, P. A. Trevisoli, R. Lello, L. G. Clemente, T. B. de Almeida, J. Petrini, L. L. Countinho, Simulation of group testing scenarios can boost COVID-19 screening power. *Scientific Reports*, Vol. 12, No. 1, 2022, no: 11854.

[9] J. Wu, Y. Cheng, D.-Z. Du, An improved zig zag approach for competitive group testing, *Discrete Optimization*, Vol. 43, Feb. 2022, no: 100687

[10] A. De Bonis, G. Di Crescenzo, Combinatorial group testing for corruption localizing hashing, In: *International Computing and Combinatorics Conference*, Springer, Berlin, 2011, pp. 579–591

[11] L. Lazic, S. Popovic, N. Mastorakis, A simultaneous application of combinatorial testing and virtualization as a method for software testing. *WSEAS Transactions on Information Science and Applications*, Vol. 6, No. 11, 2009, pp. 1802–1813.

[12] S. W. Chiu, K. K. Chen, J. C. Yang, M. H. Hwang, Deriving the Optimal Production-Shipment Policy with Imperfect Quality and an Amending Delivery Plan using Algebraic Method. *WSEAS Transactions on Systems*, Vol. 11, No. 5, 2012, pp. 163–172.

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