On a Class of Multivalent Functions With Negative Coefficients Involving $(r, q)-$ Calculus

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Abstract: - In this research, we focused on presenting a novel subclass of multivalent analytic functions situated in the open unit disk, characterized by the use of Jackson's derivative operator. Our investigation systematically establishes the requisite inclusion conditions in this class, offering detailed coefficient characterizations. The exploration encompasses an array of significant properties intrinsic to this subclass, encompassing coefficient estimates, growth and distortion theorems, identification of extreme points, and the determination of the radius of starlikeness and convexity for functions falling within this specialized category. Expanding the preliminary findings, this research extended the inquiry to delve deeper into the intriguing features and implications associated with this new subclass of multivalent analytic functions. The research concentrated the light on the nuanced intricacies of coefficient estimates, providing a comprehensive understanding of how these functions evolve within the open unit disk through exploring the growth and distortion theorems, unraveling the underlying mathematical principles governing the behavior of functions in this subclass as they extend beyond the unit disk. The findings of this research contribute to the broader understanding of multivalent analytic functions, paving the way for further exploration and applications in diverse mathematical contexts.

Key-Words: -Analytic function, Unit disk, New subclass, $p-$valent function, Quantum or (r, q)-Calculus, (r, q)-Derivative operator.

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1 Commencement and Definition

The category of all analytic functions exhibiting the following structure

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \quad \text{is complex number}$$

withen the open unit disk, denoted as $\underline{\mathcal{U}} = \{z \in \mathbb{C} : |z| < 1\}$ where $\mathbb{C}$ signifies the set of complex numbers. This class of all analytic functions is denoted as $\hat{\mathbb{A}}$. Also, let $\hat{\mathbb{A}}(p)(p \in \mathbb{N} = \{1, 2, 3, ...\})$ be the class consisting of all analytic functions f. This class is represented by a series that articulates the underlying structure of these functions.

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad a_n \quad \text{is complex number}$$

which called p-valent in the open unit disk $\underline{\mathcal{U}}$ over the complex numbers $\mathbb{C}$. It is important to observe that $\hat{\mathbb{A}}(1)$ is equivalent to $\hat{\mathbb{A}}$. Additionally, the collection of all univalent functions within the open disk $\underline{\mathcal{U}}$ is symbolized as $S(p)$ which is a subclass of $\hat{\mathbb{A}}(p)$. Furthermore, let $S_p^*(\alpha)$ and $C_p(\alpha)$ represent the classes of p- valent functions respectively starlike of order α and convex of order α, for $0 \leq \alpha < p$. Notably, $S_p^*(0)$ is synonymous with S_p^* and $C_p(0)$ corresponds to C_p, both being the well-known classes of starlike and convex p-valent functions in $\underline{\mathcal{U}}$, respectively.

Next, let us assume that $\mathcal{T}(p)(p \in \mathbb{N} = \{1, 2, 3, ...\})$ denote the subclass of $S(p)$ of analytic functions having the structure

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \qquad a_n > 0. \tag{1}$$

As it is defined on the open unit disk $\underline{\mathcal{U}} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{T}(p)$ is denoted as a p- valent function with negative coefficients. Moreover, it is evident that $S^*_{\mathcal{T},p}(\alpha)$ and $C_{\mathcal{T},p}(\alpha)$ for $0 \leq \alpha < p$, that are p- valent functions, respectively starlike of order α and convex of order α which is subclasses of $\mathcal{T}(p)$. Clearly, the class $\mathcal{T}(1) = \mathcal{T}$ in [1], he derived and investigated the subclasses of $\mathcal{T}(1)$ denoted by $S^*_{\mathcal{T},1}(\alpha) = S^*_{\mathcal{T}}(\alpha)$ and $C_{\mathcal{T},1}(\alpha) = C_{\mathcal{T}}(\alpha)$, for $0 \leq \alpha < 1$ that are respectively starlike of order α and convex of order α.

In this section, we revisit established concepts and fundamental results of (r,q)-calculus. Throughout this paper, we denote constants as let r, q be constants with $0 < q < r \leq 1$. We provide some definitions and theorems pertinent to (r,q)-calculus, which will be employed in the subsequent sections of these papers. [2], [3], [4], [5], [6], and, [7].

For $0 < q < r \leq 1$ the Jackson's (r,q)-derivative of a function $f \in \hat{\mathbb{A}}(p)$ is, by definition, given as follow

$$\mathcal{D}_{r,q} f(z) := \begin{cases} \dfrac{f(rz) - f(qz)}{(r-q)\, z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{2}$$

From (2), we have

$$\mathcal{D}_{r,q} f(z) = [p]_{r,q}\, z^{p-1} + \sum_{n=p+1}^{\infty} [n]_{r,q}\, a_n\, z^{n-1},$$

where $[p]_{r,q} = \frac{r^p - q^p}{r-q}$, $[n]_{r,q} = \frac{r^n - q^n}{r-q}$ and $0 < q < r \leq 1$.

Note that for $r = 1$, the Jackson (r,q)-derivative reduces to the Jackson q-derivative operator of the function f, $\mathcal{D}_q f(z)$ (refer to [8], [9], and, [10]). Note also that $\mathcal{D}_{1,q} f(z) \to f'(z)$ when $q \to 1-$, where f' is the classical derivative of the function f.

Clearly for a function $g(z) = z^n$, we obtain

$$\mathcal{D}_{r,q} g(z) = \mathcal{D}_{r,q} z^n = \frac{r^n - q^n}{r-q} z^{n-1} = [n]_{r,q}\, z^{n-1}.$$

And

$$\lim_{q \to 1-} \mathcal{D}_{1,q} g(z) = \lim_{q \to 1-} \frac{1-q^n}{1-q} z^{n-1} = n z^{n-1} = g'(z),$$

where g' is the ordinary derivative.

The theory of q-calculus finds application in the adaptation and resolution of various problems in applied science like ordinary fractional calculus, quantum physics, optimal control, hypergeometric series , operator theory, q-difference and q-integral equations, as well as geometric function theory of complex analysis. The application of q-calculus was started by, [11]. In, [12], have utilized the fractional q-calculus operators in examinations of specific classes of functions which are analytic in $\underline{\mathcal{U}}$. For further details on q-calculus one can refer to [13], [14], [15], and, [16], as well as the additional references mentioned therein.

Alongside the advancement of the theory and application of q-calculus, the theory of q-calculus dependent on two parameters (r,q)-integers has also been introduced and received more consideration during the last few decades. In 1991, [17], showed the (r,q)-calculus. Next, [18], investigated the fundamental theorem of (r,q)-calculus and a few (r,q)-Taylor formulas. As of late, [19], studied the (r,q)-derivative and (r,q)-integral on finite intervals. Besides, they concentrated on certain properties of (r,q)-calculus and (r,q)-associated of some important integral inequalities. The (r,q)-derivative have been considered and quickly created during this period by many creators.

Classes defined by derivative operators often arise in complex analysis or functional analysis and are crucial in characterizing specific sets of functions that satisfy certain properties related to their derivatives. These classes help classify functions based on their behavior under differentiation or specialized derivative operators. Utilizing the above defined (r,q)-calculus, certain subclasses in the class $\hat{\mathbb{A}}(p)$ have as of now been explored in geometric function theory. Some classes defined by (r,q)-calculus operators like q-starlike and q-convex functions, these classes are defined using q-calculus operators. For instance, q-starlike functions are those for which the q-derivative has positive real part in the unit disk under the q-calculus framework. Similarly, q-convex functions satisfy certain conditions related to their q-derivatives in the unit disk. Also (r,q)-starlike and (r,q)-convex functions, these classes are extensions of the q-starlike and q-convex classes, incorporating additional parameters, r and q in the calculus operators. (r,q)-starlike and (r,q)-convex functions exhibit specific properties related to their derivatives under the (r,q)-calculus framework. [20], were the first who utilized the q-derivative operator $\mathcal{D}_q$ to concentrate on the q-calculus comparable of the class

S^* of starlike functions in $\underline{\mathcal{U}}$.

Now, let $\underline{\mathcal{M}}(A,B,C)$ be the subclass of $\hat{\mathbb{A}}(1)$ consisting of functions $f \in \hat{\mathbb{A}}(1)$ which satisfy the inequality

$$\left|\frac{f'(z)-1}{Af'(z)+(1-B)}\right| < C,$$

where $0 \leq A \leq 1$, $0 \leq B < 1$ and $0 < C \leq 1$ for all $z \in \underline{\mathcal{U}}$. This class of functions was studied by, [21].

In, [21], defined the class $\underline{\mathcal{M}}^T(A,B,C)$ by $\underline{\mathcal{M}}^T(A,B,C) = \underline{\mathcal{M}}(A,B,C) \cap \mathcal{T}$.

Further we present some general subclass of analytic and multivalent functions related to (r,q)-derivative operator as follows.

Definition 1.1 For $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$, $k \geq 0$, $0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$, we let $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ consist of functions $f \in \mathcal{T}(P)$ satisfying the condition

$$\begin{aligned} &\mathrm{Re}\left(\frac{(\mathcal{D}_{r,q}f(z))'-1}{\alpha\,(\mathcal{D}_{r,q}f(z))'+(1-\gamma)}\right)\\ &> \ k\left|\frac{(\mathcal{D}_{r,q}f(z))'-1}{\alpha\,(\mathcal{D}_{r,q}f(z))'+(1-\gamma)}-1\right|+\beta. \quad (3)\end{aligned}$$

Certainly, the study outlined here appears to be focused on a specific class of mathematical functions denoted as $f \in \Upsilon(\alpha, \beta, \gamma, k, r, q, p)$. The initial finding, the coefficient estimate or determine the coefficients of functions within the specified class, for functions $f \in \Upsilon(\alpha, \beta, \gamma, k, r, q, p)$. These coefficients likely hold crucial information about the behavior and properties of these functions. Growth and distortion theorem are included, by including growth and distortion theorems, the study aims to investigate how these functions behave under certain transformations or conditions. Delving into their growth and distortion characteristics offers valuable insights into their behavior and potential practical uses. Furthermore, we obtain the extreme points, the study seeks to identify extreme points of these functions. These extreme points often hold significance in understanding the behavior and nature of the functions. After that, the radius of starlikeness and convexity, for the function in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ are determined. Understanding the boundaries within these functions possess such geometric properties is crucial in their characterization.

As a matter of first importance, let us take at the coefficient inequalities. And the technique which studied in, [22], and also in [23].

2 Coefficient Inequalities

In this section we present a fundamental and sufficient condition for the function f in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$. Our main first result as follows:

Theorem 2.1 Let $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$, $k \geq 0$, $0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$. A function f given by (1) is in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ if and only if

$$\sum_{n=p+1}^{\infty} \mu_n\, a_n \leq \mu_p + 1, \tag{4}$$

where

$$\mu_n = \left(\frac{(k+1)\,(1-\alpha)+\alpha\,(1-\beta)}{(k+\beta)\,(1-\gamma)+k+1}\right)(n-1)[n]_{r,q}. \tag{5}$$

Proof. We have $f \in \Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ if and only if the condition (3) is satisfied.
Let

$$w = \frac{(\mathcal{D}_{r,q}f(z))'-1}{\alpha\,(\mathcal{D}_{r,q}f(z))'+1-\gamma},$$

upon the fact that,

$$Re\,(\,w\,) \ \geq \ k\ |\,w\,-\,1\,|\,+\,\beta \quad \text{if and only if} \quad (k\,+\,1\,)\ |\,w\,-\,1|\ \leq\ 1\,-\beta.$$

Now

$$\begin{aligned} & (k+1)\,|w-1| \\ = \ & (k+1)\left|\left\{\left(\sum_{n=p+1}^{\infty}(\alpha-1)\,[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right)\right.\right.\\ - \ & \left.(\alpha-1)\,[p]_{r,q}(p-1)z^{p-2}-(2-\gamma)\right\}\\ / \ & \left\{\alpha[p]_{r,q}(p-1)z^{p-2}\right.\\ - \ & \left.\left.\alpha\left(\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right)-(\gamma-1)\right\}\right|\\ \leq \ & 1-\beta. \end{aligned} \tag{6}$$

The above inequality reduces to

$$\begin{aligned} & (k+1)\left(\left|\sum_{n=p+1}^{\infty}(\alpha-1)\,[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right|\right.\\ - \ & \left.\left|(\alpha-1)\,[p]_{r,q}(p-1)z^{p-2}\right|-|2-\gamma|\right)\\ / \ & \left(\left|\alpha[p]_{r,q}(p-1)z^{p-2}\right|\right.\\ - \ & \left.\alpha\left|\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right|-|\gamma-1|\right)\\ \leq \ & 1-\beta. \end{aligned} \tag{7}$$

After that,

$$\begin{aligned}
& (k+1)\left(\sum_{n=p+1}^{\infty}(1-\alpha)\,[n]_{r,q}\,(n-1)\,a_n\right. \\
- & \left.(1-\alpha)\,[p]_{r,q}(p-1)-(2-\gamma)\right) \\
/ & \left(\alpha[p]_{r,q}(p-1)\right. \\
- & \left.\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n-(1-\gamma)\right) \\
\leq & \ 1-\beta. \qquad \text{where } |z|<1.
\end{aligned}$$

then, we have

$$\begin{aligned}
& \sum_{n=p+1}^{\infty}\left((k+1)\,(1-\alpha)+\alpha\,(1-\beta)\right)(n-1)\,[n]_{r,q}a_n \\
\leq & \ \left((k+1)\,(1-\alpha)+\alpha\,(1-\beta)\right)(p-1)\,[p]_{r,q} \\
- & \ (1-\gamma)\,(1-\beta)+(k+1)\,(2-\gamma)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
& \sum_{n=p+1}^{\infty}\left((k+1)\,(1-\alpha)+\alpha\,(1-\beta)\right)(n-1)\,[n]_{r,q}a_n \\
\leq & \ \left((k+1)\,(1-\alpha)+\alpha\,(1-\beta)\right)(p-1)\,[p]_{r,q} \\
+ & \ (1-\gamma)\,(k+\beta)+(k+1)\,,
\end{aligned}$$

divide by $(1-\gamma)\,(k+\beta)+(k+1)$ for both side, which yield to (4).

Suppose that (4) holds and we have to show (6) holds. Here the inequality (4) is equivalent to (7). So it suffices to show that,

$$\begin{aligned}
& \left|\left\{\left(\sum_{n=p+1}^{\infty}(\alpha-1)\,[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right)\right.\right. \\
- & \left.(\alpha-1)\,[p]_{r,q}(p-1)z^{p-2}-(2-\gamma)\right\} \\
/ & \left\{\alpha[p]_{r,q}(p-1)z^{p-2}\right. \\
- & \left.\left.\alpha\left(\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right)-(\gamma-1)\right\}\right| \\
\leq & \left\{\left(\sum_{n=p+1}^{\infty}(1-\alpha)\,[n]_{r,q}\,(n-1)\,a_n\right)\right. \\
- & \left.(1-\alpha)\,[p]_{r,q}(p-1)-(2-\gamma)\right\} \\
/ & \left\{\alpha[p]_{r,q}(p-1)\right. \\
- & \left.\alpha\left(\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n\right)-(1-\gamma)\right\}. \qquad (8)
\end{aligned}$$

Since,

$$\begin{aligned}
& \left|\alpha[p]_{r,q}(p-1)z^{p-2}-\left(\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right)\right. \\
- & \left.(\gamma-1)\right|, \\
\geq & \left|\alpha[p]_{r,q}(p-1)z^{p-2}\right|-\left|\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right| \\
- & \ |\gamma-1|\,,
\end{aligned}$$

we have

$$\begin{aligned}
& \left|\alpha[p]_{r,q}(p-1)z^{p-2}-\left(\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n z^{n-2}\right)\right. \\
- & \left.(\gamma-1)\right|, \\
\geq & \ \alpha[p]_{r,q}(p-1)-\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\,(n-1)\,a_n-(1-\gamma),
\end{aligned}$$

where $|z|<1$, and hence, we obtain (8).

Theorem 2.2 Let $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$, $k \geq 0$, $0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1, 2, 3, ...\}$. If the function f given by (1) be in the class $\Upsilon(\alpha, \beta, \gamma, k, r, q, p)$ then

$$a_n \leq \frac{\mu_p+1}{\mu_n}, \qquad n = p+1, p+2, p+3, ..., \quad (9)$$

where μ_n is given by (5).

Equality holds for the functions f given by,

$$f(z) = z^P - \frac{(\mu_p+1)\,z^n}{\mu_n}. \qquad (10)$$

Proof. Since $f \in \Upsilon(\alpha, \beta, \gamma, k, r, q, p)$ Theorem 2.1 holds.

Now

$$\sum_{n=p+1}^{\infty}\mu_n a_n \leq \mu_p + 1,$$

we have,

$$a_n \leq \frac{\mu_p+1}{\mu_n}.$$

Clearly the function given by (10) satisfies (9) and therefore f given by (10) is in $\Upsilon(\alpha, \beta, \gamma, k, r, q, p)$ for this function, the result is clearly sharp.

3 Growth and Distortion Theorems for the Subclass $\Upsilon(\alpha, \beta, \gamma, k, r, q, p)$

The growth and distortion theorems represent foundational principles within complex analysis, focusing on comprehending how analytic functions behave, grow, and affect geometric properties particularly in the context of univalent functions. These theorems play a crucial role in understanding the mappings and transformations of complex-valued functions. Both the growth and distortion theorems are especially important when dealing with univalent functions, which are functions that are injective or one-to-one within a certain domain. These theorems provide insights into the behavior of such functions and are crucial in various areas of mathematics and its applications.

The growth and distortion theorem will be considered and the covering property for function in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ is given by the following theorems.

Theorem 3.1 Let $0 \le \alpha \le 1$, $0 \le \beta < 1$, $0 \le \gamma < 1$, $k \ge 0$, $0 < q < r \le 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$. If the function f given by (1) be in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ then for $0 < |z| = l < 1$, we have

$$l^p - \frac{\mu_p+1}{\mu_{p+1}} l^{p+1} \leqslant |f(z)| \leqslant l^p + \frac{\mu_p+1}{\mu_{p+1}} l^{p+1}. \quad (11)$$

Equality holds for the function,

$$f(z) = z^p - \frac{\mu_p+1}{\mu_{p+1}} z^{p+1}, \quad (z = \pm l,\ \pm il),$$

where μ_p and μ_{p+1} can be found by (5).

Proof. We only prove the right hand side inequality in (11), since the other inequality can be justified using similar arguments.

Since $f \in \Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ by Theorem 2.1 we have,

$$\sum_{n=p+1}^{\infty} \mu_n a_n \le \mu_p + 1.$$

Now

$$\begin{aligned}\mu_{p+1}\sum_{n=p+1}^{\infty} a_n &= \sum_{n=p+1}^{\infty} \mu_{p+1} a_n,\\ &\le \sum_{n=p+1}^{\infty} \mu_n\, a_n\\ &\le \mu_p + 1.\end{aligned}$$

And therefore

$$\sum_{n=p+1}^{\infty} a_n \leqslant \frac{\mu_p+1}{\mu_{p+1}}, \quad (12)$$

since

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n,$$

we have,

$$\begin{aligned}|f(z)| &= \left| z^p - \sum_{n=p+1}^{\infty} a_n z^n \right|,\\ &\le |z|^p + |z|^{p+1} \sum_{n=p+1}^{\infty} a_n\, |z|^{n-(p+1)},\\ &\le l^p + l^{p+1} \sum_{n=p+1}^{\infty} a_n.\end{aligned}$$

By aid of inequality (12), yields the right hand side inequality of (11).

Theorem 3.2 If the function f given by (1) is in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ for $0 < |z| = l < 1$ then, we have

$$\begin{aligned} p\, l^{p-1} - \frac{(p+1)(\mu_p+1)}{\mu_{p+1}} l^p &\leqslant \left|f'(\mathrm{z})\right| \\ &\leqslant p\, l^{p-1} + \frac{(p+1)(\mu_p+1)}{\mu_{p+1}} l^p. \end{aligned} \quad (13)$$

Equality holds for the function f given by

$$f(z) = z^p - \frac{\mu_p+1}{\mu_{p+1}} z^{p+1}, \qquad (z = \pm l,\ \pm il),$$

where μ_p and μ_{p+1} can be found by (5).

Proof. Since $f \in \Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ by Theorem 2.1 we have

$$\sum_{n=p+1}^{\infty} \mu_n\, a_n \le \mu_p + 1.$$

Now,

$$\begin{aligned}\mu_{p+1}\sum_{n=p+1}^{\infty} n\, a_n &\le (p+1)\sum_{n=p+1}^{\infty} \mu_n a_n\\ &\le (p+1)(\mu_p+1).\end{aligned}$$

Hence

$$\sum_{n=p+1}^{\infty} n a_n \leqslant \frac{(p+1)(\mu_p+1)}{\mu_{p+1}}, \quad (14)$$

since

$$f'(z) = p z^{p-1} - \sum_{n=p+1}^{\infty} n\, a_n\, z^{n-1}.$$

Then, we have

$$p\ |z|^{p-1} - |z|^p \sum_{n=p+1}^{\infty} na_n\ |z|^{n-1-p}$$

$$\leqslant \quad \left|f'(z)\right| \leqslant p\ |z|^{p-1} + |z|^p \sum_{n=p+1}^{\infty} na_n\ |z|^{n-1-p},$$

where $|z| < 1$. By using the inequality (14), we get Theorem 3.2 and this completes the proof.

Theorem 3.3 If the function f given by (1) is in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ then f is starlike of order δ, where

$$\delta = 1 - \frac{(\mu_p+1)p}{-(\mu_p+1)+\mu_{p+1}}.$$

The result is sharp with

$$f(z) = z^p - \frac{\mu_p+1}{\mu_{p+1}} z^{p+1},$$

where μ_p and μ_{p+1} can be found by (5).

Proof. It is suffices to show that (4) implies

$$\sum_{n=p+1}^{\infty} a_n(n-\delta) \ \leq\ 1-\delta. \tag{15}$$

That is,

$$\frac{n-\delta}{1-\delta} \ \leq\ \frac{\mu_n}{\mu_p+1}\ , \qquad n\ \geq\ p+1. \tag{16}$$

The above inequality is equivalent to

$$\delta \leqslant 1 - \frac{(n-1)(\mu_p+1)}{-(\mu_p+1)+\mu_n} = \psi(n),$$

where $n\ \geq\ p+1$.

And $\psi(n)\ \geq\ \psi(p+1)$, (16) holds true for any $0 \leq \alpha \leq 1,\ 0 \leq \beta < 1,\ 0 \leq \gamma < 1,\ k \geq 0,\ 0\ <\ q < r \leq 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$. This completes the proof of Theorem 3.3.

4 Extreme Points of the Class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$

In spite of its elegance, the significance of extreme point theory within complex function theory is relatively limited. Several papers authored by Brickman, Hallenbeck, Mac Gregor, and Wilken have specifically determined extreme points within traditional families of analytic functions. A comprehensive overview of their findings can be found in [24]. Notably, the availability of extreme points within the set $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$, comprising functions f that are analytic on the unit disc, possess a positive real part, and are normalized by $f(0) = 1$, holds fundamental importance. The theorem detailing the extreme points of the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ is as follows.

Theorem 4.1 Let $f_p(z) = z^p$,
and

$$f_n(z) = z^p - \frac{\mu_p+1}{\mu_n} z^n,\ \ n = p+1, p+2, p+3, ...,$$

where μ_n is given by (5).

Then $f \ \in\ \Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ if and only if it can be represented in the form

$$f(z) = \sum_{n=p}^{\infty} y_n f_n(z) \tag{17}$$

where $y_n \geq 0$ and $\sum\limits_{n=p}^{\infty} y_n\ =\ 1.$

Proof. Suppose f can be expressed as in (17). Our goal is to show that $f \in\ \Upsilon(\alpha,\beta,\gamma,k,r,q,p)$.

By (17) we have

$$\begin{aligned} f(z) &= \sum_{n=p}^{\infty} y_n f_n(z) \\ &= y_p f_p(z) + \sum_{n=p+1}^{\infty} y_n f_n(z), \\ &= y_p z^p + \sum_{n=p+1}^{\infty} y_n\Big(z^p - \frac{\mu_p+1}{\mu_n} z^n\Big), \\ &= y_p z^p + \sum_{n=p+1}^{\infty} y_n z^p - \sum_{n=p+1}^{\infty} y_n \frac{\mu_p+1}{\mu_n} z^n, \\ &= z^p - \sum_{n=p+1}^{\infty} \frac{(\mu_p+1)y_n}{\mu_n} z^n. \end{aligned}$$

After that, $f(z)\ =\ z^p - \sum\limits_{n=p+1}^{\infty} a_n z^n$ we see $a_n\ =\ \frac{(\mu_p+1)\,y_n}{\mu_n}, \qquad n \geqslant p+1.$

Now, we have $\sum\limits_{n=p}^{\infty} y_n\ =\ y_p + \sum\limits_{n=p+1}^{\infty} y_n\ =\ 1$ then $\sum\limits_{n=p+1}^{\infty} y_n\ =\ 1 - y_p\ \leq\ 1.$

Setting

$$\sum_{n=p+1}^{\infty} y_n \frac{\mu_p+1}{\mu_n} \times \frac{\mu_n}{\mu_p+1} = \sum_{n=p+1}^{\infty} y_n = 1 - y_p \leqslant 1.$$

It follows from Theorem 2.1 that the function $f \in \Upsilon(\alpha,\beta,\gamma,k,r,q,p)$.

Conversely, it suffices to show that

$$a_n = \frac{\mu_p+1}{\mu_n} y_n.$$

Now we have $f \in \Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ then by previous Theorem 2.2.

$$a_n \leqslant \frac{\mu_p+1}{\mu_n}, \quad n \geqslant p+1.$$

That is,

$$\frac{\mu_n a_n}{\mu_p+1} \leqslant 1,$$

but $y_n \le 1$.
Setting,

$$y_n = \frac{\mu_n a_n}{\mu_p+1}, \quad n \geqslant p+1.$$

Thus yields to the desired result and completes the theorem.

Corollary 4.2 The extreme point of the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ are the function

$$f_p(z) = z^p,$$

and

$$f_n(z) = z^p - \frac{\mu_p+1}{\mu_n} z^n, \quad n = p+1, p+2, p+3, ...,$$

where μ_n is given by (5).

Finally, in this paper we consider the radius of starlikeness and convexity.

5 Radius of Starlikeness and Convexity

The radius of starlikeness and convexity for the function in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$ will also be considered.

Theorem 5.1 If the function f given by (1) is in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$, then f is starlike of order δ $(0 \le \delta < p)$, in the disk $|z| < R$ where

$$R = \inf\left[\frac{\mu_n}{\mu_p+1} \times \left(\frac{p-\delta}{n-\delta}\right)\right]^{\frac{1}{n-p}}, \tag{18}$$

where $n = p+1, p+2, p+3, ...$, and μ_n is given by (5).

Proof. Here (18) implies

$$(\mu_p+1)(n-\delta)|z|^{n-P} \le \mu_n (p-\delta).$$

It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p - \delta,$$

for $|z| < R$, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum_{n=p+1}^{\infty}(n-p)a_n|z|^{n-p}}{1 - \sum_{n=p+1}^{\infty} a_n |z|^{n-p}}. \tag{19}$$

By aid of (9), we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leqslant \frac{\sum_{n=p+1}^{\infty}\frac{(\mu_p+1)(n-p)|z|^{n-p}}{\mu_n}}{1 - \sum_{n=p+1}^{\infty}\frac{(\mu_p+1)|z|^{n-p}}{\mu_n}}.$$

The last expression is bounded above by $p-\delta$ if.

$$\sum_{n=p+1}^{\infty}\frac{(\mu_p+1)(n-p)|z|^{n-p}}{\mu_n} \le \left[1 - \sum_{n=p+1}^{\infty}\frac{(\mu_p+1)|z|^{n-p}}{\mu_n}\right](p-\delta),$$

and it follows that

$$|z|^{n-p} \leqslant \left[\frac{\mu_n}{\mu_p+1}\left(\frac{p-\delta}{n-\delta}\right)\right], \quad n \geqslant p+1$$

which is equivalent to our condition (18) of the theorem.

Theorem 5.2 If the function f given by (1) is in the class $\Upsilon(\alpha,\beta,\gamma,k,r,q,p)$, then f is convex of order ε $(0 \le \varepsilon < p)$, in the disk $|z| < w$ where

$$w = \inf\left[\frac{\mu_n}{\mu_p+1} \times \left(\frac{p(p-\varepsilon)}{n(n-\varepsilon)}\right)\right]^{\frac{1}{n-p}}$$

where $n = p+1, p+2, p+3, ...,$ and μ_n is given by (5).

Proof. By using the same technique in the proof of Theorem 5.1, we can show that

$$\left| \frac{zf''(z)}{f'(z)} - (p-1) \right| \leq p - \varepsilon, \qquad \text{for} \qquad |z| \leq w,$$

with the aid of (9). Thus we have the assertion of Theorem 5.2.

6 Conclusion

In this article, our primary focus lies in investigating a newfound subclass of multivalent analytic functions within the open unit disc, delineated by the application of Jackson's derivative operator. Our exploration begins with a thorough investigation by uncovering the essential criteria for functions falling into this category through the lens of Coefficients' Characterization. This approach flattens many of intriguing features, with notable highlights encompassing coefficient estimates, theorems on growth and distortion, identification of extreme points, determination of the starlikeness radius, and exploration of convexity within this unique subclass. It is clear from the analysis in this article that the use of the Jackson derivative operator is not limited only to this specific subclass, but also opens up ways to derive broader classes of multivalent analytical functions. During the analysis, the importance of extending this approach to studying the description of parameters for these broader classes was emphasized, thus enriching our understanding of the diverse mathematical landscapes governed by the interaction of functions and their distinctive properties. We are going to branch out into uncharted territories, pushing the boundaries of knowledge in the fascinating realm of multivalent analytic functions.

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