On Generalised Hankel Functions and a Bifurcation of Their Asymptotic Expansion

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Abstract: The generalised Bessel differential equation has an extra parameter relative to the original Bessel equation and its asymptotic solutions are the generalised Hankel functions of two kinds distinct from the original Hankel functions. The generalised Bessel differential equation of order ν and degree μ reduces to the original Bessel differential equation of order ν for zero degree, $\mu = 0$. In both cases the differential equations have a regular singularity near the origin and the the point at infinity is the other singularity. The point at infinity is an irregular singularity of different degree, namely one for the original and two for the generalised Bessel differential equation. It follows that in the limit of degree being equal to zero the generalised Hankel functions do not converge to the original ones. The implication is that the generalised Bessel differential equation has a Hopf-type bifurcation for the asymptotic solution. In the case of a real variable and parameters the asymptotic solution is: (i) oscillatory when the degree of generalised Hankel function is zero (corresponding in this case to original Hankel functions); (ii) diverging hence unstable for the generalised Hankel functions with positive degree; (iii) decaying hence stable for the generalised Hankel functions with negative degree.

Key-Words: Generalised Bessel differential equation, Generalised Bessel functions, Generalised Hankel functions, Asymptotic solutions, Irregular singularities, Thomé normal integrals

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1 Introduction

The Bessel differential equation was first considered in connection with the oscillations of a heavy chain, [1], and vibrations of a circular membrane, [2], and, since the work by [3], has had a vast number of applications supported by an extensive theory, [4], [5]. The original Bessel differential equation appears in connection with several problems in mathematical physics and engineering, notably associated not only with cylindrical and spherical, [6], [7], [8], [9], geometries but also with hypercylindrical and hyperspherical, [10], geometries, including the propagation of acoustic, [11], [12], [13], and electromagnetic, [14], [15], [16], waves, heat diffusion, [17], [18], [19], quantum mechanics, [20], [21], [22], and also problems in solid mechanics, [23], [24], [25], such as oscillations of heavy chains and vibrations of circular elastic plates, [26], [27], [28].

Among the applications to waves in fluids, [29], [30], [31], are mentioned two: (i) sound in a cylindrical nozzle with uniform axial flow, corresponding to longitudinal compressive waves; (ii) vortical waves in a rotating flow with constant angular velocity, corresponding to transverse, incompressible modes. Although in both cases (i) and (ii) the pressure perturbation has a radial dependence specified by the original cylindrical Bessel differential equation, there are two differences: (a) the wave speed resulting from the dispersion relation between frequency and wavevector; (b) the polarisation relations relating the pressure perturbations to other wave variables such as velocity, density and temperature perturbations for adiabatic propagation, since entropy modes are excluded. The coupling of (i) and (ii) for acoustic-vortical waves in a compressible swirling flow with uniform axial velocity and constant angular velocity no longer satisfies the original Bessel differential equation, but rather the generalised Bessel differential equation considered in the present paper, due to the physical interaction between compressibility and rigid body rotation: (α) for small radius the swirl velocity is small and the acoustic-vortical waves resemble acoustic waves, leading to the usual regular singularity on axis; (β) when the radius becomes larger the swirl velocity also increases, consequently the irregular singularity of degree one at infinity for purely acoustic or vortical waves becomes of degree two due to coupling of rotation and compressibility in acoustic-vortical shear waves.

Since the singularities of the differential equation affect the solution everywhere, in the case of superposition of a uniform mean flow and a rigid body rotation, the physics of acoustic-vortical waves is closely related to the two singularities of the generalised Bessel differential equation, at the origin and infinity, that are examined more closely next. The generalised Bessel differential equation has two parameters, namely the order ν and the degree μ, and have several other applications that should be studied separately. It can generalise the Bessel, Neumann and Hankel functions changing some properties of the original functions.

The singularities of the original and generalised Bessel differential equations are only at the origin and infinity; the similarities arise because in both cases the singularity is regular at the origin, and the differences because the singularity at infinity is irregular with different degree. The solutions of the generalised Bessel differential equation around the regular singularity at the origin, [32], has: (i) indices that are exponents of the leading power depending only on the order; (ii) recurrence relation for the coefficients of the power series expansion depending not only on the order, but also on the degree. From (i) it follows the familiar situation that generalised Bessel functions specify the general integral for non-integer degree, and generalised Neumann functions are needed for integer degree. From (ii) it follows that the series expansion for the generalised Bessel and Neumann functions differ from the original series in having finite products multiplying each term; these finite products can be expressed as ratios of Gamma functions, whose arguments become singular for zero degree. However, the formulas of generalised Bessel and Neumann functions in the form of ascending power series are not well adapted for numerical computation when the independent variable is large because the series converge slowly and an observation to their initial values offer no conclusion to the convergent values of $J_\nu^\mu(z)$ and $Y_\nu^\mu(z)$, [5], [32]. Therefore, the aim of this work is to determine a formula which calculates, in an easier way, the numerical values of the solution of generalised Bessel equation when z is large and to compare the results obtained with the original Hankel functions.

Since the generalized Bessel differential equation has an extra parameter that is dominant when the degree is not zero, the asymptotic solutions are quite different from those of the original Bessel differential equation because the irregular singularity at infinity is of degree two and not one as in the original Bessel differential equation. The original Hankel functions scale asymptotically as an exponential of the variable, and are oscillating for real variable and monotonic for imaginary variable. The leading term of the asymptotic solutions of the generalised Bessel equation is monotonic (subsection 2.1) both for real and imaginary variable: (i) there is one solution without exponential factor, designated generalised Hankel function of first kind; (ii) all other solutions involve the generalised Hankel function of the second kind (subsection 2.2) that scales an an exponential of the square of the variable. To obtain a solution of the generalised Bessel differential equation, the Frobenius-Fuchs, [33], [34], method can be used to specify power series around the regular singularity at the origin and also can be used the normal integrals, [35], to specify asymptotic expansions in the neighbourhood of the irregular singularity at the infinity (subsection 2.3).

When the degree tends to zero, the generalised Hankel functions of first and second kinds do not converge to the original functions for zero degree (subsection 3.1), in contrast with the generalised Bessel and Neumann functions that do tend to their original functions. The reason is that the origin is a regular singularity both for the original and generalised Bessel differential equations, whereas the point at infinity is an irregular singularity of different degree, namely 1 for the original and 2 for the generalised Bessel differential equation (subsection 3.2). It can be shown that it is impossible to obtain an asymptotic solution of the generalised Bessel differential equation that is continuous and converges to the original Hankel function after taking the limit of the degree to zero (subsection 3.3). The degree appears as a Hopf-type bifurcation in the asymptotic solution of the generalised Bessel differential equation because the solution has different behaviour depending on the sign for real value of degree: for negative degree, the solution is decaying, for zero degree the solution is oscillatory and for positive degree the solution becomes divergent.

2 Asymptotic solutions of the generalised Bessel differential equation

The origin is a regular singularity of the generalised Bessel differential equation and consequently the solutions are ascending power series which are convergent, with some terms having logarithms for integer degree, [32]. Otherwise, the point at infinity of the generalised Bessel differential equation is an irregular singularity, hence the solution (subsection 2.3) is a linear combination of generalised Hankel functions of two kinds; they consist on descending asymptotic expansions and for generalised Hankel functions of first kind they do not have an exponential factor (subsection 2.1) while the generalised Hankel functions of second kind have an exponential factor (subsection 2.2).

2.1 Asymptotic series for the Hankel function of the first kind

The generalised Bessel differential equation is defined in order to proceed to its asymptotic expansions around the point at infinity.

Definition 1. Having the complex order $\nu \in \mathbb{C}$ and degree $\mu \in \mathbb{C}$ as parameters and with the independent variable $z \in \mathbb{C}$ also complex, the generalised Bessel differential equation is defined by

$$z^2 Q'' + z\left(1 - \frac{\mu}{2} z^2\right) Q' + \left(z^2 - \nu^2\right) Q = 0. \quad (1)$$

Remark 1. The original Bessel differential equation corresponds to zero degree with $\mu = 0$.

The equation (1) has only two singularities: one at the origin and the other at the infinity. The asymptotic solutions around the singularity at infinity are obtained next.

Definition 2. Th generalised Bessel differential equation has an asymptotic solution as an expansion around the point at infinity that corresponds to the inversion of the origin:

$$\zeta = \frac{1}{z}. \quad (2a)$$

Defining $\Phi(\zeta) \equiv Q(z)$, the differential equation (1) leads to

$$\zeta^2 \Phi'' + \zeta\left(1 + \frac{\mu}{2\zeta^2}\right) \Phi' + \left(\frac{1}{\zeta^2} - \nu^2\right) \Phi = 0. \quad (2b)$$

Remark 2. If the point $\zeta = 0$ of the equation (2b) is a regular singularity then the point at infinity $z = \infty$ of the generalised Bessel differential equation (1) is also a regular singularity. However, since the factors in curved brackets have double poles and are not analytic at $\zeta = 0$, this last point is not a regular singularity like the point $z = \infty$. Thus, two linearly independent solutions of (2b), assuming they are ascending power series of ζ or equivalently descending power series of z, cannot exist in the form

$$\Phi_\vartheta(\zeta) = \sum_{j=0}^{\infty} d_j(\vartheta) \zeta^{j+\vartheta} = \sum_{j=0}^{\infty} d_j(\vartheta) z^{-j-\vartheta} = Q(z), \quad (3)$$

where the coefficients d_j depend on ϑ. However, it is possible that (i) one power series solution (3) exists at most or (ii) maybe even none. It is demonstrated next that the former case (i) is the correct option.

Theorem 1. *The asymptotic solution of the generalised Bessel differential equation* (1) *generalises the Hankel function,*

$$H^{(1)}_{\mu,\nu}(z) \sim z^{2/\mu} \left\{ 1 + \sum_{j=1}^{N-1} \frac{\left(-\mu z^2/4\right)^{-j}}{j!} \times \prod_{l=0}^{j-1} \left[\left(l - \frac{1}{\mu}\right)^2 - \frac{\nu^2}{4} \right] + O\left(z^{-2N}\right) \right\}, \quad (4)$$

which is of the first kind, valid for non-zero degree, $\mu \neq 0$, because the descending asymptotic expansion has not an exponential factor.

Proof. The first series of (3) is substituted in (2b) or the second series of (3) in (1). Both substitutions result to the same recurrence equation for the coefficients:

$$\left[(j+\vartheta)^2 - \nu^2\right] d_j(\vartheta) + \left[\frac{\mu}{2}(j+\vartheta+2) + 1\right] d_{j+2}(\vartheta) = 0. \quad (5)$$

If $j = -2$ the indicial equation is $\vartheta = -2/\mu$ which has only one root resulting in

$$d_{2j}\left(-\frac{2}{\mu}\right) = -4 \frac{(j-1-1/\mu)^2 - (\nu/2)^2}{\mu j} d_{2j-2}\left(-\frac{2}{\mu}\right). \quad (6a)$$

Applying the relation (6a) n-times leads to

$$d_{2j}\left(-\frac{2}{\mu}\right) = \frac{(-4/\mu)^j}{j!} \prod_{l=0}^{j-1} \left[\left(l - \frac{1}{\mu}\right)^2 - \frac{\nu^2}{4} \right] \quad (6b)$$

assuming

$$d_0\left(-\frac{2}{\mu}\right) = 1. \quad (6c)$$

After substituting (6b) in (3) the first solution (4) in the neighbourhood of the irregular singularity at infinity of the generalised Bessel differential equation (1) is obtained. □

Corollary 1. *Replacing the products on the right-hand side (r.h.s.) of* (4) *by*

$$\prod_{l=0}^{j-1} \left(l - \frac{1}{\mu} \pm \frac{\nu}{2}\right) = \frac{\Gamma(j \pm \nu/2 - 1/\mu)}{\Gamma(\pm\nu/2 - 1/\mu)}, \quad (7)$$

the generalised Hankel function can also be written *as*

$$H_{\mu,\nu}^{(1)}(z) \sim z^{2/\mu} \left\{ 1 + \overline{d}_0(\mu,\nu) \sum_{j=1}^{N-1} \frac{\left(-\mu z^2/4\right)^{-j}}{j!} \times \Gamma\left(j + \frac{\nu}{2} - \frac{1}{\mu}\right) \Gamma\left(j - \frac{\nu}{2} - \frac{1}{\mu}\right) + O\left(z^{-2N}\right) \right\}, \tag{8a}$$

choosing a different leading constant factor:

$$\overline{d}_0(\mu,\nu) = \left[\Gamma\left(\frac{\nu}{2} - \frac{1}{\mu}\right) \Gamma\left(-\frac{\nu}{2} - \frac{1}{\mu}\right)\right]^{-1}. \tag{8b}$$

The solution (4), or equivalently (8a), is called the generalised Hankel function of the first kind. This function has coefficients increasing with j because $d_{j+2}/d_j \sim O(j)$ in (6a), therefore the expansion is not convergent as a series when $j \to \infty$; nonetheless, it is an asymptotic expansion that can be evaluated with a finite number of terms, $j < \infty$, as $|z| \to \infty$. To guarantee the convergence of the Frobenius series, [36], the Fuchs theorem, [37], cannot be applied around an irregular singularity because the theorem in this case leads to an asymptotic expansion and not a convergent series. When $|z| \to \infty$, the generalised Hankel function of the first kind (4) decays for $\Re(\mu) < 0$ and diverges for $\Re(\mu) > 0$. The terms $-\mu z^2/4$ exist in these Hankel functions (4); for real z the term is negative and so the factor $\left(-\mu z^2/4\right)^{-j}$ has alternating sign, otherwise for pure imaginary z the term is positive and consequently the same factor is positive for any j.

The Fig. 1 shows the effect of increasing the number of terms in the series expansion of the generalised Hankel function of the first kind and, for values greater than $N = 10$, the new terms of summation don't lead to noticeable difference because the term $(\mu z^2/4)^{-j}/j!$ decreases for greater values of j. Therefore, $N = 15$ is established for all other plots. The Fig. 1 also shows, for $N = 15$, the differences induced by varying the value of ν. For all values of ν and μ, the Hankel function "explodes" at $z = 0$ (this effect is visible in the plot for $\nu = 15$) and for that reason the plots don't show the function for values near the origin (also, the objective is to study the behavior of functions for large values through asymptotic expansions). The greater the value of the parameter ν, the greater the value of generalised Hankel function is, for all values of z. In the Fig. 1, all the plots diverge because μ is a real positive number and the greater the value of ν the more quickly the function diverges. The Hankel function is even with respect to

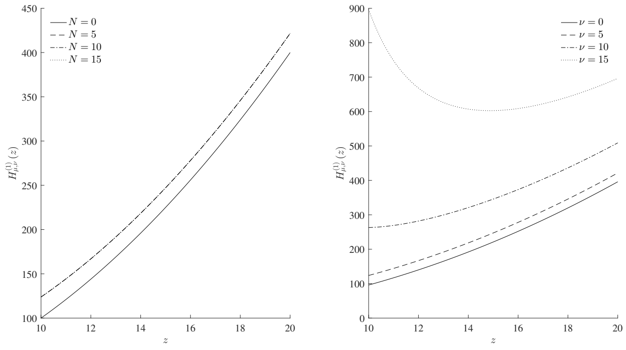

Fig. 1: Generalised Hankel function of the first kind, setting $\mu = 1$ and $\nu = 5$, for different values of the number of terms in the series expansion (left), or setting $\mu = 1$ and $N = 15$, for different values of ν (right).

z and that is the reason why there is no need to illustrate the function for negative values of z (the plots are symmetric regarding the vertical axis) and is also even with respect to ν, that is, the plots are exactly the same for symmetric values of ν (for instance, the plots obtained with $\nu = 5$ and $\nu = -5$ are identical). All the parameters and values of the function are real values in the Fig. 1 to not appear any complex numbers and to illustrate the function in a single 2D-plot to make easier the observations, and because the parameters are usually real in the differential equations deduced from the physics subjects.

The Fig. 2 shows the strong dependence of the values of the generalised Hankel function of the first kind on the parameter μ. The generalised Hankel function is not valid for $\mu = 0$. It diverges for real positive values of μ, but decays for negative real values and the step of increasing or decreasing is higher for lower values of μ. As previously in the Fig. 1, to facilitate the comparison of plots resulting from different conditions, all the parameters and the variables are real numbers to not plot complex numbers. Again, the plots show that the function always diverges at $z = 0$ for all values of μ. Lastly, the Hankel function is even with respect to z so the Fig. 2 only illustrates the function for positive values of z.

2.2 Asymptotic expansion for the generalised Hankel function of the second kind

The factor $\exp\left[A(z)\right]$, where $A(z)$ is a polynomial of z, has an essential singularity at infinity, [38], [39], the asymptotic scaling of the original Hankel functions, [40], is

$$H_\nu^{(1,2)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left[\pm \mathrm{i}\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right]. \tag{9}$$

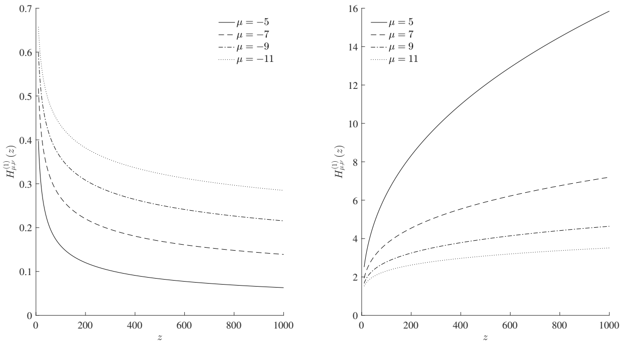

Fig. 2: Generalised Hankel function of the first kind, setting $N = 15$ and $\nu = 1$, for different negative values of the parameter μ (left) or positive values of the parameter μ (right).

When $\zeta \to 0$ the coefficient of Φ' in (2b) is analytic if $\mu = 0$, but has a double pole if $\mu \neq 0$. Consequently, in spite of the original Hankel function corresponding to the generalised Hankel function with zero degree, when the degree tends to zero $\mu \to 0$ the limit is not continuous. The degree of the irregular singularity at infinity in the generalised Bessel differential equation (1) is two, as will be proved next, and therefore higher than the degree of the singularity at infinity in the original Bessel differential equation which is one. Consequently the limit when $\mu \to 0$ is descontinouous (section 3).

The generalised Bessel differential equation (1) has an asymptotic solution which is linearly independent of the generalised Hankel function of first kind (4). The reason is that the asymptotic solution cannot be written as in (3) because: (i) the corresponding indicial equation, $\vartheta = -2/\mu$, has only one solution; (ii) if there is such a solution then the point at infinity would be a regular singularity instead of an irregular singularity. Nonetheless, an essential singularity must be present in the asymptotic solution of the generalised Bessel differential equation (1) linearly independent from the generalised Hankel function of first kind. The essential singularity can be written in the form of a normal integral, [41].

$$\Phi(\zeta) = \exp\left[A\left(\frac{1}{\zeta}\right)\right]\Psi(\zeta). \tag{10a}$$

Ihe degree of the polynomial $A(z)$ determines the degree of the essential singularity at infinity, provided that the remaining factor Ψ is a Frobenius series or at least an asymptotic expansion:

$$\Psi_\chi(\zeta) = \sum_{j=0}^{\infty} e_j(\chi)\zeta^{j+\chi}. \tag{10b}$$

The coefficients e_j depend on χ. This leads to the theorem 2.

Theorem 2. *An asymptotic solution of the generalised Bessel differential equation* (1) *is the generalised Hankel function of the second kind*

$$H^{(2)}_{\mu,\nu}(z) \sim \exp\left(\frac{\mu z^2}{4}\right) z^{-2-2/\mu} \times \left\{1 + \sum_{j=1}^{N-1} \frac{\left(\mu z^2/4\right)^{-j}}{j!} \times \prod_{l=1}^{j}\left[\left(l+\frac{1}{\mu}\right)^2 - \frac{\nu^2}{4}\right] + O\left(z^{-2N}\right)\right\} \tag{11}$$

consisting of a descending asymptotic series multiplied by an exponential term. This series solution exists only for non-zero degree, $\mu \neq 0$.

Proof. The asymptotic normal integral (10a) is substituted in (2b) leading to

$$\zeta^2\Psi'' + \zeta\left(1 + \frac{\mu}{2\zeta^2} + 2A'\zeta\right)\Psi' + \left[\zeta^2 A'^2 + \zeta^2 A'' + \zeta A'\left(1+\frac{\mu}{2\zeta^2}\right) + \frac{1}{\zeta^2} - \nu^2\right]\Psi = 0. \tag{12}$$

to obtain a second solution of the generalised Bessel differential equation (1) that, unlike the first (4), is a normal integral, and hence linearly independent. The polynomial A' in (12) should be chosen such that the solution for Ψ in the form (10b) exists. The simplest choice is an inverse power $A' \sim \zeta^{-n}$. For $n = 1$, the relation $A' \sim \zeta^{-1}$ leads to $A \sim \vartheta \ln \zeta$ which is equal to $e^A \sim \zeta^\vartheta$ being this the Frobenius solution (3). For $n = 2$, the relation $A' \sim \zeta^{-2}$ leads to $A \sim \zeta^{-1}$ and $\mathrm{e}^A \sim \exp(1/\zeta) \sim \exp z$. This case corresponds to zero degree $\mu = 0$ and leads to the asymptotic scaling of the original Hankel function (9) and therefore a solution of the original Bessel differential equation that has an irregular singularity at infinity of degree one. It follows that the generalised Bessel differential equation with non-zero degree has a stronger irregular singularity at infinity and hence the lowest degree must be two leading to $A \sim \zeta^{-2}$ or $A' \sim \zeta^{-3}$. To confirm the degree of the irregular singularity at infinity of the generalised Bessel differential equation, the equality

$$A'\left(\frac{1}{\zeta}\right) = \frac{b}{\zeta^3} \tag{13a}$$

is substituted in (12) leading to the differential equation (13b)

$$\zeta^2\Psi'' + \zeta\left(1 + \frac{2b+\mu/2}{\zeta^2}\right)\Psi' + \left[\frac{b(b+\mu/2)}{\zeta^4} + \frac{1-2b}{\zeta^2} - \nu^2\right]\Psi = 0. \tag{13b}$$

The fourth-order pole in the square bracketed term can be eliminated by choosing

$$b = -\frac{\mu}{2} \tag{14a}$$

and consequently the differential equation (13b) is simplified to

$$\zeta^2\Psi'' + \zeta\left(1 - \frac{\mu}{2\zeta^2}\right)\Psi' + \left(\frac{1+\mu}{\zeta^2} - \nu^2\right)\Psi = 0. \tag{14b}$$

It is impossible to have two linearly independent solutions of the form (10b) because the point at infinity is still an irregular singularity of (14b). Nonetheless, knowing already one asymptotic solution (4) of the generalized Bessel differential equation (1), only one more solution is needed.

Substituting of (10b) in (14b) results in the recurrence formula

$$\left[(j+\chi)\frac{\mu}{2} - 1\right]e_{j+2}(\chi) = \left[(j+\chi)^2 - \nu^2\right]e_j(\chi). \tag{15}$$

When $j = -2$, the indicial equation is

$$\left[\mu\left(\frac{\chi}{2} - 1\right) - 1\right]e_0(\chi) = 0, \tag{16a}$$

and noting that $e_0(\chi) \neq 0$, it has one root:

$$\chi = 2 + \frac{2}{\mu}. \tag{16b}$$

Assuming that

$$e_0\left(2 + \frac{2}{\mu}\right) = 1 \tag{17a}$$

and substituting (16a) in (15), the coefficients of the asymptotic expansion, needed for the equation (10b), are specified:

$$e_{2j}\left(2+\frac{2}{\mu}\right) = \frac{(2j+2/\mu)^2 - \nu^2}{\mu j} e_{2j-2}\left(2+\frac{2}{\mu}\right) = \frac{(\mu/4)^{-j}}{j!}\prod_{l=1}^{j}\left[\left(j+\frac{1}{\mu}\right)^2 - \frac{\nu^2}{4}\right]. \tag{17b}$$

The equation above is substituted first in the Frobenius series (10b) and after in the asymptotic solution (10a) that is multiplied by an exponential term. Regarding (13a) and (14a), the argument of the exponential satisfies

$$A' = -\frac{\mu}{2\zeta^3}, \tag{18a}$$

implying

$$A = \frac{\mu}{4\zeta^2} \tag{18b}$$

and consequently

$$\exp\left[A\left(\frac{1}{\zeta}\right)\right] = \exp\left(\frac{\mu}{4\zeta^2}\right) = \exp\left(\frac{\mu z^2}{4}\right). \tag{18c}$$

Substituting (17b) and (18c) in (10b) and (10a) respectively leads to the second asymptotic solution of the generalised Bessel differential equation that is called the generalised Hankel function of the second kind (11). □

Corollary 2. *If in the last term on the r.h.s. of* (11) *a pair of relations similar to* (7) *is used, but with* $-1/\mu$ *replaced by* $1/\mu$*, an alternate expression for the generalised Hankel function of second kind is obtained, specifically*

$$H^{(2)}_{\mu,\nu} \sim \exp\left(\frac{\mu z^2}{4}\right) z^{-2-2/\mu}\left\{1 + \overline{e}_0(\mu,\nu) \times \sum_{j=1}^{N-1}\frac{(\mu z^2/4)^{-j}}{j!}\Gamma\left(j+1+\frac{\nu}{2}+\frac{1}{\mu}\right) \times \Gamma\left(j+1-\frac{\nu}{2}+\frac{1}{\mu}\right) + O\left(z^{-2N}\right)\right\}. \tag{19a}$$

The leading constant factor $\overline{e}_0$ *is replaced by*

$$\overline{e}_0(\mu,\nu) = \left[\Gamma\left(1+\frac{\nu}{2}+\frac{1}{\mu}\right)\Gamma\left(1-\frac{\nu}{2}+\frac{1}{\mu}\right)\right]^{-1}. \tag{19b}$$

The Fig. 3 illustrates the generalised Hankel function of the second kind, setting again $N = 15$ and $\mu = 1$, and leading to the same conclusions as in the Fig. 1. However, in opposition to the example of Fig. 1, the values of the function increase more quickly for lower values of ν. But the main difference is that, for the same value of z, the function of the second kind reaches greater values than the function of the first kind, due to the exponential function that is present only in the function of the second kind.

The Fig. 4 illustrates the generalised Hankel function of the second kind, setting $N = 15$ and $\nu = 1$,

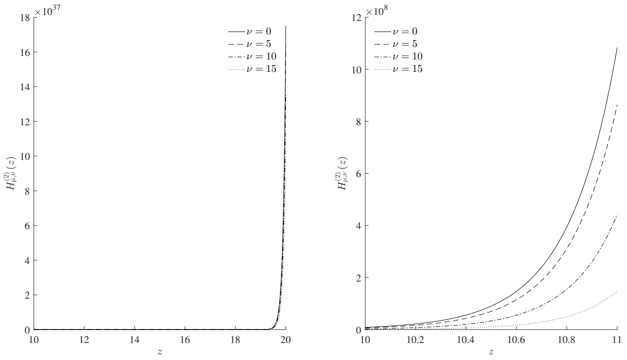

Fig. 3: Generalised Hankel function of the second kind, setting $\mu = 1$ and $N = 15$, for different values of ν, plotted in the range $10 \leq z \leq 20$ (left) or in the range $10 \leq z \leq 11$ (right).

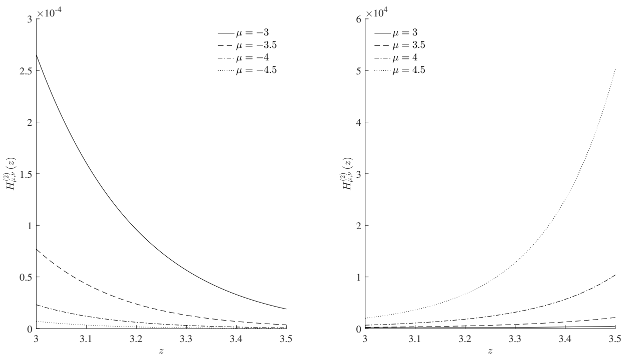

Fig. 4: Generalised Hankel function of the second kind, setting $N = 15$ and $\nu = 1$, for different negative values of the parameter μ (left) or positive values of the parameter μ (right).

plotted for different values of μ. The observations are the same as the Fig. 2, however there is two big differences: for the same values of the parameters and z, the function of the second kind reaches greater values than the function of the first kind, due to the presence of an exponential function in the Fig. 4; the parameter μ appears in the exponential term of the Hankel function of the second kind and consequently the values of that function are strongly dependent of the parameter μ, more than in case of the function of the first kind. As it can be seen from the Fig. 4, for $z = 3.5$, changing the value of μ only 0.5 units (for instance, from 3.5 to 4) can lead to the difference in the Hankel function of 40000 units.

2.3 Wronskian for linearly independent asymptotic solutions and general integral

It is shown next that the Wronskian between the generalised Hankel functions of first (4) and second (11) kinds is non-zero if the degree is also non-zero. In that case, both asymptotic solutions of the generalised Bessel differential equation (1) are linearly independent and their linear combination determines the general integral. A preliminary lemma on the Wronskian of any two solutions of the Bessel differential equation, [32], is used in a subsequent lemma to specify the Wronskian of the generalised Hankel functions of two kinds.

Lemma 3. *Two particular solutions* (Q_1) *and* (Q_2) *of the generalised Bessel differential equation* (1) *have the Wronskian*

$$W\left[Q_1(z), Q_2(z)\right] \equiv Q_1(z)Q_2'(z) - Q_1'(z)Q_2(z) = \frac{W(0)}{z}\exp\left(\frac{1}{4}\mu z^2\right). \qquad (20)$$

Lemma 4. *The Wrosnkian of the generalised Hankel functions of first* (4) *and second* (11) *kinds, for non-zero degree,* $\mu \neq 0$*, has the asymptotic expansion equal to*

$$W\left[H^{(1)}_{\mu,\nu}(z), H^{(2)}_{\mu,\nu}(z)\right] \sim \frac{\mu}{2z}\exp\left(\frac{1}{4}\mu z^2\right). \qquad (21)$$

The two functions are linearly independent because the Wronskian is non-zero.

Proof. The asymptotic solutions (4) and (11) have the leading terms

$$H^{(1)}_{\mu,\nu}(z) \sim z^{2/\mu} \qquad (22a)$$

and

$$H^{(2)}_{\mu,\nu}(z) \sim z^{-2-2/\mu}\exp\left(\frac{1}{4}\mu z^2\right), \qquad (22b)$$

leading to the Wronskian (21) that is equal to (20) if

$$W(0) = \frac{\mu}{2}. \qquad (22c)$$

□

From this follows immediately the next theorem.

Theorem 5. *The general asymptotic solution* (2b) *of the generalised Bessel differential equation* (1) *is a linear combination with arbitrary constants* (C_1) *and* (C_2) *of the generalised Hankel functions of first* (4) *and second* (11) *kinds,*

$$Q(z) \sim C_1 H^{(1)}_{\mu,\nu}(z) + C_2 H^{(2)}_{\mu,\nu}(z). \qquad (23)$$

This linear combination is valid for any value of degree, including zero degree corresponding to original Hankel functions.

The general solution of the generalised Bessel differential equation (1) in the neighbourhood of the regular singularity at the origin for non-integer order involves generalised Bessel functions while for integer order the Neumann functions are present in the solution, [32]. In both cases, the solutions hold for finite z. Hence, both solutions overlap with the generalised Hankel function that also holds for non-zero degree μ and large variable z. Consequently, it follows the theorem 6.

Theorem 6. *The asymptotic expansion of the generalised Bessel functions with non-zero degree, $\mu \neq 0$, has the form*

$$J_{\nu}^{\mu}(z) = D_{+}(\mu,\nu) H_{\mu,\nu}^{(1)}(z) + D_{-}(\mu,\nu) H_{\mu,\nu}^{(2)}(z). \tag{24}$$

Furthermore the coefficients $D_{+}(\mu,\nu)$ and $D_{-}(\mu,\nu)$ are functions of the degree μ and order ν satisfying the relations

$$-\frac{4\sin(\pi\nu)}{\pi\mu} = D_{+}(\mu,\nu) D_{-}(\mu,-\nu) - D_{+}(\mu,-\nu) D_{-}(\mu,\nu). \tag{25}$$

Proof. The general integral (23) for the generalised Bessel function applies to any order and non-zero degree. Hence the generalised Bessel function must be expressible in the form (24) with coefficients $D_{+}(\mu,\nu)$ and $D_{-}(\mu,\nu)$ determined by comparing the solutions around the origin for $|z| < \infty$ and around the point at infinity for $|z| > 0$ in the region of overlap $0 < |z| < \infty$. In the situation where $mu = 0$ corresponding to the original Bessel functions, the asymptotic expansion is typically found using an integral representation or using the method of Wronskians, [42], from the asymptotic solution of the original Bessel differential equation. The asymptotic solutions (4) and (11) of the generalised Hankel functions are the same when changing the sign of the order nu,

$$H_{\mu,-\nu}^{(1,2)}(z) = H_{\mu,\nu}^{(1,2)}(z), \tag{26a}$$

and therefore

$$J_{-\nu}^{\mu}(z) = D_{+}(\mu,-\nu) H_{\mu,\nu}^{(1)}(z) + D_{-}(\mu,-\nu) H_{\mu,\nu}^{(2)}(z). \tag{26b}$$

Using (24) and (26a) the Wronskian of the generalised Bessel functions of orders $\pm\nu$ is related to the Wronskian of the Hankel functions of two kinds and order ν, with all four functions having the same degree μ:

$$W\left[J_{+\nu}^{\mu}(z), J_{-\nu}^{\mu}(z)\right] = \left[D_{+}(\mu,\nu) D_{-}(\mu,-\nu) - D_{+}(\mu,-\nu) D_{-}(\mu,\nu)\right] \times W\left[H_{\mu,\nu}^{(1)}(z), H_{\mu,\nu}^{(2)}(z)\right]; \tag{27}$$

using the Wronskian of the generalised Bessel functions (lemma 2 in [32])

$$W\left[J_{+\nu}^{\mu}(z), J_{-\nu}^{\mu}(z)\right] = -\frac{2}{\pi z}\sin(\nu\pi)\exp\left(\frac{1}{4}\mu z^{2}\right) \tag{28}$$

and of the generalised Hankel functions of first and second kinds (21) in (27) demonstrates (25). □

Thus for non-integer order the coefficients relating (24) the initial and asymptotic solutions of the generalised Bessel equation must satisfy the relation (25). Therefore, the method of Wronskian leads to only one relation between the terms $D_{+}(\mu,\nu)$ and $D_{-}(\mu,\nu)$. The determination of both coefficients uses instead the relation between generalised Bessel and confluent hypergeometric functions presented in a follow-on paper.

The general solution of the generalised Bessel differential equation (1) for integer order includes generalised Bessel functions

$$J_{n}^{\mu}(z) = D_{+}(\mu,n) H_{\mu,n}^{(1)}(z) + D_{-}(\mu,n) H_{\mu,n}^{(2)}(z) \tag{29a}$$

and generalised Neumann functions

$$Y_{n}^{\mu}(z) = E_{+}(\mu,n) H_{\mu,n}^{(1)}(z) + E_{-}(\mu,n) H_{\mu,n}^{(2)}(z), \tag{29b}$$

where the coefficients $E_{+}(\mu,n)$ and $E_{-}(\mu,n)$ are functions of the complex degree μ and integer order n, and whose asymptotic forms are similar to (24) because the latter holds for all orders, leading therefore to the next theorem.

Theorem 7. *For the same integer order $\nu = n$ and complex non-zero degree $\mu \neq 0$, the generalised Hankel function of the first* (4) *and second* (11) *kinds are linear combinations of generalised Bessel* (29a) *and Neumann* (29b) *functions with coefficients related by:*

$$D_{+}(\mu,n) E_{-}(\mu,n) - D_{-}(\mu,n) E_{+}(\mu,n) = \frac{4C}{\pi\mu} = 4\frac{(-1)^{n}}{\pi}\mu^{-1-n}\frac{\Gamma(-n/2-1/\mu)}{\Gamma(n/2-1/\mu)}. \tag{30}$$

Proof. From (29a) and (29b) follows the relation between the Wronskians:

$$W\left[J_{n}^{\mu}(z), Y_{n}^{\mu}(z)\right] = \left[D_{+}(\mu,n) E_{-}(\mu,n) - E_{+}(\mu,n) D_{-}(\mu,n)\right] \times W\left[H_{\mu,n}^{(1)}(z), H_{\mu,n}^{(2)}(z)\right]. \tag{31}$$

The generalised Bessel and Neumann functions of arbitrary degree and integer order (lemma 3 in [32]) have the Wrosnkian

$$W\left[J_n^\mu(z), Y_n^\mu(z)\right] = \frac{2}{\pi z} C(\mu) \exp\left(\frac{1}{4}\mu z^2\right) \quad (32a)$$

involving the coefficient

$$\begin{aligned}\frac{1}{C(\mu)} &\equiv \prod_{l=0}^{n-1} \left[1 - \mu\left(l - \frac{n}{2}\right)\right] \\ &= (-\mu)^n \prod_{l=0}^{n-1} \left(l - \frac{n}{2} - \frac{1}{\mu}\right) \\ &= (-\mu)^n \frac{\Gamma(n/2 - 1/\mu)}{\Gamma(-n/2 - 1/\mu)}. \quad (32b)\end{aligned}$$

Replacing (32b) and (21), valid for $\mu \neq 0$, in (31) proves (30). □

The method of Wronskian leads to one relation between the four coefficients - $D_+(\mu, n)$, $D_-(\mu, n)$, $E_+(\mu, n)$ and $E_-(\mu, n)$ - whereas the use of confluent hypergeometric functions leads to explicit functions of $E_+(\mu, n)$ and $E_-(\mu, n)$ as shown in a subsequent paper.

3 Bifurcation of the asymptotic solutions around the point at infinity for zero degree

The reason the asymptotic solutions of the generalised Bessel equation and the original Bessel equation stop working as the degree approaches to zero is because the essential singularity at infinity changes from degree two to degree one. It will be shown in two other ways that a solution that is continuous in μ near the origin cannot exist (subsection 3.2). This is like a Hopf-type bifurcation because the asymptotic solution changes its behaviour depending on the sign of degree: if the degree is negative, the solution decays, if the real degree is zero the solution is oscillatory and finally if the degree is positive the solution diverges. (See section 3. 3)

3.1 Discontinuity in the degree between the original and generalised Hankel functions

The singularity at the origin of the generalised Bessel differential equation (1) is regular regardless the value of degree μ. The indices are determined only by the order ν while the degree μ appears only in the coefficients of the Frobenius-Fuchs series expansion. As a consequence, when the degree approaches the value zero, the generalised Bessel (33a) and Neumann (33b) functions tend respectively to the original Bessel and Neumann functions:

$$\lim_{\mu \to 0} J_\nu^\mu(z) = J_\nu(z); \quad (33a)$$

$$\lim_{\mu \to 0} Y_n^\mu(z) = Y_n(z). \quad (33b)$$

The situation is different when considering the asymptotic solutions near the irregular singularity at infinity: (i) the generalised Hankel function of first (4) and second (11) kinds become invalid when the degree is zero; (ii) the original Hankel functions of two kinds (9) have an essential singularity of degree one, whereas the degree of the singularity in the generalised Hankel function of second kind (11) is two while the generalised Hankel function of first kind (4) does not have an essential singularity. So it is not possible to satisfy the continuity in the parameter μ:

$$\lim_{\mu \to 0} H_{\mu,\nu}^{(1,2)}(z) \neq H_\nu^{(1,2)}(z). \quad (34)$$

This could be predicted from the generalised Bessel differential equation (1) since the degree μ appears only in the factor $\mu z^3 Q'$: (i) when $z \to 0$, this term tends to zero and therefore preserves the continuity of the ascending power series specifying the generalised and original Bessel functions (33a), and likewise for the generalised and original Neumann functions (33b), that involve a similar logarithmic term; (ii) as $z \to \infty$, the term $\mu z^3 Q' = 0$ vanishes for $\mu = 0$, but diverges for $\mu \neq 0$ because $\mu z^3 Q' \to \infty$, and thus the original and generalised Hankel functions are discontinuous, as stated in (34), and are represented by asymptotic expansions in descending powers of z.

The solutions of the generalised and original Bessel differential equations are: (i) continuous with regard to the degree near the origin as $z \to 0$ because μ does not appear to leading order in the ascending power series solutions near the regular singularity at the origin; (ii) discontinuous with regard to the degree asymptotically at infinity as $z \to \infty$ because μ appears in the leading term of the descending power asymptotic expansions and normal integrals near the irregular singularity at infinity that has degree 1 for $\mu = 0$ and degree 2 for $\mu \neq 0$. Concerning the ascending power series solutions around the regular singularity at the origin only the Neumann but not the Bessel function has a logarithmic term. Concerning the descending power asymptotic expansions near the irregular singularity at the point at infinity, there is an essential singularity associated with the exponential factor in the normal integral in all cases, including the generalised Hankel functions of the second kind, with the exception being only the generalised Hankel function of the first kind, in which no such factor appears.

This is not the case for the original Hankel functions of two kinds, which are asymptotic solutions of the original Bessel differential equation, specified by a normal integral with essential singularity of degree unity, leading to oscillatory solutions for real variable. In contrast, the generalised Bessel equation with non-zero degree has an asymptotic solution (23) with an essential singularity of degree two (18c), leading to a monotonic response for real μ and z. The irregular singularities of the original (equation (9)) and generalised (equations (4) and (11)) Bessel differential equations are of different degree, specifically one and two for the original and generalised Bessel equations, and consequently the two differential equations have distinct asymptotic expansions. This shows that the solution of an ordinary differential equation around a singularity may be discontinuous with regard to a parameter, in this example the degree of the generalised Bessel differential equation.

The main difference between the original, $\mu = 0$, and generalised, $\mu \neq 0$, Bessel equations is the irregular singularity at infinity. In the original equation, the singularity has degree one, while in the generalised equation, it has degree two. TThis change is shown in the way the original Hankel functions (9) scale asymptotically compared to the generalised Hankel functions of second kind (11). The question could be raised if there is one or two asymptotic solutions of the generalised Bessel equation (9) such that: (i) are valid for all values of the degree μ; (ii) for non-zero values of the degree, $\mu \neq 0$, the asymptotic solution scales as in (18c); (iii) for zero degree, $\mu = 0$, when the asymptotic factor (18c) is equal to one, the asymptotic factor is given by $\mathrm{e}^{\pm \mathrm{i} z}$ as for the original Hankel functions (9). It will be proved in more than one way that it is impossible to meet all the conditions (i) to (iii). Consequently, there is no solution in the previous form.

The original and generalised Hankel functions are distinct and they demonstrate a discontinuity with regard to the degree μ of the same differential equation; the irregular singularity at infinity is of different degree – two for non-zero degree and one for zero degree in the generalised Bessel differential equation (1) – leading to the discontinuity of the asymptotic solution. The common factor (18c) that appears in all Wronskians – specifically in the equations (21), (28) and (32a) – between all pairs of linearly independent solutions leads to asymptotic expansions distinct from those of the original Bessel equation (9). It will be proved next that it is impossible to have one or two asymptotic solutions of the generalised Bessel differential equation that become the original Hankel functions when the degree tends to zero. Two independent ways (subsections 3.2 and 3.3) will be used to prove the preceding statement.

3.2 Non-existence of an asymptotic normal integral continuous for zero degree

Theorem 8. *The generalised Bessel differential equation* (1) *with non-zero degree does not have an asymptotic solution in the form of a normal integral* (10a) *continuous with respect to the degree μ as it approaches to zero.*

Proof. Starting with the generalised Bessel differential equation (1), to seek a solution as a first normal integral (10a) with leading term (18c) leads to the differential equation (14b). When the degree is zero, $\mu = 0$, the leading term (18c) is equal to one, and it possible that the normal integral for the differential equation (14b) would result in the original Hankel functions (9). To find out if this is possible, a second normal integral is assumed to be part of the solution of the differential equation (14b),

$$\Psi(\zeta) = \mathrm{e}^{B(1/\zeta)}\Theta(\zeta), \tag{35a}$$

so that multiplied with the first normal integral, stated in (13a), the asymptotic solution of the generalised Bessel differential equation is equal to

$$\Phi(\zeta) = \mathrm{e}^{A(1/\zeta)}\mathrm{e}^{B(1/\zeta)}\Theta(\zeta) \tag{35b}$$

where the function Θ must satisfy the following differential equation

$$\begin{aligned}\zeta^2\Theta'' &+ \zeta\left(1 - \frac{\mu}{2\zeta^2} + 2B'\zeta\right)\Theta' \\ &+ \left[\zeta^2\left(B'^2 + B''\right) + \left(\zeta - \frac{\mu}{2\zeta}\right)B' \right.\\ &\left. + \frac{1+\mu}{\zeta^2} - \nu^2\right]\Theta = 0,\end{aligned} \tag{36}$$

obtained substituting (35a) in (14b).

The first normal integral was found with a triple pole for A', as in (13a); an asymptotic expansion without essential singularity can be obtained with a simple pole for B'. For that reason B' should have a double pole,

$$B' = \frac{g}{\zeta^2}, \tag{37a}$$

leading by substitution in (36) to the differential equation

$$\begin{aligned}\zeta^2\Theta'' &+ \zeta\left(1 - \frac{\mu}{2\zeta^2} + \frac{2g}{\zeta}\right)\Theta' \\ &+ \left[-\frac{\mu g}{2\zeta^3} + \frac{1+\mu+g^2}{\zeta^2} - \frac{g}{\zeta} - \nu^2\right]\Theta = 0.\end{aligned} \tag{37b}$$

The term in square brackets has a triple pole, that could be suppressed by setting $g = 0$ resulting in a trivial differential equation $B' = 0$ in (37a). This may demonstrate that trying to find the normal integral might be impossible.

The arbitrary constant g in (37a) can be specified to suppress, in the differential equation (37b), the double pole in square brackets in the coefficient of Θ, by setting

$$g^2 + 1 + \mu = 0; \tag{38a}$$

the roots

$$g = \mp \mathrm{i}\sqrt{1+\mu} \tag{38b}$$

lead, using the equation (37a), to

$$B = -\frac{g}{\zeta} = \pm \mathrm{i}\frac{\sqrt{1+\mu}}{\zeta}. \tag{38c}$$

Using (18b) and (38c) in (37b) leads to

$$\begin{aligned}\Phi(\zeta) &= \exp\left(\frac{\mu}{4\zeta^2} \pm \mathrm{i}\frac{\sqrt{1+\mu}}{\zeta}\right)\Theta(\zeta) \\ &= \exp\left(\frac{1}{4}\mu z^2\right)\exp\left(\pm \mathrm{i}z\sqrt{1+\mu}\right)\Theta\left(\frac{1}{z}\right)\end{aligned} \tag{39}$$

where: (i) the dominant term is (18c) as for the generalised Hankel function of second kind (11) with nonzero degree, $\mu \neq 0$; (ii) the next leading term reduces to $\exp(\pm \mathrm{i}z)$ for zero degree, $\mu = 0$ (in this case, the leading term is unity), and actually coincides with the asymptotic scaling of the original Hankel functions (9); (iii) the function $\Theta(\zeta)$ must be an asymptotic expansion in Frobenius form,

$$\Theta_\phi(\zeta) \sim \sum_{j=0}^{\infty} f_j(\phi)\zeta^{\phi+j} = \sum_{j=0}^{\infty} f_j(\phi) z^{-\phi-j}, \tag{40}$$

so that the function Φ in (39) represents the two asymptotic solutions of the generalised Bessel differential equation. To make this possible, only one index ϕ must exist; since $\zeta = 0$ is an irregular singularity of the differential equation (37b), there cannot be two indices.

Substituting (38b), the differential equation (37b) becomes

$$\begin{aligned}&\zeta^2\Theta'' + \zeta\left(1 - \frac{\mu}{2\zeta^2} \mp \frac{2\mathrm{i}\sqrt{1+\mu}}{\zeta}\right)\Theta' \\ &\quad - \left[\nu^2 \mp \frac{\mathrm{i}}{\zeta}\left(1 + \frac{\mu}{2\zeta^2}\right)\sqrt{1+\mu}\right]\Theta = 0.\end{aligned} \tag{41}$$

To obtain the coefficients, the substitution of (40) in the equation (41) leads to the following recurrence equation:

$$\begin{aligned}&\pm \mathrm{i}\frac{\mu}{2}\sqrt{1+\mu} f_{j+3}(\phi) - \frac{\mu}{2}(\phi + j + 2) f_{j+2}(\phi) \\ &\mp \mathrm{i}\sqrt{1+\mu}(2j + 2\phi + 1) f_{j+1}(\phi) \\ &+ \left[(\phi + j)^2 - \nu^2\right] f_j(\phi) = 0.\end{aligned} \tag{42}$$

Setting $j = -3$ leads to

$$\mu\sqrt{1+\mu} f_0(\phi) = 0, \tag{43a}$$

and from $f_0(\phi) = 0$ follows $f_j(\phi) = 0$, showing that there is no index ϕ and only a trivial solution exists:

$$\Theta(\zeta) = 0. \tag{43b}$$

The reason for the existence of only a trivial solution (43b) is that the indicial equation has no roots, because it is specified by the triple pole in the coefficient of Θ in the differential equation (37b). Consequently, it has been shown that it is impossible to find an asymptotic solution of the generalised Bessel differential equation (1) when the degree μ is not zero in such a way that provides also an asymptotic solution of the original Bessel differential equation after setting the limit $\mu \to 0$. □

3.3 Alternative proof of discontinuity of Hankel functions for zero degree

It is possible to use the method of "reductio ad absurdum" to demonstrate in an independent and simple way that the generalised Bessel differential equation has no solution of the form (39).

Theorem 9. *There is no asymptotic solution of the form* (35a) *and* (35b) *involving a descending asymptotic expansion* (40) *which satisfies the generalised Bessel differential equation* (1).

Proof. The proof is made by "reductio ad absurdum". Let's assume that there is a solution like (39) with Θ being a descending asymptotic expansion as in (40). The solution (39) has to be a linear combination of generalised Hankel functions of first (4) and second (11) kinds:

$$\begin{aligned}&\exp\left(\frac{1}{4}\mu z^2 \pm \mathrm{i}z\sqrt{1+\mu}\right)\Theta\left(\frac{1}{z}\right) \\ &= G_+ H^{(1)}_{\mu,\nu}(z) + G_- H^{(2)}_{\mu,\nu}(z).\end{aligned} \tag{44}$$

The equation (44) shows that the r.h.s. only has even powers of z while the left-hand side (l.h.s.) has both even and odd powers of z. The even powers in the r.h.s. come from the equations (4) and (11); the even and odd powers in the other side come from the exponential term. For that reason, the equality is impossible. □

Remark 3. If the exponential argument on the l.h.s. did not have the second term, then both sides of the equation would have only even powers of z and the equality in (44) would be possible. However, that case does not include the original Hankel functions.

To conclude: (i) the simplest asymptotic solution of the generalised Bessel differential equation (1) is the generalised Hankel function of first kind (4), which is the only solution that has no essential singularity; (ii) any other solution has to include the exponential factor (18c) leading to an essential singularity, as in the case of the generalised Hankel function of second kind (11); (iii) the general solution is therefore a linear combination of the preceding two and thus cannot be of the form

$$\Phi\left(\frac{1}{z}\right) \neq \exp\left(\frac{1}{4}\mu z^2\right) \exp\left[\pm \mathrm{i} z h(\mu)\right] \Theta\left(\frac{1}{z}\right), \tag{45}$$

that would be continuous with the original Hankel functions; (iv) the discontinuity between the generalised and original Hankel functions, stated in (34), is associated with different asymptotic solution of the original and generalised Bessel differential equation that has at infinity an irregular singularity of different degree, respectively 1 for $\mu = 0$ in the former case and 2 for $\mu \neq 0$ in the latter case.

A very simple example of bifurcation of the solution of an ordinary differential equation is given by

$$\Phi'' + \mu\Phi = 0, \tag{46}$$

with real constant coefficient μ. The solution is: (i) oscillatory,

$$\Phi(x) = C_1 \cos(\nu x) + C_2 \sin(\nu x), \tag{47a}$$

for positive $\mu > 0$ hence real μ; (ii) decaying or diverging,

$$\Phi(x) = C_1 \exp(\nu x) + C_2 \exp(-\nu x), \tag{47b}$$

for negative $\mu < 0$; (iii) linearly diverging,

$$\Phi(x) = C_1 + C_2 x, \tag{47c}$$

in the intermediate case $\mu = 0$ between (i) and (ii), where $\nu \equiv |\mu|^{1/2}$ and (C_1, C_2) are arbitrary constants of integration. The bifurcation at $\mu = 0$ separates sinusoidal oscillations in (47a) for $\mu > 0$ for all real x, from decay or instability for $\mu \leq 0$ in (47b) and (47c). For $\mu = 0$ in (47c) there is linear instability. For $\mu < 0$ in (47b): (i) for positive $x > 0$ there is exponential decay for $C_1 = 0 \neq C_2$ and exponential instability for $C_1 \neq 0$; (ii) for negative $x < 0$ there is exponential decay for $C_1 \neq 0 = C_2$ and exponential instability for $C_2 \neq 0$; (iii) for the full range of real values of x there is always exponential instability for $(C_1, C_2) \neq (0, 0)$.

4 Conclusion

Since the generalised Bessel differential equation (1) has only two singularities, one at the origin $z = 0$ and the other at infinity $z = \infty$, the method of expansion in powers of z adopted in the present paper has the advantage that the solutions apply over the full domain of the independent variable $0 < |z| < \infty$ outside the singularities $z \neq \{0, \infty\}$. Because the singularity is regular at the origin, the Frobenius-Fuchs method, [36], [37], [39], [40], specifies power series solutions convergent for $|z| > 0$ involving: (i) only generalised Bessel functions of orders $\pm\nu$ and degree μ if ν is not an integer; (ii) generalised Bessel and Neumann functions of order ν and degree μ if ν is an integer. In both cases the original Bessel and Neumann functions correspond to the case of zero degree $\mu = 0$. Another situation is for the solution in the neighbourhood of the singularity at infinity because: (i) it is an irregular singularity and the method of normal integrals leads to two Hankel functions specified by asymptotic expansions that provide a good approximation at order N for large $|z|$ but do not converge as $N \to \infty$; (ii) the irregular singularity at infinity is of distinct degree – one for the original and two for the generalised Bessel equations – leading to asymptotic expansions in powers of z with negative integer exponents, that are multiplied by an exponential of the form $\exp(\alpha z)$ with constant α or $\left[\exp\left(\beta z^2\right)\right]$ with constant β respectively for the original and generalised Hankel functions.

The asymptotic solution of the generalised Bessel differential equation (1) is discontinuous when the degree becomes zero. That discontinuity is like a Hopf-type bifurcation because for the real variable z and order ν, the asymptotic solution is: (i) oscillatory for zero degree, $\mu = 0$, the same behaviour as in the original Hankel functions for zero degree (9); (ii) unstable for positive degree, $\mu > 0$ since both the generalised Hankel functions of first (4) and second (11) kinds diverge as $z \to \infty$; (iii) stable for negative degree, $\mu < 0$, since both the generalised Hankel functions of first (4) and second (11) kinds decay as $z \to \infty$. The original Hopf bifurcation, [43], is about an autonomous differential system relating to a non-linear second-order differential equation with constant coefficients, [44]. The present example studies a linear second-order differential equation with variable coefficients (1) that can be transformed to an autonomous system of two first-order differential equations. The present method has proved that it is possible to get explicit solutions of a differential equation around a bifurcation with the method of normal integrals, [35], [36], that generalises the method of regular integrals with branch-points, [33], [34], [37].

The generalised Bessel differential equation can have the form of a confluent hypergeometric differential equation, leading to relations between the Kummer functions, [45], [46], and the generalised Bessel, Neumann and Hankel functions that will be explored in a future work. The direct solutions of the generalised Bessel differential equation given here tries to keep as close as possible to the original Bessel, Neumann and Hankel functions, to highlight more clearly the differences associated with the generalization.

The generalised Bessel functions are an alternative to the Kummer confluent hypergeometric functions, [45], and the Whittaker functions, [39], with the advantage that the connection with the original Bessel functions is much simpler: (i) the original Bessel functions of order ν are the generalised Bessel functions of order ν and degree μ zero,

$$J_\nu(z) = J_\nu^0(z), \tag{48}$$

and thus are identified by a single parameter with $\mu = 0$; (ii) the relation between confluent hypergeometric functions and the original Bessel functions is a limit,

$$J_\nu(z) = \lim_{a\to\infty} \left(\frac{z}{2}\right)^\nu F\left(a, \nu; -\frac{z}{a}\right), \tag{49}$$

that is much less obvious than (48); (iii) the relation of Whittaker functions is

$$J_\nu(z) = 2^{-\nu} z^{\nu/2} \lim_{a\to\infty} (-a)^{-\nu/2}\, \mathrm{e}^{-z/(2a)} \times W_{\nu/2-a,\nu/2-1/2}\left(-\frac{z}{a}\right), \tag{50}$$

which is again a limit more complex than (48). Thus the generalised Bessel functions are equivalent to Kummer's confluent hypergeometric functions and to Whittaker functions that have the simplest relation (48) to the original Bessel functions. The simple relation (48) also applies to the generalised and original Neumann functions,

$$Y_\nu(z) = Y_\nu^0(z), \tag{51}$$

since, like the generalised Bessel functions, they are solutions of the generalised Bessel differential equation (1) around the origin where as a regular singularity. The relations (48) and (51) do not extend to Hankel functions because the original/generalised Hankel functions are asymptotic solutions of respectively the original/generalised Bessel differential equation that contains at infinity irregular singularities with different degrees. The original Bessel functions have numerous applications to problems in physics and engineering problems, [4], [5], [40], [47], [48], involving special functions, [39], [49], [50], [51], that are both a classical subject, [52], [53], [54], [55], and a topic of current research, [56], [57], [58], [59], some recent examples concern, [60], [61], Bessel, [42], [62], [63], [64], and Gaussian and confluent hypergeometric, [65], functions and various physical and engineering problems, [66], [67], concerning specifically the generalised Bessel differential equation; solutions (i) were obtained previously as Frobenius-Fuchs series around the regular singularity at the origin specifying generalised Bessel and Neumann functions, complemented (ii) in the present paper by asymptotic expansions near the irregular singularity at infinity specifying generalised Hankel functions. Possible future research directions include: (iii) relating generalised Hankel functions to generalised Bessel and Neumann functions through confluent hypergeometric functions; (iv) solving the generalised Bessel differential equation using the Laplace transform in the complex plane to obtain integral representations for the generalised Bessel, Neumann and Hankel functions; (v) obtaining representations of the latter as differintegrations of elementary functions using the Riemann-Liouville fractional derivatives in the complex plane.

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