A survey of De Casteljau algorithms and regular iterative constructions of Bézier curves with control mass points

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Abstract: Drawing a curve on a computer actually involves approximating it by a set of segments. The De Casteljau algorithm allows to construct these piecewise linear curves which approximate polynomial Bézier curves using convex combinations. However, for rational Bézier curves, the construction no longer admits regular sampling. To solve this problem, we propose a generalization of the De Casteljau algorithm that addresses this issue and is applicable to Bézier curves with mass points (a weighted point or a vector) as control points and using a homographic parameter change dividing the interval $[0,1]$ into two equal-length intervals $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$. If the initial Bézier curve is in standard form, we obtain two curves in standard form, unless the mass endpoint of the curve is a vector. This homographic parameter change also allows transforming curves defined over an interval $[\alpha,+\infty]$, $\alpha\in\mathbb{R}$, into Bézier curves, which then enables the use of the De Casteljau algorithm. Some examples are given: three-quart of circle, semicircle and a branch of a hyperbola (degree 2), cubic curve on $[0;+\infty]$ and loop of a Descartes Folium (degree 3) and a loop of a Bernouilli Lemniscate (degree 4).

Key-Words: De Casteljau algorithms, Bézier curves, mass points, homographic parameter change, regular sampling iterative construction.

Received: October 16, 2023. Revised: February 18, 2024. Accepted: March 9, 2024. Published: April 3, 2024.

1 Introduction

Bézier curves are the simplest curves defined by control points and were invented by [1] at Renault and [2] at Citroën. More details about their uses in C.A.G.D. can be found in [3], [4], [5], [6], [7], [8], [9], [10], [11] and [12]. Initially, these curves were defined as the barycentric locus of weighted points, with the weights computed using the appropriate Bernstein polynomials. In a second step, adding weights to the control points allows for obtaining more curves such as conic arcs with center, but there is no way to obtain semi-ellipses [12]. A mass point that is either a weighted point or a vector with a weight equal to 0 is a generalization of the concept of barycenter. For example

$$1\,\overrightarrow{GB}-1\,\overrightarrow{GA}=\overrightarrow{GB}-\left(\overrightarrow{GB}+\overrightarrow{BA}\right)=\overrightarrow{AB}$$

and the barycenter of the weighted points $(A,-1)$ and $(B,1)$ does not exist, but the calculus leads to the vector $\overrightarrow{AB}$. For any natural number greater than or equal to 2, the conversion of the parametric equation curve $\left(t,\frac{1}{t^n}\right)$ on $[0,1]$ from the canonical basis to the Bernstein basis yields a rational Bézier curve of degree $n+1$ with $n-1$ control vectors. Furthermore, by converting the parametric curve $t\mapsto(t,t^n)$ with $n>2$, after the variable change $t=\frac{u}{1-u}$ in the appropriate Bernstein basis, the null vector $\overrightarrow{0}$ appears $n-2$ times.

To solve the problem of constructing semi-ellipses or hyperbola branches, it is sufficient to add control vectors by using mass points as control points. Furthermore, it is possible to model Descartes Folium or Bernoulli Lemniscate loops. The De Casteljau algorithm is generalized using a specific homographic parameter change, which allows for obtaining regular curve subdivision algorithms that can be applied to quadratic curves usable in the usual Euclidean space or in the (non-Euclidean) Minkowski-Lorentz space for Dupin cyclides [13] cubic for Descartes Folium, quartic for Bernoulli Lemniscate.

In 2004 the fractal nature of Bézier curves is demonstrated in [14], that is, their self-similarities, based on the works [15], particularly focusing on the concepts of attractors and iterative processes. The purpose of this paper is to construct a Bézier curve iteratively with regular sampling. Figure 1 provides a visual representation of the difference between the projective De Casteljau algorithm

[7], [12], Figure 1 and our generalized De Casteljau algorithm, Figure 2, to construct a circle arc as a rational quadratic Bézier. The first algorithm does not yield a regular distribution of constructed points, Figure 1, whereas our generalized De Casteljau algorithm do, Figure 2.

In Figure 1, the angles are given in the Table 1. In the Figure 2, the angle, in degree, between two consecutive constructed points and the center O of the circle equals to

$$\widehat{P_0 O B_0} = \widehat{B_i O B_{i+1}} = \widehat{B_6 O P_2} = 33.75 = \frac{270}{8}$$

where $i \in [\![0;5]\!]$. Considering a quarter of a circle, the calculations are performed to show that the De Casteljau algorithm does not allow obtaining a regular construction.

Table 1: Angles, in degree, between two consecutive points and the center of the circle in the Figure 1.

Angle	$\widehat{P_0OA_0}$	$\widehat{A_0OA_1}$	$\widehat{A_1OA_2}$	$\widehat{A_2OA_3}$
Value	12.82	21.46	38.50	62.23
Angle	$\widehat{A_3OA_4}$	$\widehat{A_4OA_5}$	$\widehat{A_5OA_6}$	$\widehat{A_6OP_2}$
Value	62.23	38.50	21.46	12.82

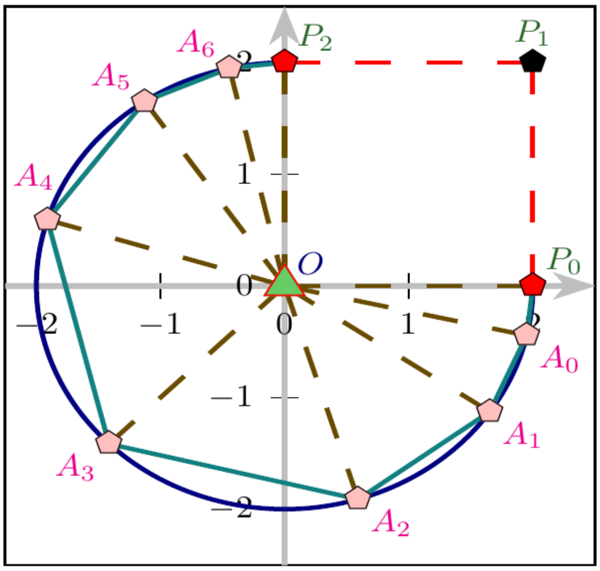

Figure 1: Distribution of points constructed by the projective De Casteljau algorithm.

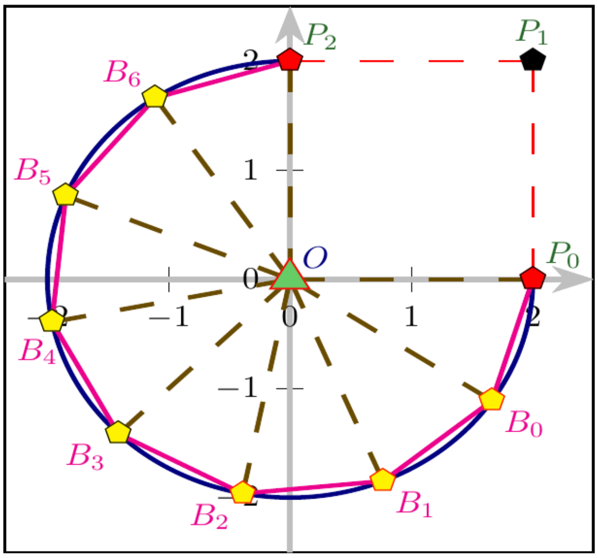

Figure 2: Distribution of points constructed by our algorithm based on the change of homographic parameter.

Moreover, let γ be the rational quadratic Bézier curve in standard form of control points $(P_0, 1)$, $\left(P_1, -\frac{\sqrt{2}}{2}\right)$ and $(P_2, 1)$ representing the three quarters of circle in Figure 1. Then $B_3 = \gamma\left(\frac{1}{2}\right)$. Let γ_1 one of the two subcurve of γ, in standard form, of endpoints $(P_0, 1)$ and $(B_3, 1)$. Note that the tangents to the Bézier curve at P_0 and B_3 are known thanks to the De Casteljau algorithm. Unlike polynomial curves, in the rational case, it is necessary to perform an iterative construction because

$$\gamma_1\left(\frac{1}{2}\right) \neq \gamma\left(\left(\frac{1}{2}\right)^2\right) = \gamma\left(\frac{1}{4}\right) \tag{1}$$

In [7], it is written, regarding the projective version of the De Casteljau algorithm: "The intermediate points could also be close to being vectors i.e. having a very small third component. This may cause numerical problems.". The use of mass points allows solving this problem since, in the case of a rational curve, we do not divide by the weights when obtaining a vector (including the null vector $\overrightarrow{0}$), see Formula (8).

The article is structured as follows. In the second section, we provide a brief overview of the classical De Casteljau algorithm for degrees 2 and 3. Before concluding and discussing future prospects, we extend the De Casteljau algorithm to rational Bézier curves with control mass points by using a homographic parameter change function.

2 Classical De Casteljau Algorithm

The De Casteljau algorithm for a polynomial Bézier curve of degree n is given by the Algorithm 1. The algorithm performs iterative calculations to obtain the Bézier curve point tab$[n][0]$. It starts by linearly interpolating between adjacent control points to generate intermediate points tabP$[k]$. These intermediate points are then linearly interpolated to obtain further intermediate points tab$[j][k]$. Finally, the last step involves linearly interpolating between the remaining intermediate points to obtain the final point tab$[n][0]$ on the Bézier curve.

The use of a table allows for the iterative construction of a Bézier curve by replacing, at each iteration, a Bézier curve with two Bézier curves of the same degree. The control points of these new curves are either the first elements of each column or the last elements of each column (the table forms an upper triangular table). Note that at each iteration, the tangents at the endpoints of the Bézier curves are defined either by the first two control points or by the last two control points.

Algorithm 1 De Casteljau Algorithm for polynomial Bézier curves of degree n.

Input : let tabP be a table of $n+1$ points and Ω an other point.
Definition of the table tab of dimension $(n+1)\times(n+1)$
For $k=0$ To n
  tab$[0][k]\leftarrow$tabP$[k]$
For $j=1$ To n
  For $k=0$ To $n-j$

$$\begin{aligned}\overrightarrow{\Omega\,\mathrm{tab}[j][k]} &\leftarrow (1-t)\ \overrightarrow{\Omega\,\mathrm{tab}[j-1][k]} \\ &\quad + \ t\,\overrightarrow{\Omega\,\mathrm{tab}[j-1][k+1]}\end{aligned} \tag{2}$$

Output : the point tab$[n][0]$ of the polynomial Bézier curve with $n+1$ control points in tabP. The tangent to the curve at the point tab$[n][0]$ is the line (tab$[n-1][0]$tab$[n-1][1]$).

Let us detail the Agorithm 1 for degrees 2 and 3 by explaining the iterative constructions.

2.1 Quadratic case

2.1.1 De Casteljau algorithm

With $n=2$, the Algorithm 1 allows building a parabola arc $\mathcal{C}_P$ using a polynomial quadratic Bézier curve with control points P_0, P_1 and P_2. To simplify the figures, let us denote tab$[1][0]=Q_0(t)$, tab$[1][1]=Q_1(t)$ and tab$[2][0]=R_0(t)$ which is a point of the curve $\mathcal{C}_P$, Figure 3 for $t=\frac{1}{2}$.

2.1.2 Iterative construction

The polynomial quadratic Bézier curve with control points P_0, P_1 and P_2 is cut into two polynomial quadratic Bézier curves γ_0 with control points P_0, $Q_0(t)$ and $R_0(t)$ on the one hand and γ_1 with control points $R_0(t)$, $Q_1(t)$ and P_2 on the other hand, Figure 3. Moreover, the line $(Q_0(t)\,Q_1(t))$ is the tangent to the curves γ_0 and γ_1 at the point $R_0(t)$. If Ω equals O, the elements of the array tab are

$$\begin{array}{lll} P_0 & Q_0(t) & R_0(t) \\ P_1 & Q_1(t) & \\ P_2 & & \end{array} \tag{3}$$

and the points of the first line of tab correspond to the control points of the first sub-curve of Bézier, whereas the points on the diagonal of tab correspond to the control points of the second sub-curve of Bézier. The common point belonging to the two sub-curves is $R_0(t)$ and the tangent to these sub-curves at $R_0(t)$ is the line $(Q_0(t)\,Q_1(t))$.

2.1.3 Comparison between De Casteljau algorithm and iterative construction

Consider the example of a Bézier curve of control points $P_0(-4,0)$, $P_1(0,4)$ and $P_2(4,0)$. The points $A\left(-3,\dfrac{7}{8}\right)$, $B\left(-2,\dfrac{3}{2}\right)$, $C\left(-1,\dfrac{15}{8}\right)$, $D(0,2)$, $E\left(1,\dfrac{15}{8}\right)$, $F\left(2,\dfrac{3}{2}\right)$ and $G\left(3,\dfrac{7}{8}\right)$ are computed by De Casteljau algorithm or the iterative construction, Figures 4. The Table 2 provides the different points constructed by the two algorithms at different steps. Seven iterations are needed with the De Casteljau algorithm whereas only three are needed with the iterative method (with four threads). In the Figure 4:

- the square point designates the points which is built at first with De Casteljau algorithm;
- the triangle points designate the points which are built in the same time with De Casteljau algorithm and the iterative construction;
- the pentagon points designates the points which are built at first with the iterative construction.

In the Figure 4, the point A is computed by the De Casteljau algorithm and the point D is built using the iterative construction.

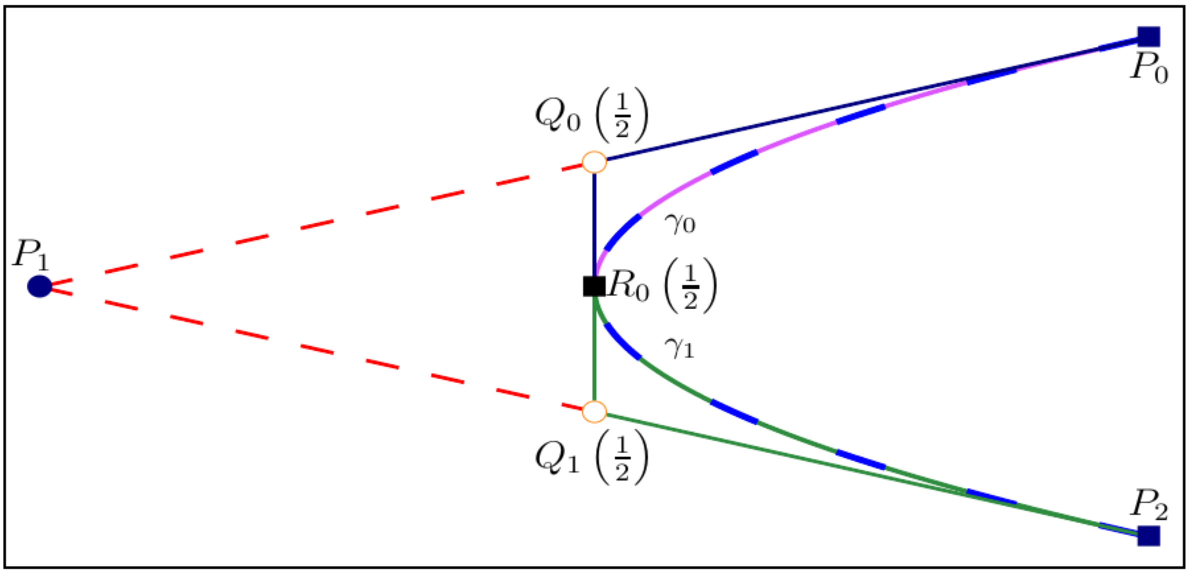

Figure 3: Iterative construction of a Bézier curve based on De Casteljau method, Algorithm 1 with $n = 2$ and $t = \frac{1}{2}$, $\mathcal{C}_P = \gamma_0 \cup \gamma_1$ and $\left\{R_0\left(\frac{1}{2}\right)\right\} = \gamma_0 \cap \gamma_1$.

Table 2: Comparison between the steps : sequential De Casteljau Algorithm or the iterative construction with four threads.

Constructed point on the Bézier curve	A	B	C	D	E	F	G
Step of De Casteljau algorithm	1	2	3	4	5	6	7
Step of the iterative construction	3	2	3	1	3	2	3

The point B is computed by the De Casteljau algorithm and by the iterative construction whereas the point F is built using the iterative construction.

The point C is computed by the De Casteljau algorithm and by the iterative construction whereas the points E and G are built using the iterative construction.

2.2 Cubic case

With $n = 3$, the Algorithm 1 allows building a polynomial cubic Bézier curve γ with control points P_0, P_1, P_2 and P_3. To simplify the figures, let us notate tab $[1][0] = Q_0(t)$, tab $[1][1] = Q_1(t)$, tab $[1][2] = Q_2(t)$, tab $[2][0] = R_0(t)$, tab $[2][1] = R_1(t)$, and tab $[3][0] = S_0(t)$ which is a point of the curve γ, Figure 5 for $t = \frac{1}{2}$.

The polynomial cubic Bézier curve with control points P_0, P_1, P_2 and P_3 is cut into two polynomial cubic Bézier curves γ_0 with control points P_0, $Q_0(t)$, $R_0(t)$ and $S_0(t)$ on the one hand and γ_1 with control points $S_0(t)$, $R_1(t)$, $Q_2(t)$ and P_3 on the other hand. Moreover, the line $(R_0(t)\,R_1(t))$ is the tangent to the curves γ_0 and γ_1 at the point $S_0(t)$. Let Ω be the point O, the elements of the array tab are

$$\begin{array}{llll} P_0 & Q_0(t) & R_0(t) & S_0(t) \\ P_1 & Q_1(t) & R_1(t) & \\ P_2 & Q_2(t) & & \\ P_3 & & & \end{array} \tag{4}$$

and the points of the first line of tab correspond to the control points of the first sub-curve of Bézier γ_0, whereas the points on the diagonal of tab correspond to the control points of the second sub-curve of Bézier γ_1. The common point belonging to the two sub-curve is $S_0(t)$ and the tangent to these sub-curves at $S_0(t)$ is the line $(R_0(t)\,R_1(t))$.

Unfortunately, regular construction for rational Bézier curves is not possible using the projective De Casteljau algorithm. One solution could be to recalculate the weights at each step [7], which is computationally heavy and time-consuming. So, another method, which generalises the usual De Casteljau algorithm must be developed.

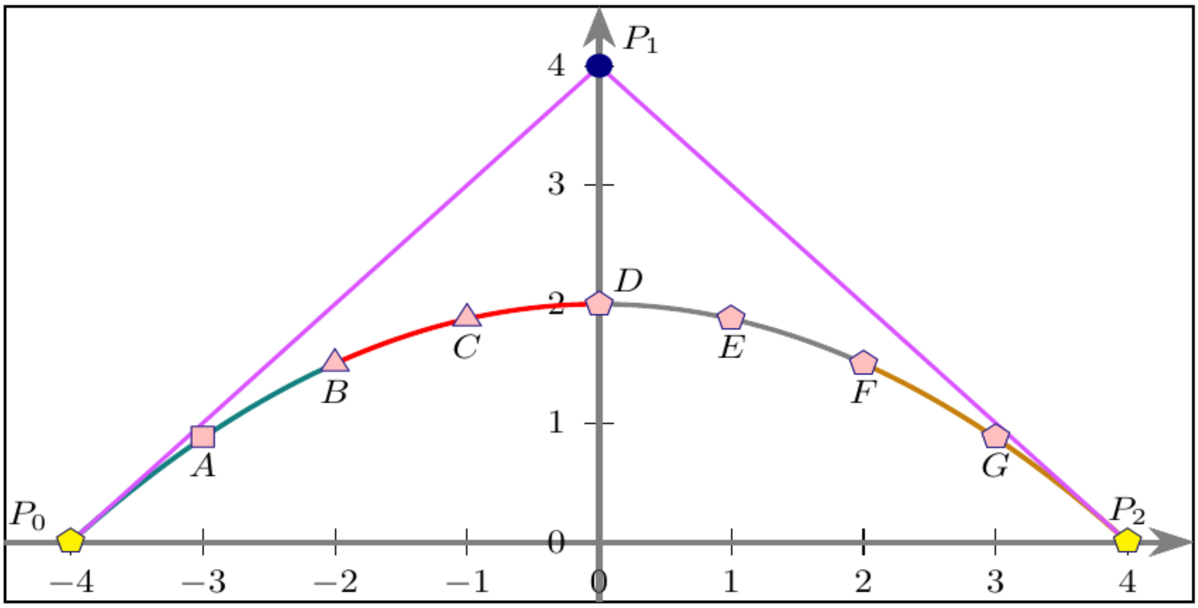

Figure 4: Comparison between sequential De Casteljau algorithm and iterative construction of a Bézier curve, third iteration.

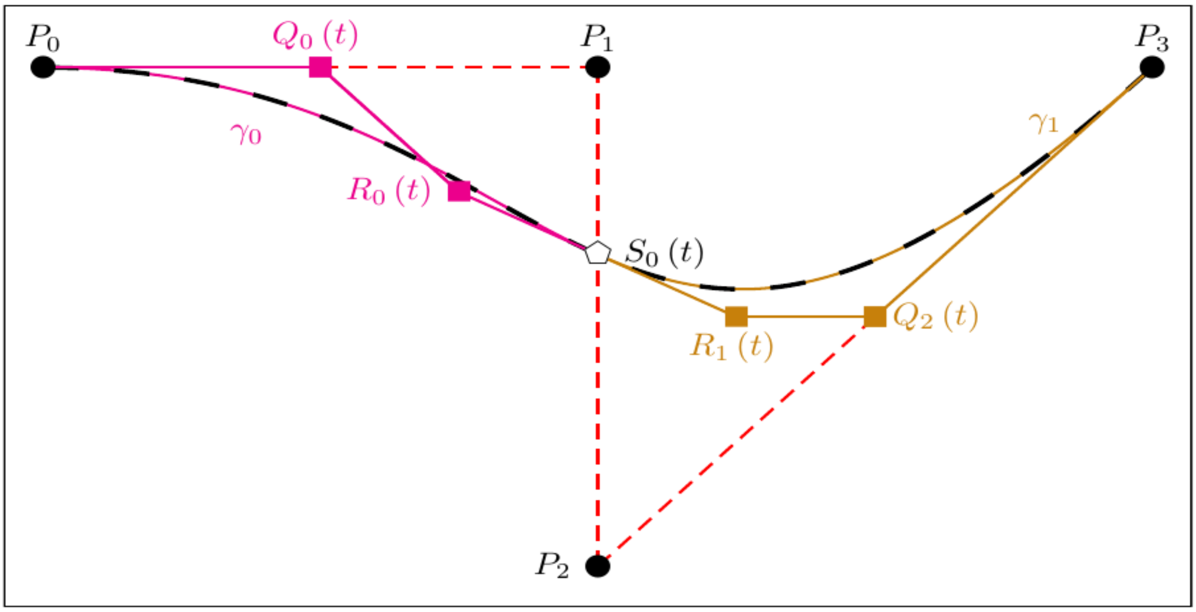

Figure 5: Iterative construction of a Bézier curve based on De Casteljau method, Algorithm 1 with $n = 3$ and $t = \frac{1}{2}$.

3 Homographic parameter change for the De Casteljau algorithm

The purpose of this section is to construct, iteratively and with regular sampling, points on a Bézier curve. To achieve this, the formula (1) gives the expression of the irregularity of the subdivision in the case of a rational Bézier curve, we replace the original Bézier curve with two Bézier curves of the same degree. If the initial Bézier curve is in standard form, we obtain two curves in standard form, unless the mass endpoint of the curve is a vector.

3.1 Rational Bézier curves in $\widetilde{\mathcal{P}}$

In the following $(O;\overrightarrow{\imath};\overrightarrow{\jmath})$ designates a direct reference frame in the usual Euclidean affine plane $\mathcal{P}$ and $\overrightarrow{\mathcal{P}}$ is the set of vectors of the plane. The set of mass points is defined by

$$\widetilde{\mathcal{P}} = (\mathcal{P}\times\mathbb{R}^*)\cup\left(\overrightarrow{\mathcal{P}}\times\{0\}\right)$$

On the mass point space, the addition, denoted $\oplus$, is defined as follows

- $\omega\neq 0 \Longrightarrow (M;\omega)\oplus(N;-\omega) = \left(\omega\ \overrightarrow{NM};0\right)$;
- $\omega\,\mu\,(\omega+\mu)\neq 0 \Longrightarrow (M;\omega)\oplus(N;\mu) = \left(\text{Bar}\left\{(M;\omega);(N;\mu)\right\};\omega+\mu\right)$ where $\text{Bar}\left\{(M;\omega);(N;\mu)\right\}$ denotes the barycenter of the weighted points $(M;\omega)$ and $(N;\mu)$;
- $(\overrightarrow{u};0)\oplus(\overrightarrow{v};0) = (\overrightarrow{u}+\overrightarrow{v};0)$;
- $\omega\neq 0 \Longrightarrow (M;\omega)\oplus(\overrightarrow{u};0) = \left(\mathcal{T}_{\frac{1}{\omega}\overrightarrow{u}}(M);\omega\right)$ where $\mathcal{T}_{\overrightarrow{W}}$ is the translation of $\mathcal{P}$ of vector $\overrightarrow{W}$.

In the same way, on the space $\widetilde{\mathcal{P}}$, the multiplication by a scalar, denoted $\odot$, is defined as follows

- $\omega\,\alpha\neq 0 \Longrightarrow \alpha\odot(M;\omega) = (M;\alpha\,\omega)$
- $\omega\neq 0 \Longrightarrow 0\odot(M;\omega) = \left(\overrightarrow{0};0\right)$
- $\alpha\odot(\overrightarrow{u};0) = (\alpha\overrightarrow{u};0)$

One can note that $\left(\widetilde{\mathcal{P}},\oplus,\odot\right)$ is a vector space [16]. So, a mass point is a weighted point (M,ω) with $\omega\neq 0$ or a vector $(\overrightarrow{u},0)$. The Bernstein polynomials of degree n are defined by

$$B_{i,n}(t) = \begin{pmatrix} n \\ i \end{pmatrix}(1-t)^{n-i}\ t^i \tag{5}$$

These Bernstein polynomials provide the definition of rational Bézier curve (BR curve) in $\widetilde{\mathcal{P}}$ given below.

Definition 1 Rational Bézier curve (BR curve) in $\widetilde{\mathcal{P}}$

Let $(P_i;\omega_i)_{i\in[\![0;n]\!]}$ $n+1$ mass points in $\widetilde{\mathcal{P}}$. Define two sets

$$I = \{i \mid \omega_i\neq 0\} \quad\text{and}\quad J = \{i \mid \omega_i = 0\}$$

Define the weight function ω_f as follows

$$\begin{array}{rccl} \omega_f: & [0;1] & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \omega_f(t) = \displaystyle\sum_{i\in I}\omega_i\times B_i(t) \end{array} \tag{6}$$

A mass point $(M;\omega)$ or $(\overrightarrow{u};0)$ lays to the rational Bézier curve defined by the control mass points $(P_i;\omega_i)_{i\in[\![0;n]\!]}$ if there is a real t_0 in $[0;1]$ such that:

- if $\omega_f(t_0)\neq 0$ then

$$\left\{\begin{array}{rcl} \overrightarrow{OM} & = & \dfrac{1}{\omega_f(t_0)}\left(\displaystyle\sum_{i\in I}\omega_i\,B_i(t_0)\ \overrightarrow{OP_i}\right) \\ & + & \dfrac{1}{\omega_f(t_0)}\left(\displaystyle\sum_{i\in J}B_i(t_0)\ \overrightarrow{P_i}\right) \\ \omega & = & \omega_f(t_0) \end{array}\right. \tag{7}$$

- if $\omega_f(t_0) = 0$ then

$$\overrightarrow{u} = \sum_{i\in I}\omega_i B_i(t_0)\,\overrightarrow{OP_i} + \sum_{i\in J}B_i(t_0)\,\overrightarrow{P_i} \tag{8}$$

Such a curve is denoted $BR\left\{(P_i;\omega_i)_{i\in[\![0;n]\!]}\right\}$

If $J=\emptyset$, this definition leads to the usual rational Bézier curve.

The Algorithm 1 can be generalized to use mass points: the Equation (2) is replaced by

$$\begin{array}{rcl} \text{tab}[j][k] & \leftarrow & (1-t)\odot\text{tab}[j-1][k] \\ & & \oplus\quad t\odot\text{tab}[j-1][k+1] \end{array} \tag{9}$$

but the issue regarding the weight of the constructed point remains.

3.2 Homographic parameter change

To achieve regular constructions, the weights of mass endpoints equal 0 in the case of a vector and 1 in the case of a weighted point.

3.2.1 Definition and fundamental theorem

Let α and β be two distinct reals. The homographic change allows obtaining, without increasing the degree of the curve, the portion of the curve γ defined on the interval $[\alpha;\beta]$, $[\alpha,+\infty]$ or $[-\infty,\beta]$ by using the Bézier curve $\gamma\circ h$ over the interval $[0;1]$ i.e. $h([0,1]) = [\alpha,\beta]$, $h([0,1]) = [\alpha,+\infty]$ or $h([0,1]) = [-\infty,\alpha]$.

Theorem 1 : Homographic parameter change

Let γ be a Bézier curve of degree n of control mass points $((P_i;\omega_i))_{i\in[\![0;n]\!]}$.

Let h be the homographic function from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ defined by

$$h(u) = \frac{a\ (1-u)+b\,u}{c\ (1-u)+d\,u} \tag{10}$$

Then $\gamma \circ h$ is a Bézier curve of degree n of control mass points $((Q_i;\varpi_i))_{i\in[\![0;n]\!]}$ and their expressions depend on the control points $((P_i;\omega_i))_{i\in[\![0;n]\!]}$ and on the values a, b, c and d, see [17].

One can note that h is monotone since the sign of $h'(u)$ is

$$\begin{vmatrix} b-a & a \\ d-c & c \end{vmatrix} = \begin{vmatrix} b & a \\ d & c \end{vmatrix} = bc - ad \neq 0$$

In this paper, to obtain a regular construction, the condition are

$$h([0,1]) = \left[0, \frac{1}{2}\right] \tag{11}$$

and

$$h([0,1]) = \left[\frac{1}{2}, 1\right] \tag{12}$$

The restriction of h to the interval $\left[0, \frac{1}{2}\right]$ is denoted as h_1, while the restriction of h to the interval $\left[\frac{1}{2}, 1\right]$ is denoted as h_2.

Corollary 1 (h_1 defined by Formula (11))

Let $((P_i;\omega_i))_{i\in[\![0;n]\!]}$ be $n+1$ control mass points of a Bézier curve γ of degree n.

Let b and c be two non null real numbers. Let h_1 be defined by

$$h_1 : \begin{array}{ccc} \overline{\mathbb{R}} & \longrightarrow & \overline{\mathbb{R}} \\ u & \longmapsto & \dfrac{b\,u}{c\,(1-u)+2\,b\,u} \end{array} \tag{13}$$

Then $\gamma \circ h_1$ is a Bézier curve of degree n of control mass points $((Q_i;\varpi_i))_{i\in[\![0;n]\!]}$ and their expressions depend on the control points $((P_i;\omega_i))_{i\in[\![0;n]\!]}$ and on the values b and c, see Tables 3, 5 and 7.

Moreover, the function h_1 is monotonically increasing if $b \times c > 0$.

Proof: Using the fonction h_1 defined by the Equation (10).

$$h_1(0) = \frac{a}{c} = 0 \Longrightarrow a = 0$$

$$h_1(1) = \frac{b}{d} = \frac{1}{2} \Longrightarrow d = 2b$$

Moreover

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = \begin{vmatrix} b & 0 \\ 2b & c \end{vmatrix} = b \times c$$

■

Corollary 2 (h_2 defined by Formula (12))

Let $((P_i;\omega_i))_{i\in[\![0;n]\!]}$ be $n+1$ control mass points of a Bézier curve γ of degree n.

Let a and d be two non null real numbers. Let h_2 be defined by

$$h_2 : \begin{array}{ccc} \overline{\mathbb{R}} & \longrightarrow & \overline{\mathbb{R}} \\ u & \longmapsto & \dfrac{a\,(1-u)+d\,u}{2\,a\,(1-u)+d\,u} \end{array} \tag{14}$$

Then $\gamma \circ h_2$ is a Bézier curve of degree n of control mass points $((Q_i;\varpi_i))_{i\in[\![0;n]\!]}$ and their expressions depend on the control points $((P_i;\omega_i))_{i\in[\![0;n]\!]}$ and on the values a and d, see Tables 4, 6 and 8.

Moreover, the function h_2 is monotonically increasing if $a \times d > 0$.

Proof: Using the fonction h_2 defined by the Equation (10).

$$h_2(0) = \frac{a}{c} = \frac{1}{2} \Longrightarrow c = 2a$$

$$h_2(1) = \frac{b}{d} = 1 \Longrightarrow b = d$$

Moreover,

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = \begin{vmatrix} d & a \\ d & 2a \end{vmatrix} = \begin{vmatrix} d & a \\ 0 & a \end{vmatrix} = a \times d$$

■

3.2.2 Degree 2 case

The Table 3 defines the control mass points using the function h_1 defined by the Equation (13) with $a = 0$, $d = 2b$ and $c = 1$ to keep the first control mass point. The value of b is calculated so that the weight of the last weighted point is equal to 1 or the vector is unchanged.

The Table 4 defines the control mass points using the function h_2 defined by the Equation (14) with $c = 2a$ and $b = d = 1$ to keep the last control mass point. The value of a is calculated so that the weight of the first weighted point is equal to 1 or the vector is unchanged.

First example : three-quarters of circle of center $O(0,0)$, of radius $r = 2$

The control mass points of the quadratic Bézier curve are $P_0(2,0)$, $\omega_0 = 1$, $P_1(2,2)$, $\omega_1 = -\frac{\sqrt{2}}{2}$ and $P_2(0,0)$, $\omega_2 = 1$, Figure 6.

Table 3: Control mass points for the quadratic case for h_1.

$$\begin{aligned} (Q_0;\varpi_0) =& \quad (P_0,\omega_0) \\ (Q_1;\varpi_1) =& \quad b \odot ((P_0,\omega_0) \oplus (P_1,\omega_1)) \\ (Q_2;\varpi_2) =& \quad b^2 \odot ((P_0,\omega_0) \oplus 2 \odot (P_1,\omega_1) \oplus (P_2,\omega_2)) \end{aligned} \tag{15}$$

Table 4: Control mass points for the quadratic case for h_2.

$$\begin{aligned} (Q_0;\varpi_0) =& \quad a^2 \odot ((P_0,\omega_0) \oplus 2 \odot (P_1,\omega_1) \oplus (P_2,\omega_2)) \\ (Q_1;\varpi_1) =& \quad a \odot ((P_1,\omega_1) \oplus (P_2,\omega_2)) \\ (Q_2;\varpi_2) =& \quad (P_2,\omega_2) \end{aligned} \tag{16}$$

First iteration, function h_1

Directely, $(Q_0;1) = (P_0;1)$, Figure 6. The value of b must be determined such that the last weight is equal to 1 which leads to the equation

$$b^2 \left(1 - 2 \times \frac{\sqrt{2}}{2} + 1\right) = 1$$

and the positive solution is

$$b = \sqrt{\frac{1}{2-\sqrt{2}}} = \sqrt{\frac{2+\sqrt{2}}{2}}$$

Let I_1 be the midpoint of the segment $[P_0P_2]$. $(P_0,1) \oplus 2 \odot \left(P_1, -\frac{\sqrt{2}}{2}\right) \oplus (P_2,1) = (I_1,2) \oplus \left(P_1,-\sqrt{2}\right) = \left(G_1, 2-\sqrt{2}\right)$ where $G_1\left(-\sqrt{2},-\sqrt{2}\right)$ and then

$$(Q_2;\varpi_2) = (G_1;1)$$

$(P_0,\omega_0) \oplus (P_1,\omega_1) = (P_0,1) \oplus \left(P_1, -\frac{\sqrt{2}}{2}\right) = \left(G_2, \frac{2-\sqrt{2}}{2}\right)$ with the point $G_2\left(2; -2\left(\sqrt{2}+1\right)\right)$ and then

$$(Q_1;\varpi_1) = \left(G_2; \frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

First iteration, function h_2

The points of this curve are changed into the points R_0, R_1 and R_2 instead of Q_0, Q_1 and Q_2. The value of a must be determined such that the first weight is equal to 1 which leads to the equation

$$a^2 \left(1 - 2 \times \frac{\sqrt{2}}{2} + 1\right) = 1$$

and the positive solution[1] is

$$a = b$$

and then

$$(R_0;\varpi_0) = (G_1;1)$$

Concerning the point P_1:

$$\begin{aligned} & (P_1,\omega_1) \oplus (P_2,\omega_2) \\ =& \left(P_1, -\frac{\sqrt{2}}{2}\right) \oplus (P_2,1) \\ =& \left(G_3, \frac{2-\sqrt{2}}{2}\right) \end{aligned}$$

with the point $G_3\left(-2\left(\sqrt{2}+1\right), 2\right)$ and then

$$(R_1;\varpi_1) = \left(G_3; \frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

and $(R_2;\varpi_2) = (P_2;1)$, Figure 6.

Second iteration

The quadratic Bézier curve with control mass points (Q_i,ω_i), $i \in [\![0,2]\!]$, is split into two quadratic Bézier curves with control mass points (S_i,ϖ_i), $i \in [\![0,2]\!]$ on the one hand, and (T_i,υ_i), $i \in [\![0,2]\!]$ on the other hand, Figure 6.

- The value of b is the positive solution of the equation

$$b^2 \left(1 + 2 \times \frac{\sqrt{2-\sqrt{2}}}{2} + 1\right) = 1$$

[1] Always, a is equal to b.

which leads to

$$b=\sqrt{\frac{\left(2-\sqrt{2-\sqrt{2}}\right)\left(2-\sqrt{2}\right)}{2}}\simeq 0.601$$

The other coefficients of the homographic parameter change function equal to $a=0$, $c=1$ equal to $d\simeq 1.203$.

The points are $S_0\,(2,0)$, $S_1\,(2,;-1.336)$ and $S_2\,(0.765,-1.848)$. The weights are $\varpi_0=1$, $\varpi_1=0.831$ and $\varpi_2=1$.

- The coefficients of the homographic parameter change function equal to $a\simeq 0.542$, $b=d=1$, $c\simeq 1.085$. The points are $T_0\,(0.765,-1.848)$, $T_1\,(-0.469,-2.359)$ and $T_2\,(-1.414,-1.414)$. The weights are $\upsilon_0=1$, $\upsilon_1=0.831$ and $\upsilon_2=1$.

The quadratic Bézier curve with control mass points (R_i,ω_i), $i\in[\![0,2]\!]$, is separeted into two quadratic Bézier curves with control mass points (U_i,ϖ_i), $i\in[\![0,2]\!]$ on the one hand, and (V_i,υ_i), $i\in[\![0,2]\!]$ on the other hand, Figure 6.

- The coefficients of the homographic parameter change function equal to $a=0$, $b\simeq 0.601$, $c=1$ $d\simeq 1.203$.

  The points are $U_0\,(-1.414,-1.414)$, $U_1\,(-2.359,-0.469)$ and $U_2\,(-1.848,0.765)$. The weights are $\varpi_0=1$, $\varpi_1=0.831$ and $\varpi_2=1$.

- The coefficients of the homographic parameter change function equal to $a\simeq 0.601$, $d=b=1$ $c\simeq 1.203$. The points are $V_0\,(-1.848,0.765)$, $V_1\,(-1.336,2)$ and $V_2\,(0,2)$. The weights are $\upsilon_0=1$, $\upsilon_1=0.831$ and $\upsilon_2=1$.

Second example : semi-circle of center $O\,(0,0)$, of radius $r=1$

The control mass points of the quadratic Bézier curve are chosen as $P_0\,(1,0)$, $\omega_0=1$, $\overrightarrow{P_1}\,(0,1)$, $\omega_1=0$ and $P_2\,(-1,0)$, $\omega_2=1$, Figure 7.

First iteration, function h_1

Directely, $(Q_0;1)=(P_0;1)$, Figure 7. The value of b must be determined such that the last weight is equal to 1 which leads to the equation

$$b^2\,(1+2\times 0+1)=1$$

and the positive solution equals to

$$b=\frac{\sqrt{2}}{2}$$

Let I_1 be the midpoint of the segment $[P_0P_2]$.

$$\begin{aligned} & (P_0,1)\oplus 2\odot\left(\overrightarrow{P_1},0\right)\oplus(P_2,1)\\ = & (I_1,2)\oplus\left(2\overrightarrow{P_1},0\right)\\ = & (G_1,2)\end{aligned}$$

where $G_1\,(0,1)$ and then

$$(Q_2;\varpi_2)=(G_1;1)$$

$$(P_0,\omega_0)\oplus(P_1,\omega_1)=(P_0,1)\oplus\left(\overrightarrow{P_1},0\right)=(G_2,1)$$

where $G_2\,(1,1)$ and then

$$(Q_1;\varpi_1)=\left(G_2;\frac{\sqrt{2}}{2}\right)$$

First iteration, function h_2

The points of this curve are changed into R_0, R_1 and R_2 instead of Q_0, Q_1 and Q_2. The value of $a=b$ leads to

$$(R_0;\varpi_0)=(G_1;1)$$

$$(P_1,\omega_1)\oplus(P_2,\omega_2)=\left(\overrightarrow{P_1},0\right)\oplus(P_2,1)=(G_3,1)$$

where $G_3\,(-1;1)$ and then

$$(R_1;\varpi_1)=\left(G_3;\frac{\sqrt{2}}{2}\right)$$

and $(R_2;\varpi_2)=(P_2;1)$, Figure 7.

Second iteration

The quadratic Bézier curve with control mass points (Q_i,ω_i), $i\in[\![0,2]\!]$, is decomposed into two quadratic Bézier curves with control mass points (S_i,ϖ_i), $i\in[\![0,2]\!]$ on the one hand, and (T_i,υ_i), $i\in[\![0,2]\!]$ on the other hand, Figure 7.

- The value of b is the positive solution of the equation

  $$b^2\,(1+2\times 0+1)=1$$

  which leads to

  $$b=\frac{\sqrt{2}}{2}$$

  The other coefficients of the homographic parameter change function are $a=0$, $c=1$ and $d=\sqrt{2}$. The points are $S_0\,(1,0)$, $S_1\,(1,0.414)$ and $S_2\,(0.707,0.707)$. The weights are $\varpi_0=1$, $\varpi_1\simeq 0.924$ and $\varpi_2=1$.

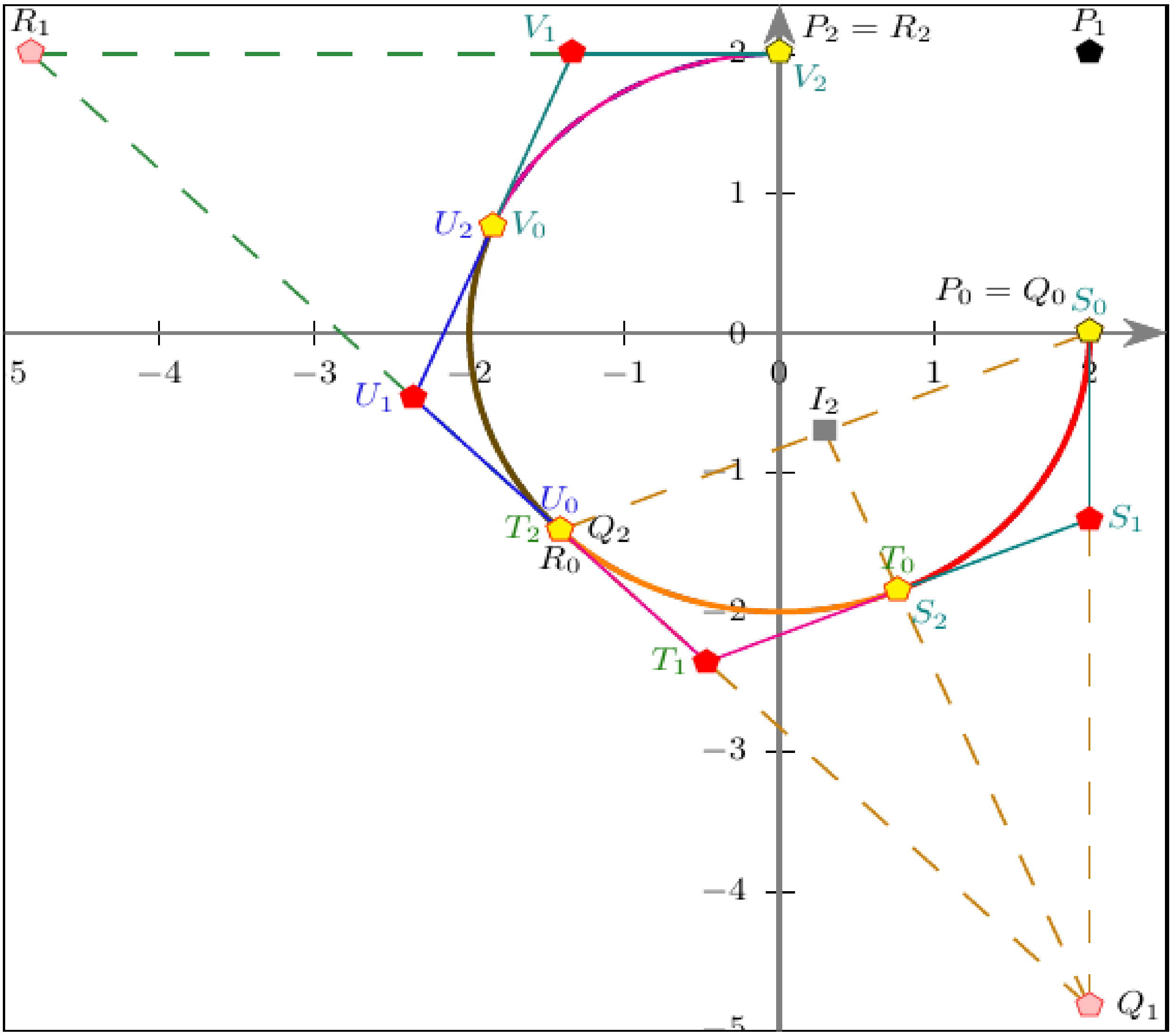

Figure 6: Second step of an iterative construction of a rational quadratic Bézier curve with control weighted points with homographic parameter change based on De Casteljau algorithm for a three-quarters of circle.

- The coefficients of the homographic parameter change function are $a = \frac{\sqrt{2}}{2}$, $b = d = 1$, $c = \sqrt{2}$. The points are $T_0\,(0.707, 0.707)$, $T_1\,(0.414, 1)$ and $T_2\,(0, 1)$. The weights are $\upsilon_0 = 1$, $\upsilon_1 \simeq 0.924$ and $\upsilon_2 = 1$.

The quadratic Bézier curve with control mass points (R_i, ω_i), $i \in [\![0, 2]\!]$, is decomposed into two quadratic Bézier curves with control mass points (U_i, ϖ_i), $i \in [\![0, 2]\!]$ on the one hand, and (V_i, υ_i), $i \in [\![0, 2]\!]$ on the other hand, Figure 7.

- The coefficients of the homographic parameter change function are $a = 0$, $b \simeq 0.541$, $c = 1$ $d \simeq 1.082$. The points are $U_0\,(0, 1)$, $U_1\,(-0.414; 1)$ and $U_2\,(-0.707; 0.707)$. The weights are $\varpi_0 = 1$, $\varpi_1 \simeq 0.924$ and $\varpi_2 = 1$.

- The coefficients of the homographic parameter change function are $a \simeq 0.541$, $d = b = 1$ $c \simeq 1.082$. The points are $V_0\,(-0.707; 0.707)$, $V_1\,(-1; 0.414)$ and $V_2\,(-1; 0)$. The weights are $\upsilon_0 = 1$, $\upsilon_1 \simeq 0.924$ and $\upsilon_2 = 1$, Figure 7.

Third example : a branch of a hyperbola

The control mass points of the quadratic Bézier curve are $\overrightarrow{P_0}\,(1, 1)$, $\omega_0 = 0$, $P_1\,(0, 0)$, $\omega_1 = 1$ and $\overrightarrow{P_2}\,(1, -1)$, $\omega_2 = 0$, Figure 8. The point P_1 is the center of the hyberbola, the directions of the vec-

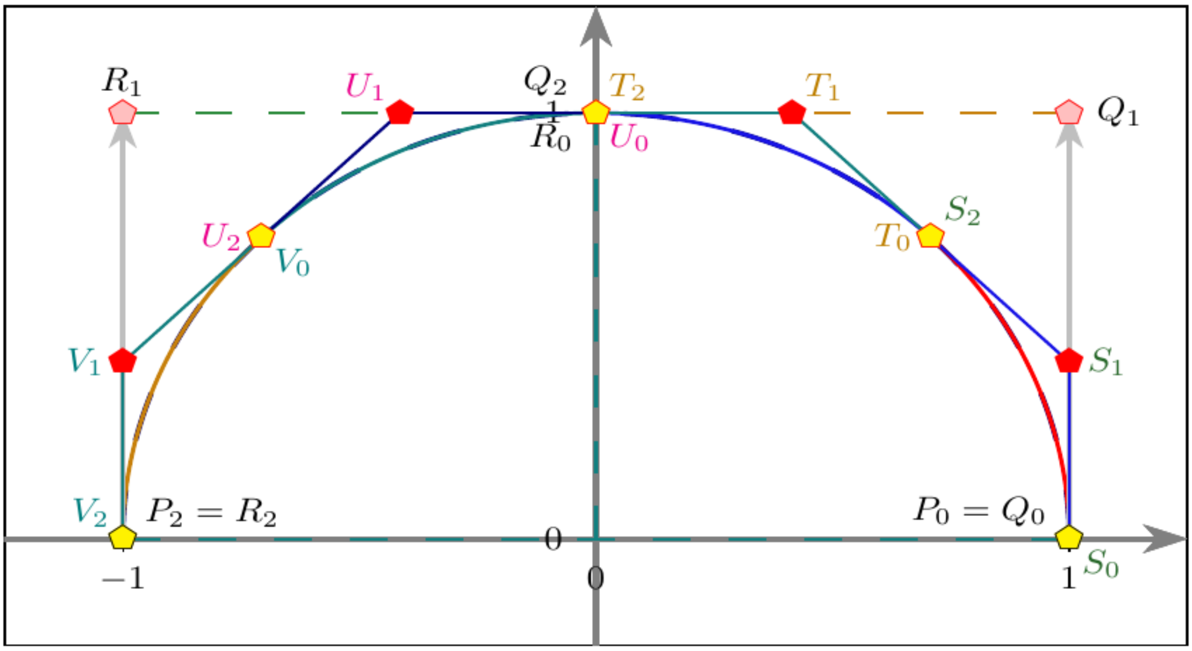

Figure 7: Iterative construction of a rational quadratic Bézier curve with control mass points with homographic parameter change based on De Casteljau algorithm for a semicircle.

tors $\overrightarrow{P_0}$ and $\overrightarrow{P_2}$ are the directions of the asymptotic lines to the hyperbola.

Function h_1

Directely, $\left(\overrightarrow{Q_0};0\right) = \left(\overrightarrow{P_0};0\right)$, Figure 8. The value of b must be determined such that the last weight is equal to 1 which leads to the equation

$$b^2\,(0+2+0) = 1$$

and the positive solution is

$$b = \frac{\sqrt{2}}{2}$$

$\left(\overrightarrow{P_0},0\right) \oplus 2 \odot (P_1,1) \oplus \left(\overrightarrow{P_2},0\right) = \left(\overrightarrow{P_0}+\overrightarrow{P_2},0\right) \oplus (P_1,2) = (2\overrightarrow{\imath},0) \oplus (P_1,2) = (G_1,2)$ where $G_1\,(1,0)$ and then

$$(Q_2;\varpi_2) = (G_1;1)$$

$(P_0,\omega_0) \oplus (P_1,\omega_1) = \left(\overrightarrow{P_0},0\right) \oplus (P_1,1) = (G_2,1)$ where $G_2\,(1,1)$ and then

$$(Q_1;\varpi_1) = \left(G_2;\frac{\sqrt{2}}{2}\right)$$

Function h_2

The points of this curve are changed into R_0, R_1 and R_2 instead of Q_0, Q_1 and Q_2. The value of $a = b$ leads to

$$(R_0;\varpi_0) = (G_1;1)$$

$(P_1,\omega_1) \oplus (P_2,\omega_2) = (P_1,1) \oplus \left(\overrightarrow{P_2},0\right) = (G_3,1)$ where $G_3\,(1;-1)$ and then

$$(R_1;\varpi_1) = \left(G_3;\frac{\sqrt{2}}{2}\right)$$

and $(R_2;\varpi_2) = (P_2;1)$, Figure 8. The tangents to the curves at the point $Q_0 = R_0$ is the line (Q_1R_1). This last property will always hold true in future constructions.

Second iteration

The quadratic Bézier curve with control mass points (Q_i,ω_i), $i \in [\![0,2]\!]$, is decomposed into two quadratic Bézier curves with control mass points (S_i,ϖ_i), $i \in [\![0,2]\!]$ on the one hand, and (T_i,υ_i), $i \in [\![0,2]\!]$ on the other hand, Figure 8.

- The value of b is the positive solution of the equation

$$b^2\left(0+2\times\frac{\sqrt{2}}{2}+1\right) = 1$$

which leads to

$$b = \sqrt{\sqrt{2} - 1} \simeq 0.644$$

The other coefficients of the homographic parameter change function are $a = 0$, $c = 1$ and $d \simeq 1.287$. The points are $\overrightarrow{S}_0\,(1;1)$, $S_1\,(2.414; 2.414)$ and $S_2\,(1.414, 1.000)$. The weights are $\varpi_0 = 1$, $\varpi_1 \simeq 0.455$ and $\varpi_2 = 1$.

- The coefficients of the homographic parameter change function are $a = \sqrt{\sqrt{2} - 1} \simeq 0.644$, $b = d = 1$, $c \simeq 1.287$. The points are $T_0\,(1.414, 1)$, $T_1\,(1, 0.414)$ et $T_2\,(1;0)$. The weights are $\upsilon_0 = 1$, $\upsilon_1 \simeq 1.099$ and $\upsilon_2 = 1$.

The quadratic Bézier curve with control mass points (R_i, ω_i), $i \in [\![0, 2]\!]$, is decomposed into two quadratic Bézier curves with control mass points (U_i, ϖ_i), $i \in [\![0, 2]\!]$ on the one hand, and (V_i, υ_i), $i \in [\![0, 2]\!]$ on the other hand, Figure 8.

- The coefficients of the homographic parameter change function are $a = 0$, $b \simeq 0.644$, $c = 1$ $d \simeq 1.287$. The points are $U_0\,(1, 0)$, $U_1\,(1, -0.414)$ and $U_2\,(1.414, -1.000)$. The weights are $\varpi_0 = 1$, $\varpi_1 = 1.099$ and $\varpi_2 = 1$.
- The coefficients of the homographic parameter change function are $a \simeq 0.644$, $d = b = 1$ $c \simeq 1.287$. The points are $V_0\,(1.414; -1.000)$, $V_1\,(2.414; -2.414)$ and $\overrightarrow{V}_2\,(1; -1)$. The weights are $\upsilon_0 = 1$, $\upsilon_1 = 0.455$ and $\upsilon_2 = 1$, Figure 8.

Applications in 5-dimensional Minkowski-Lorentz space

The 5-dimensional Minkowski-Lorentz space is a generalization of the space of relativity used by A. Einstein. In this Minkowski-Lorentz space, a Dupin cyclide is represented by two conics [13]: a circle that appears as an ellipse or as a hyperbola, or an isometric parabola with respect to a line. The points on the curve are spheres with the Dupin cyclide being their envelope. The tangent to the conic at a given point defines a sphere, known as the derived sphere, and the intersection of these two spheres is a circle of curvature of the Dupin cyclide [13]. The singular points of a Dupin cyclide correspond to isotropic vectors. By using homographic parameter transformations, it is possible to iteratively construct circles of curvature for Dupin cyclides and patches of these surfaces [18], [19].

3.2.3 Degree 3 case

For quadratic Bézier curves with control points $(P_0, 1)$, (P_1, ω_1), and $(P_2, 1)$, the concept of regular construction arises from the fact that the weighted constructed point $(R_0, 1)$ lies on the median of the triangle formed by P_1. This means that we replace a Bézier curve in standard form with two Bézier curves in standard form. From degree 3, the notion of regularity means that the standard form is preserved.

The Table 5 defines the control mass points using the function h_1 defined by the Equation (13) with $a = 0$, $d = 2b$, and $c = -1$ to keep the first control mass point.

The Table 6 defines the control mass points using the function h_2 defined by the Equation (14) with $c = 2a$ and $b = d = -1$ to keep the first control mass point.

First example : function $x \mapsto x^3$ on $\overline{\mathbb{R}}^+$

The homographic transformation allows us to use Bézier curves defined over the interval $[0, 1]$ instead of a curve parameterized over an unbounded interval, where one of the bounds is $-\infty$ or $+\infty$. First, the conversion from the canonical basis to the appropriate Bernstein basis is performed, where at least one control mass point is a vector. Then, the homographic parameter change is applied, resulting in control mass points. Finally, we use our generalized version of the De Casteljau algorithm.

For example, the control mass points of the cubic Bézier curve which represents the curve (t, t^3), $t \in [0, +\infty]$, using the changement of parameter

$$t = \frac{u}{1 - u}$$

are $P_0\,(0, 0)$ with $\omega_0 = 1$, $\overrightarrow{P_1}\left(\frac{1}{3}, 0\right)$ with $\omega_1 = 0$, $\overrightarrow{P_2} = \overrightarrow{0}$ with $\omega_2 = 0$ and $\overrightarrow{P_3}\,(0, 1)$ with $\omega_3 = 0$. The generalized De Casteljau algorithm is applied to this Bézier curve, Figures 9 and 11.

First iteration, function h_1 to obtain the curve γ_Q

Directly, $(Q_0; 1) = (P_0; 1)$, Figure 9. The value of b must be determined such that the last weight is equal to 1 which leads to the equation

$$-b^3\,(1 + 3 \times 0 + 3 \times 0 + 0) = 1$$

and the solution is

$$b = -1$$

$(P_0; 1) \oplus \left(\overrightarrow{P_1}, 0\right) = (G_1, 1)$ where $G_1\left(\frac{1}{3}, 0\right)$ and then

$$(Q_1; \varpi_1) = (G_1; 1)$$

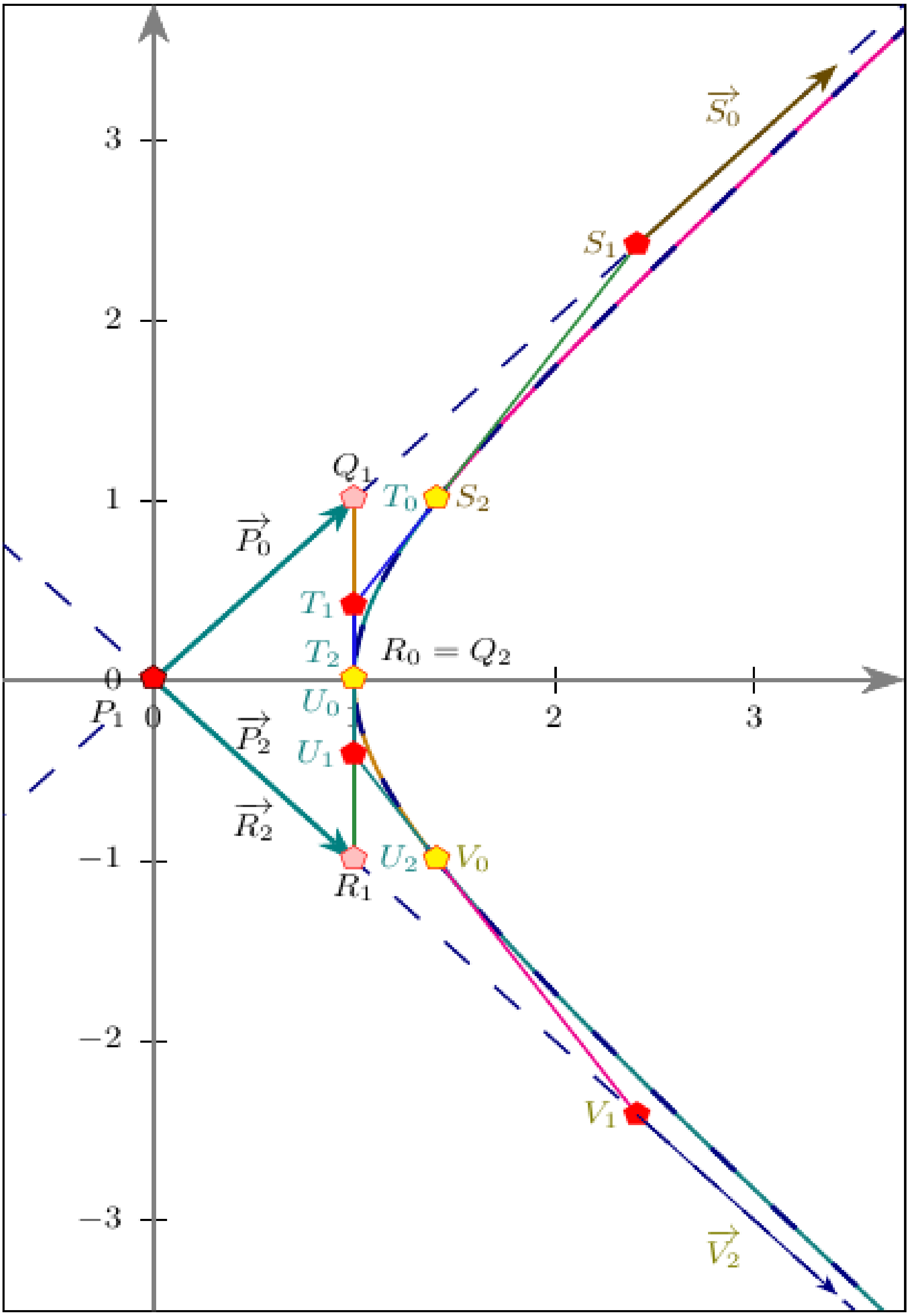

Figure 8: Iterative construction of a rational quadratic Bézier curve with control mass points with homographic parameter change based on De Casteljau algorithm for a branch of a hyperbola.

$$\left\{\begin{array}{rl} (Q_0;\varpi_0) = & (P_0;\omega_0) \\ (Q_1;\varpi_1) = & -b\odot((P_0;\omega_0)\oplus(P_1;\omega_1)) \\ (Q_2;\varpi_2) = & b^2\odot((P_0;\omega_0)\oplus 2\odot(P_1;\omega_1)\oplus(P_2;\omega_2)) \\ (Q_3;\varpi_3) = & -b^3\odot((P_0;\omega_0)\oplus 3\odot(P_1;\omega_1)) \\ \oplus & -b^3\odot(3\odot(P_2;\omega_2)\oplus(P_3;\omega_3)) \end{array}\right. \quad (17)$$

Table 5: Control mass points for the cubic case for h_1.

$$\left\{\begin{array}{rl} (Q_0;\varpi_0) = & -a^3\odot((P_0;\omega_0)\oplus 3\odot(P_1;\omega_1)) \\ \oplus & -a^3(3\odot(P_2;\omega_2)\oplus(P_3;\omega_3)) \\ (Q_1;\varpi_1) = & a^2\odot((P_1;\omega_1)\oplus 2\odot(P_2;\omega_2)\oplus(P_3;\omega_3)) \\ (Q_2;\varpi_2) = & -a\odot((P_2;\omega_2)\oplus(P_3;\omega_3)) \\ (Q_3;\varpi_3) = & (P_3;\omega_3) \end{array}\right. \quad (18)$$

Table 6: Control mass points for the cubic case for h_2.

$$(P_0,1)\oplus 2\odot\left(\overrightarrow{P_1},0\right)\oplus\left(\overrightarrow{P_2},0\right)=(P_0,1)\oplus(\overrightarrow{u},0)$$

where $\overrightarrow{u}\left(\frac{2}{3},0\right)$ and then $\left(\overrightarrow{P_1},0\right)\oplus(\overrightarrow{u},0)=(G_2,1)$ where $G_2\left(\frac{2}{3},0\right)$ and then

$$(Q_2;\varpi_2)=(G_2;1)$$

$$\begin{aligned} &=(P_0,1)\oplus 3\odot\left(\overrightarrow{P_1},0\right)\oplus 3\odot\left(\overrightarrow{P_2},0\right)\oplus\left(\overrightarrow{P_3},1\right) \\ &=(P_0,1)\oplus\left(3\overrightarrow{P_1}+\overrightarrow{P_3},0\right) \\ &=(P_0,2)\oplus(\overrightarrow{v},0) \end{aligned}$$

where $\overrightarrow{v}(1,1)$ and then $(P_0,1)\oplus(\overrightarrow{v},0)=(G_3,1)$ where $G_3(1,1)$ and then

$$(Q_3;\varpi_3)=(G_3;1)$$

First iteration, function h_2 to obtain the curve γ_R

The points of this curve are changed into R_0, R_1, R_2 and R_3 instead of Q_0, Q_1, Q_2 and Q_3. The value of $a=b$ leads to $(R_0;1)=(Q_3;1)$, Figure 9.

$\left(\overrightarrow{P_1},0\right)\oplus 2\odot\left(\overrightarrow{P_2},0\right)\oplus\left(\overrightarrow{P_3},0\right)=\left(\overrightarrow{P_3},0\right)\oplus\left(\overrightarrow{P_1},0\right)=\left(\overrightarrow{G_3},0\right)$ where $\overrightarrow{G_3}\left(\frac{1}{3},1\right)$ and then

$$\left(\overrightarrow{R_1};\varpi_1\right)=\left(\overrightarrow{G_3};0\right)$$

$\left(\overrightarrow{P_2},0\right)\oplus\left(\overrightarrow{P_3};0\right)=\left(\overrightarrow{P_3};0\right)$ and then

$$\left(\overrightarrow{R_2};\varpi_2\right)=\left(\overrightarrow{P_3};0\right)$$

Second iteration

The cubic Bézier curve with control mass points (Q_i,ω_i), $i\in[\![0,3]\!]$, is decomposed into two cubic Bézier curves with control mass points (S_i,ϖ_i), $i\in[\![0,3]\!]$ on the one hand, and (T_i,υ_i), $i\in[\![0,3]\!]$ on the other hand, Figure 10.

- The coefficients of the homographic parameter change function are $a=0$, $b=-\frac{1}{2}$, $c=-1$ and $d=-1$. The points are $S_0(0,0)$, $S_1(0.167,0)$, $S_2(0.333,0)$ and $S_3(0.5,0.125)$. The weights are $\varpi_0=1$, $\varpi_1=1$, $\varpi_2=1$ and $\varpi_3=1$.
- The coefficients of the homographic parameter change function are $a=-\frac{1}{2}$, $b=c=d=-1$. The points are $T_0(0.5,0.125)$, $T_1(0.667,0.25)$, $T_2(0.833,0.5)$ and $T_3(1,1)$. The weights are $\upsilon_0=1$, $\upsilon_1=1$, $\upsilon_2=1$ and $\upsilon_3=1$.

The cubic Bézier curve with control mass points (R_i,ω_i), $i\in[\![0,3]\!]$, is decomposed into two cubic Bézier curves with control mass points (U_i,ϖ_i), $i\in[\![0,3]\!]$ on the one hand, and (V_i,υ_i), $i\in[\![0,3]\!]$ on the other hand, Figure 11.

- The coefficients of the homographic parameter change function are $a=0$, $b=-1$, $c=-1$ $d=-2$. The points are $U_0(1,1)$, $U_1(1.333,2)$, $U_2(1.667,4)$ and $U_3(2,8)$. The weights are $\varpi_0=1$, $\varpi_1=1$, $\varpi_2=1$ and $\varpi_3=1$

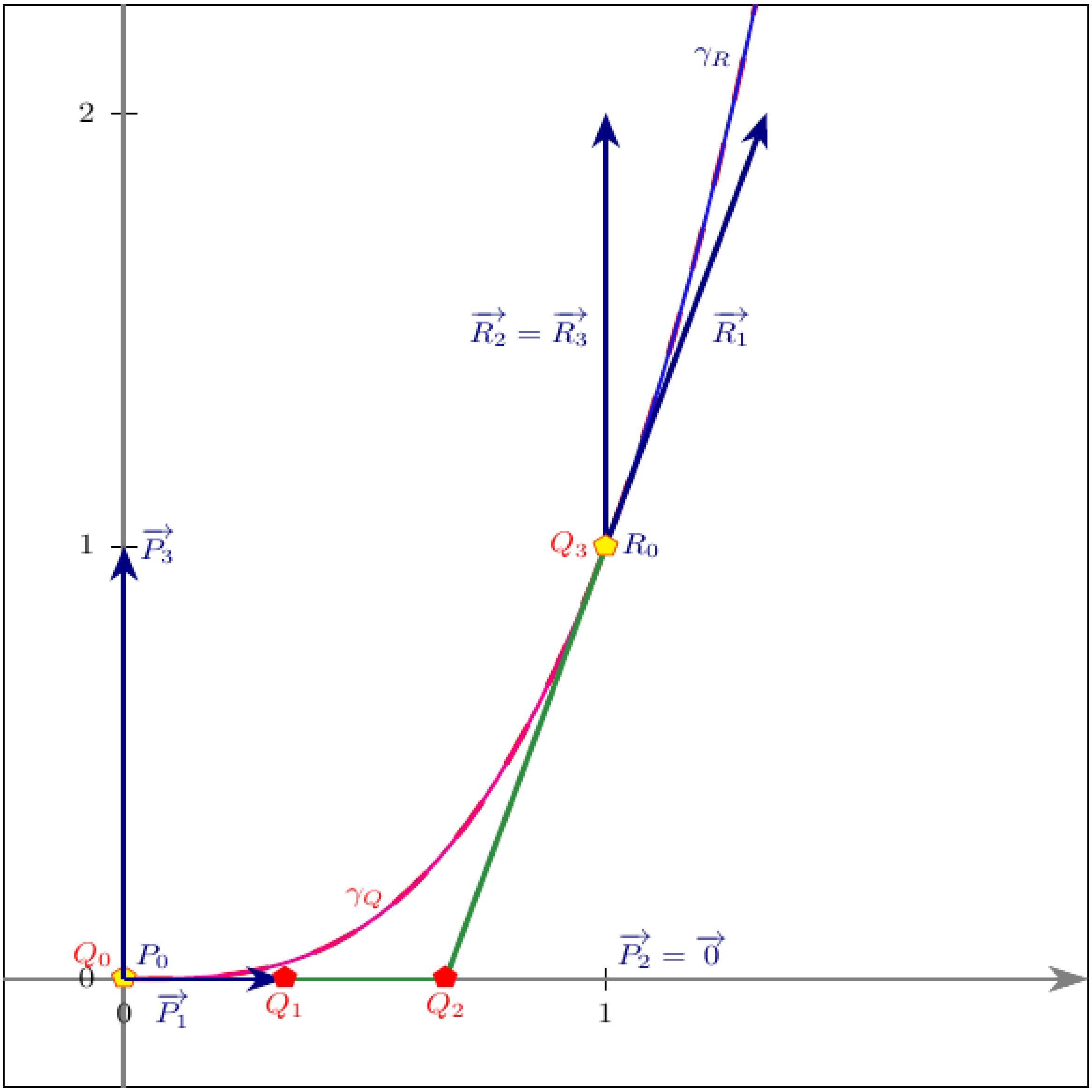

Figure 9: Iterative construction of a rational cubic Bézier curve with control mass points with homographic parameter change based on De Casteljau algorithm for the cubic curve $x \mapsto x^3$ on $[0, +\infty]$.

- The coefficients of the homographic parameter change function are $a = -1$, $d = b = -1$ $c = -2$. The points are $V_0\,(2;8)$, $\overrightarrow{V}_1\,(0.333, 4)$, $\overrightarrow{V}_2\,(0;2)$ and $\overrightarrow{V}_3\,(0;1)$. The weights are $\upsilon_0 = 1$, $\upsilon_1 = 0$, $\upsilon_2 = 0$ and $\upsilon_3 = 0$.

Second example : loop of a Descartes Folium

The generalized De Casteljau algorithm is applied to the loop of the Descartes Folium with parameter $a = 2$. This loop is modeled by the cubic rational Bézier curve γ with control mass points $P_0\,(0;0)$, $\omega_0 = 1$, $\overrightarrow{P_1}\,(2;0)$, $\omega_1 = 0$, $\overrightarrow{P_2}\,(0;2)$, $\omega_2 = 0$ and $P_3 = P_0$, $\omega_3 = 1$, Figure 12.

First iteration, function h_1

Directely, $(Q_0;1) = (P_0;1)$, Figure 12.

The value of b must be determined such that the last weight is equal to 1 which leads to the equation

$$-b^3\,(1 + 3 \times 0 + 3 \times 0 + 1) = 1$$

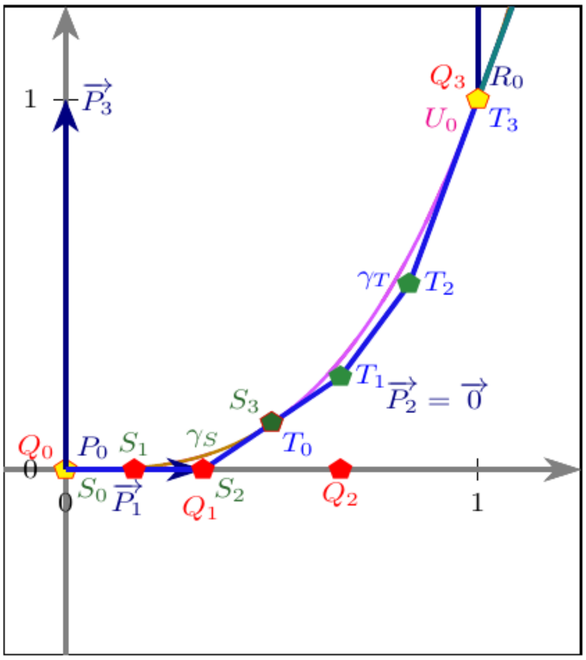

Figure 10: Second iterative construction of the rational cubic Bézier curve γ_Q with control mass points with homographic parameter change based on De Casteljau algorithm for the function $x \mapsto x^3$ on $[0, +\infty]$.

and the solution is

$$b = \frac{-1}{\sqrt[3]{2}}$$

$(P_0; 1) \oplus \left(\overrightarrow{P_1}, 0\right) = (G_1, 1)$ where $G_1\,(2, 0)$ and then

$$(Q_1; \varpi_1) = \left(G_1; \frac{-1}{\sqrt[3]{2}}\right)$$

$(P_0, 1) \oplus 2 \odot \left(\overrightarrow{P_1}, 0\right) \oplus \left(\overrightarrow{P_2}, 0\right) = (P_0, 1) \oplus (\overrightarrow{u}, 0)$ where $\overrightarrow{u}\,(4, 2)$ and then $\left(\overrightarrow{P_1}, 0\right) \oplus (\overrightarrow{u}, 0) = (G_2, 1)$ where $G_2\,(4, 2)$ and then

$$(Q_2; \varpi_2) = \left(G_2; \frac{1}{\sqrt[3]{4}}\right)$$

$(P_0, 1) \oplus 3 \odot \left(\overrightarrow{P_1}, 0\right) \oplus 3 \odot \left(\overrightarrow{P_2}, 0\right) \oplus (P_3, 1) = (P_0, 2) \oplus \left(3\overrightarrow{P_1} + 3\overrightarrow{P_2}, 0\right) = (P_0, 2) \oplus (\overrightarrow{v}, 0)$ where $\overrightarrow{v}\,(6, 6)$ and then $(P_0, 2) \oplus (\overrightarrow{v}, 0) = (G_3, 2)$ where $G_3\,(3, 3)$ and then

$$(Q_3; \varpi_3) = (G_3; 1)$$

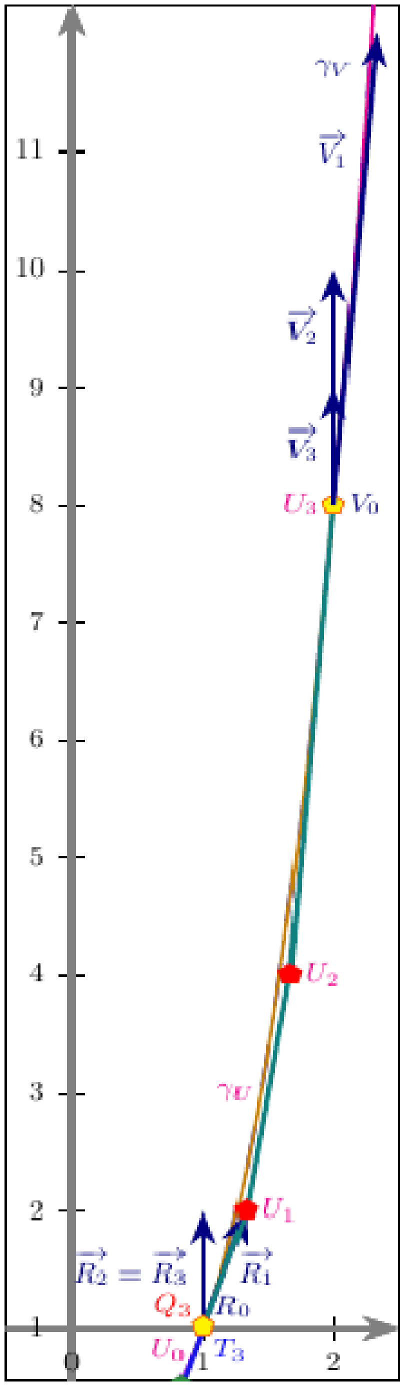

Figure 11: Second iterative construction of the rational cubic Bézier curve γ_R with control mass points with homographic parameter change based on De Casteljau algorithm for the function $x \mapsto x^3$ on $[0, +\infty]$.

First iteration, function h_2

The points of this curve are changed into R_0, R_1, R_2 and R_3 instead of Q_0, Q_1, Q_2 and Q_3. The value of $a = b$ leads to $(R_0; 1) = (Q_3; 1)$, Figure 12.

$\left(\overrightarrow{P_1},0\right)\oplus 2\odot\left(\overrightarrow{P_2},0\right)\oplus(P_3,1)=(P_3,1)\oplus(\overrightarrow{v},0)$ where $\overrightarrow{v}\,(2,4)$ and then $(P_3,1)\oplus(\overrightarrow{v},0)=(G_3,1)$ where $G_3\,(2,4)$ and then

$$(R_1;\varpi_1)=\left(G_3;\frac{1}{\sqrt[3]{4}}\right)$$

$\left(\overrightarrow{P_2},0\right)\oplus(P_3;1)=(G_4,1)$ where $G_4\,(0,2)$ and then

$$(R_2;\varpi_2)=\left(G_4;\frac{-1}{\sqrt[3]{2}}\right)$$

Second iteration

The cubic Bézier curve with control mass points (Q_i,ω_i), $i\in[\![0,3]\!]$, is decomposed into two cubic Bézier curves with control mass points (S_i,ϖ_i), $i\in[\![0,3]\!]$ on the one hand, and (T_i,υ_i), $i\in[\![0,3]\!]$ on the other hand, Figure 12.

- The coefficients of the homographic parameter change function are $a=0$, $b\simeq 0.542$, $c=1$ and $d\simeq 1.085$. The points are $S_0\,(0,0)$, $S_1\,(0.885,0)$, $S_2\,(1.770,0.392)$ and $S_3\,(2.443,1.081)$. The weights are $\varpi_0=1$, $\varpi_1\simeq 0.973$, $\varpi_2\simeq 0.946$ and $\varpi_3=1$.

- The coefficients of the homographic parameter change function are $a\simeq 0.542$, $b=d=1$, $c\simeq 1.085$. The points are $T_0\,(2.443,1.081)$, $T_1\,(3.153,1.808)$, $T_2\,(3.386,2.614)$ and $T_3\,(3;3)$. The weights are $\upsilon_0=1$, $\upsilon_1\simeq 0.898$, $\upsilon_2\simeq 0.884$ and $\upsilon_3=1$.

The cubic Bézier curve with control mass points (R_i,ω_i), $i\in[\![0,3]\!]$, is decomposed into two cubic Bézier curves with control mass points (U_i,ϖ_i), $i\in[\![0,3]\!]$ on one hand, and (V_i,υ_i), $i\in[\![0,3]\!]$ on the other hand, Figure 12.

- The coefficients of the homographic parameter change function are $a=0$, $b\simeq -0.542$, $c=-1$ $d\simeq -1.085$. The points are $U_0\,(3;3)$, $U_1\,(2.614,3.386)$, $U_2\,(1.808,3.153)$ and $U_3\,(1.081,2.443)$. The weights are $\varpi_0=1$, $\varpi_1\simeq 0.884$, $\varpi_2\simeq 0.898$ and $\varpi_3=1$.

- The coefficients of the homographic parameter change function are $a\simeq -0.542$, $d=b=-1$ $c\simeq -1.085$. The points are $V_0\,(1.081,2.443)$, $V_1\,(0.391,1.770)$, $V_2\,(0;0.885)$ and $V_3\,(0;0)$. The weights are $\upsilon_0=1$, $\upsilon_1=0.946$, $\upsilon_2=0.973$ and $\upsilon_3=1$.

3.2.4 Degree 4 case

The Table 7 defines the control mass points using the function h_1 defined by the Equation (13) with $a=0$, $d=2b$ and $c=1$ to keep the first control mass point.

The Table 8 defines the control mass points using the function h_2 defined by the defined by the Equation (14) with $c=2a$ and $b=d=1$ to keep the first control mass point.

The generalized De Casteljau algorithm is applied to the loop of the Bernouilli Lemniscate. This loop is modeled by the quartic rational Bézier curve γ with control mass points $P_0\,(0;0)$, $\omega=1$, $\overrightarrow{P_1}\left(\frac{1}{4};\frac{1}{4}\right)$, $\omega_1=0$, $\overrightarrow{P_2}=\overrightarrow{0}$, $\omega_2=0$, $\overrightarrow{P_3}\left(\frac{1}{4};-\frac{1}{4}\right)$, $\omega_3=0$ and $P_4=P_0$, $\omega_4=1$, Figure 13.

First iteration

Function h_1

Directely, $(Q_0;1)=(P_0;1)$, Figure 13.

The value of b must be determined such that the last weight is equal to 1 which leads to the equation

$$b^4\,(1+4\times 0+6\times 0+4\times 0+1)=1$$

and the positive solution is

$$b=\frac{1}{\sqrt[4]{2}}$$

$(P_0;1)\oplus\left(\overrightarrow{P_1},0\right)=(G_1,1)$ where $G_1\left(\frac{1}{4};\frac{1}{4}\right)$ and then

$$(Q_1;\varpi_1)=\left(G_1;\frac{1}{\sqrt[4]{2}}\right)$$

$(P_0,1)\oplus 2\odot\left(\overrightarrow{P_1},0\right)\oplus\left(\overrightarrow{P_2},0\right)=(P_0,1)\oplus(\overrightarrow{u},0)$ where $\overrightarrow{u}\left(\frac{1}{2};\frac{1}{2}\right)$ and then $\left(\overrightarrow{P_1},0\right)\oplus(\overrightarrow{u},0)=(G_2,1)$ where $G_2\left(\frac{1}{2};\frac{1}{2}\right)$ and then

$$(Q_2;\varpi_2)=\left(G_2;\frac{1}{\sqrt{2}}\right)$$

$$\begin{aligned}&=(P_0,1)\oplus 3\odot\left(\overrightarrow{P_1},0\right)\oplus 3\odot\left(\overrightarrow{P_2},0\right)\oplus\left(\overrightarrow{P_3},0\right)\\&=(P_0,1)\oplus\left(3\overrightarrow{P_1}+\overrightarrow{P_2},0\right)\\&=(P_0,1)\oplus(\overrightarrow{v},0)\end{aligned}$$

where $\overrightarrow{v}\left(1,\frac{1}{2}\right)$ and then $(P_0,1)\oplus(\overrightarrow{v},0)=(G_3,1)$ where $G_3\left(1,\frac{1}{2}\right)$ and then

$$(Q_3;\varpi_3)=\left(G_3;\frac{1}{\sqrt[4]{8}}\right)$$

Then

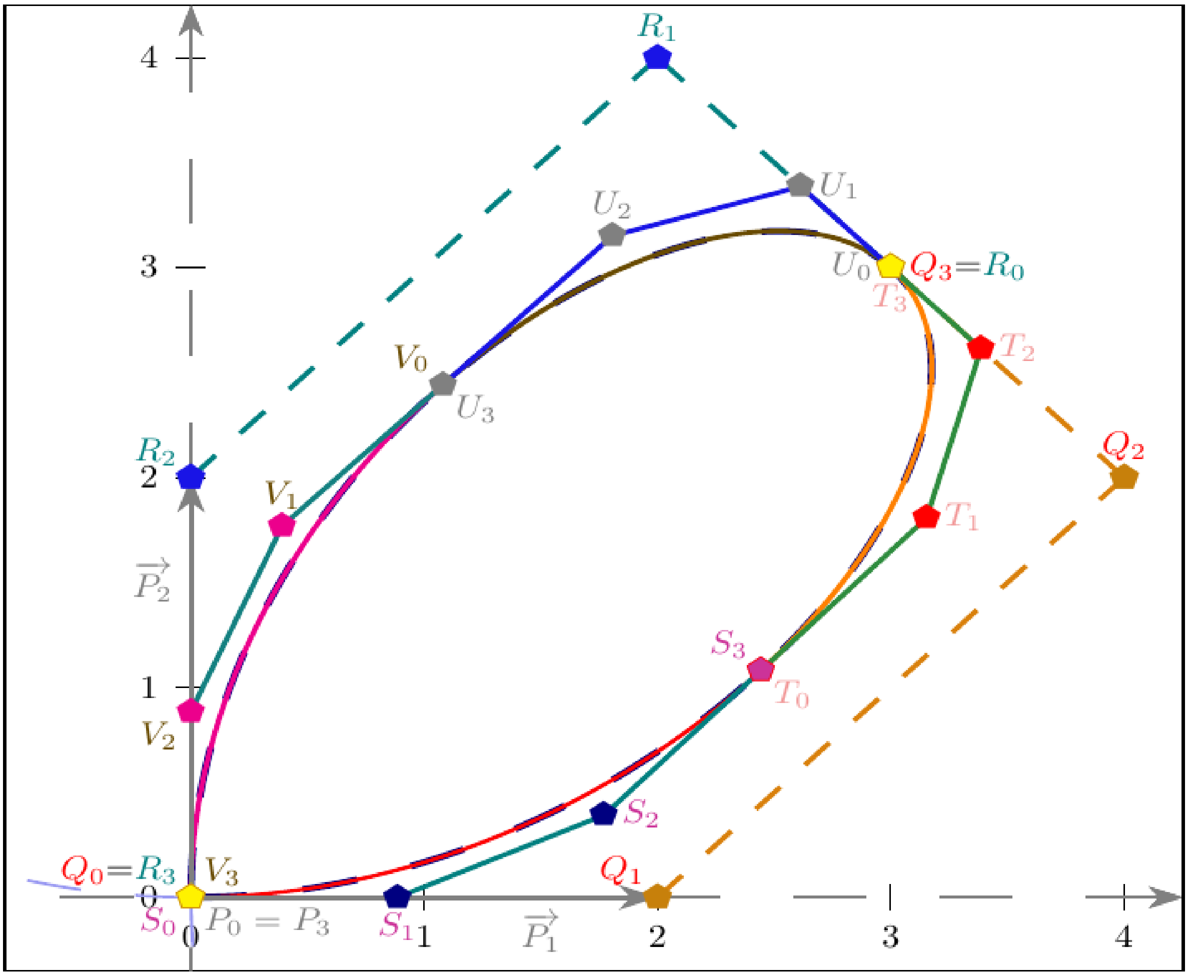

Figure 12: Second iterative construction of a rational cubic Bézier curve with control mass points with homographic parameter change based on De Casteljau algorithm for the loop of a Descartes Folium.

$$\begin{aligned} & (P_0,1)\oplus 4\odot\left(\overrightarrow{P_1},0\right)\oplus 6\odot\left(\overrightarrow{P_2},0\right)\\ \oplus\quad & 4\odot\left(\overrightarrow{P_3},0\right)\oplus(P_0,1)\\ =\quad & (P_0,2)\oplus\left(3\overrightarrow{P_1}+3\overrightarrow{P_3},0\right)\\ =\quad & (P_0,2)\oplus(2\overrightarrow{\imath},0)\\ =\quad & (G_4,1)\end{aligned}$$

where $G_4\,(1,0)$ and then

$$(Q_4;\varpi_4)=(G_4;1)$$

Function h_2

The points of this curve are changed into R_0, R_1, R_2, R_3 and R_4 instead of Q_0, Q_1, Q_2, Q_3 and Q_4. Then

$$a=\frac{1}{\sqrt[4]{2}}$$

We have $(R_0;1)=(Q_4;1)$, Figure 13.

$\left(\overrightarrow{P_1},0\right)\oplus 3\odot\left(\overrightarrow{P_2},0\right)\oplus 3\odot\left(\overrightarrow{P_3},0\right)\oplus(P_4,1)=(P_4,1)\oplus(\overrightarrow{v},0)$ where $\overrightarrow{v}\,(1,\frac{1}{2})$ and then $(P_3,1)\oplus(\overrightarrow{v},0)=(G_5,1)$ where $G_5\,(1,\frac{1}{2})$ and then

$$(R_1;\varpi_1)=\left(G_5;\frac{1}{\sqrt[4]{8}}\right)$$

$\left(\overrightarrow{P_2},0\right)\oplus 2\odot\left(\overrightarrow{P_3},0\right)\oplus(P_4;1)=(G_6,1)$ where $G_6\,(\frac{1}{2},-\frac{1}{2})$ and then

$$(R_2;\varpi_2)=\left(G_6;\frac{1}{\sqrt{2}}\right)$$

$\left(\overrightarrow{P_3},0\right)\oplus(P_4;1)=(G_7,1)$ where $G_7\,(\frac{1}{4},-\frac{1}{4})$ and then

$$(R_3;\varpi_3)=\left(G_7;\frac{1}{\sqrt[4]{2}}\right)$$

Table 7: Control mass points for the quartic case for h_1.

$$\left\{\begin{array}{rl}(Q_0;\varpi_0)= & (P_0;\omega_0)\\ (Q_1;\varpi_1)= & b\odot((P_0;\omega_0)\oplus(P_1;\omega_1))\\ (Q_2;\varpi_2)= & b^2\odot((P_0;\omega_0)\oplus 2\odot(P_1;\omega_1)\oplus(P_2;\omega_2))\\ (Q_3;\varpi_3)= & b^3\odot((P_0;\omega_0)\oplus 3\odot(P_1;\omega_1)\oplus 3\odot(P_2;\omega_2))\\ \oplus & b^3\odot(P_3;\omega_3)\\ (Q_4;\varpi_4)= & b^4\odot((P_0;\omega_0)\oplus 4\odot(P_1;\omega_1)\oplus 6\odot(P_2;\omega_2))\\ \oplus & b^4\odot(4\odot(P_3;\omega_3)\oplus(P_4;\omega_4))\end{array}\right. \tag{19}$$

Table 8: Control mass points for the quartic case for h_2.

$$\left\{\begin{array}{rl}(Q_0;\varpi_0)= & a^4\odot((P_0;\omega_0)\oplus 4\odot(P_1;\omega_1)\oplus 6\odot(P_2;\omega_2))\\ \oplus & a^4\odot(4\odot(P_3;\omega_3)\oplus(P_4;\omega_4))\\ (Q_1;\varpi_1)= & a^3\odot((P_1;\omega_1)\oplus 3\odot(P_2;\omega_2)\oplus 3\odot(P_3;\omega_3)\oplus)\\ \oplus & a^3\odot(P_4;\omega_4)\\ (Q_2;\varpi_2)= & a^2\odot((P_2;\omega_2)\oplus 2\odot(P_3;\omega_3)\oplus(P_4;\omega_4))\\ (Q_3;\varpi_3)= & a\odot((P_3;\omega_3)\oplus(P_4;\omega_4))\\ (Q_4;\varpi_4)= & (P_4;\omega_4)\end{array}\right. \tag{20}$$

and

$$(R_4;\varpi_4)=(P_4;1)$$

Second iteration

The quartic Bézier curve with control mass points (Q_i,ω_i), $i\in[\![0,4]\!]$, is decomposed into two quartic Bézier curves with control mass points (S_i,ϖ_i), $i\in[\![0,4]\!]$ on the one hand, and (T_i,υ_i), $i\in[\![0,4]\!]$ on the other hand, Figure 13.

- The coefficients of the homographic parameter change function are $a=0$, $b\simeq 0.537$, $c=1$ and $d\simeq 1.075$. The points are $S_0\,(0;0)$, $S_1\,(0.114,0.114)$, $S_2\,(0.228,0.228)$, $S_3\,(0.366,0.319)$ and $S_4\,(0.529,0.346)$. The five weights are $\varpi_0=1$, $\varpi_1\simeq 0.989$, $\varpi_2\simeq 0.979$,$\varpi_3\simeq 0.969$ and $\varpi_4=1$.
- The coefficients of the homographic parameter change function are $a\simeq 0.537$, $b=d=1$, $c\simeq 1.075$. The points are $T_0\,(0.529,0.346)$, $T_1\,(0.706,0.376)$, $T_2\,(0.878,0.327)$, $T_3\,(1;0.186)$ and $T_4\,(1;0)$. The five weights are $\upsilon_0=1$, $\upsilon_1\simeq 0.892$, $\upsilon_2\simeq 0.837$, $\upsilon_3\simeq 0.857$ and $\upsilon_4=1$.

The quartic Bézier curve with control mass points (R_i,ω_i), $i\in[\![0,4]\!]$, is decomposed into two quartic Bézier curves with control mass points (U_i,ϖ_i), $i\in[\![0,4]\!]$ on the one hand, and (V_i,υ_i), $i\in[\![0,4]\!]$ on the other hand, Figure 13.

- The coefficients of the homographic parameter change function equals to

  $a=0$, $b\simeq 0.538$, $c=1$ and $d\simeq 1.075$. The points are $U_0\,(1,0)$, $U_1\,(1,-0.186)$, $U_2\,(0.879,-0.327)$, $U_3\,(0.706,-0.376)$ and $U_4\,(0.529,-0.346)$. The weights are $\varpi_0=1$, $\varpi_1\simeq 0.857$, $\varpi_2\simeq 0.837$, $\varpi_3\simeq 0.892$ and $\varpi_3=1$
- The coefficients of the homographic parameter change function are $a\simeq 0.537$, $d=b=1$ and $c\simeq 1.075$. The points are $V_0\,(0.530,-0.346)$, $V_1\,(0.366,-0.319)$, $V_2\,(0.228,-0.228)$, $V_3\,(0.114,-0.114)$ and $V_4\,(0;0)$. The weights are $\upsilon_0=1$, $\upsilon_1\simeq 0.967$, $\upsilon_2\simeq 0.979$, $\upsilon_3\simeq 0.989$ and $\upsilon_4=1$.

4 Conclusion and outlook

In this article, firstly, we generalized the De Casteljau algorithm to (rational) Bézier curves with control mass points. Secondly, we utilized a homographic change theorem to subdivide a

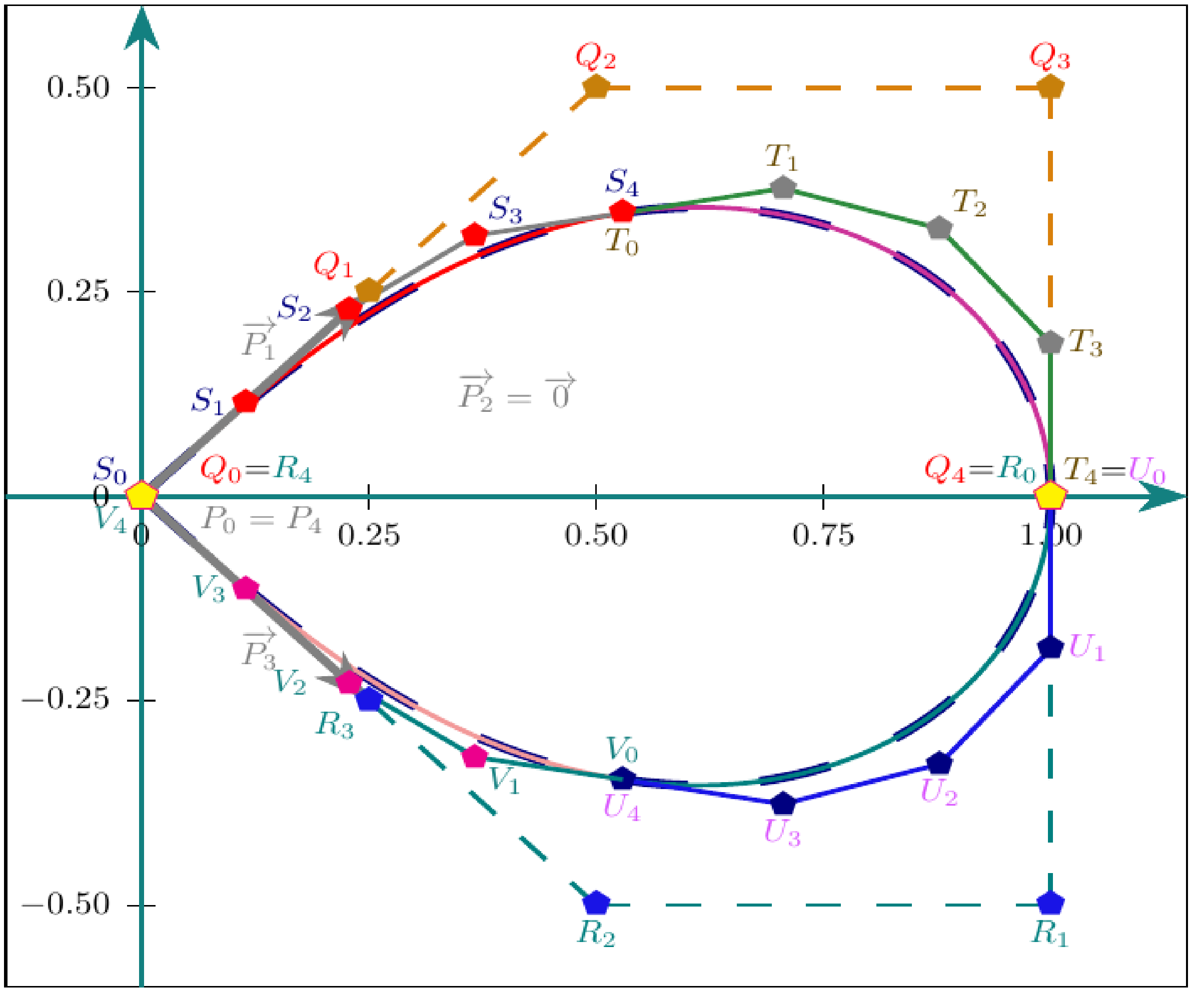

Figure 13: Iterative construction of a rational quartic Bézier curve with control mass points with homographic parameter change based on De Casteljau algorithm for the loop of a Bernouilli Lemniscate.

Bézier curve into two Bézier curves of the same degree. To achieve this, we mapped the interval $[0,1]$ to $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, respectively. We applied these subdivisions to centered conics, the Descartes Folium loop, and the Bernoulli Lemniscate loop, with one weighted control point being the null vector.

In the future, we plan to work on the kinematics of Bézier curves by controlling velocity vectors at the endpoints of the curve using a quadratic parameter change. Additionally, we intend to explore Bézier curves in the plane with complex weights.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.