Global Mapping Properties of Some Functions of Class S

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Abstract: The *Lemma of Schwarz* is one of the most surprising results in complex analysis in the sense that some very weak conditions on an analytic function in the unit disk $|z| < 1$ imply a very strict behavior of that function in the respective disk. What about the behavior of the function outside the unit disk? This is the question we deal with in this paper. The theory we presented in some previous publications was about univalent functions, not necessarily in the unit disk, but in the most general setting, namely in the *fundamental domains* of arbitrary analytic functions. Naturally, connections can be expected between the two fields of complex analysis. The purpose of this paper is to explore these connections and take advantage of the well established theory of univalent functions in order to advance the theory of fundamental domains.

Key-Words: univalent functions, Schwarz lemma, fundamental domains, Bieberbach theorem, conformal mapping, analytic continuation

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1 Introduction

Let $f(s)$ be an analytic function in $\overline{\mathbb{C}}$ with the exception of isolated singular points, which can be poles or essential singular points. Let Ω be a fundamental domain of $f(s)$, i.e., a domain which is conformally mapped by f onto the whole complex plane with a *slit* L. [1], has pointed out the importance of the study of these domains in the Riemann surface theory attached to the respective analytic functions. It is known that for any such function we have

$$\overline{\mathbb{C}} = \cup_{k=1}^{n\leq\infty}\overline{\Omega_k},$$

where Ω_k are open connected sets, $\Omega_k \cap \Omega_j = \varnothing$, for $j \neq k$ and every Ω_k is a fundamental domain of f.

If f is a rational function of a degree n, then f has exactly n fundamental domains, otherwise f has infinitely many fundamental domains. Based on the theorem of boundary correspondence in the conformal mapping, [2] and [3], the function f, which is defined also on the boundary $\partial\Omega_k$ of every fundamental domain Ω_k, maps $\partial\Omega_k$ onto a slit L_k. Reciprocally, if L_k is seen as having two sides, then since this mapping is one to one and hence the inverse function $f^{-1}_{|\Omega}$ exists, it can be extended by continuity to the sides of L_k. We will continue to denote by $f_{|\Omega_k}$ and $f^{-1}_{|\Omega_k}$ these extended functions.

If $M(z)$ is a Möbius transformation and Ω is a fundamental domain of the analytic function f, then the function

$$\chi_M(s) = f^{-1}_{|\Omega} \circ M \circ f(s),$$

where it is defined, is a conformal mapping. Indeed, the function f maps conformally the domain Ω onto the complex plane with a slit L. The Möbius transformation M carries L into a slit L', which is the image by f of a slit L_M of Ω. Also, M^{-1} carries L into a slit L'', which is the image by f of a slit $L_{M^{-1}}$ of Ω. We have proved in [4] the following:

Proposition 1. The function χ_M is a conformal mapping of $\Omega\backslash L_{M^{-1}}$ onto $\Omega\backslash L_M$. The boundaries $\partial\Omega\cup L_M$ and $\partial\Omega\cup L_{M^{-1}}$ of the two double connected domains correspond to each other through χ_M in the following way: $\partial\Omega$ is carried onto L_M and $L_{M^{-1}}$ is carried onto $\partial\Omega$.

In what follows, we will choose M to be the following Möbius transformation

$$M(z) = \frac{z-a}{1-\overline{a}z}e^{i\theta}, \tag{1}$$

for $|a| < 1$, $\theta \in \mathbb{R}$, which maps the unit disk onto itself, the exterior of the unit disk onto itself and the unit circle onto itself.

Proposition 2 (Schwarz lemma). If $f(z)$ is analytic for $|z| < 1$ and satisfies the conditions

$$|f(z)| \leq 1, \quad f(0) = 0,$$

then

$$|f(z)| \leq |z| \text{ and } |f'(0)| \leq 1.$$

If $|f(z)| = |z|$ for some $z \neq 0$, then

$$f(z) = e^{i\varphi}z,$$

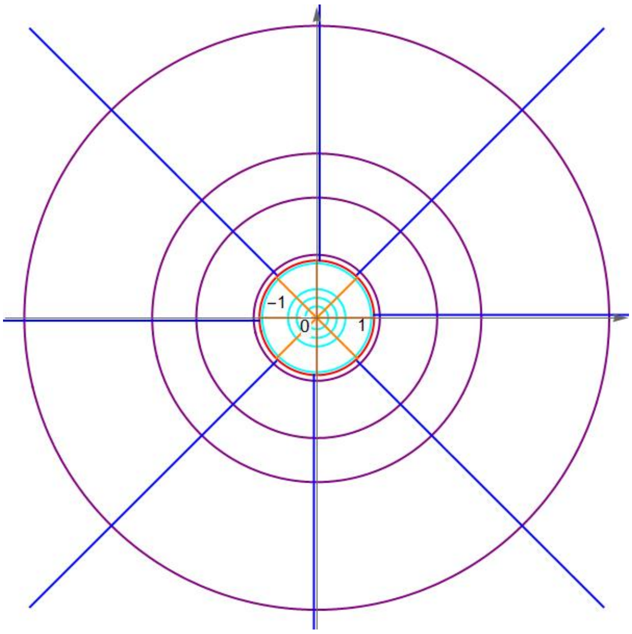

Figure 1: An orthogonal net of circles and rays which generates orthogonal nets for analytic functions in their fundamental domains

where $\varphi \in \mathbb{R}$.

It is now obvious that we preferred the Möbius transformation (1) since it preserves the unit disk, in which the Schwarz lemma holds.

We will use this property, illustrated in Fig. 1 above, to draw the graphs in the next figures illustrating various properties of the univalent functions.

Theorem 1. If $z = f(s)$ satisfies the conditions of Schwarz lemma and $M(z)$ is the Möbius transformation (1) then for the function $\varphi(s) = M(f(s))$ we have:

$$|\varphi(s)| \le \frac{|s| + |a|}{1 - |as|}$$

and

$$|\varphi'(0)| \le 1 - |a|^2.$$

Moreover,

$$\varphi'(0) = 1 - |a|^2$$

if and only if $f'(0) = 1$.

Proof: Indeed,

$$|\varphi(s)| = \left|\frac{f(s) - a}{1 - \overline{a}f(s)}\right| \le \frac{|f(s)| + |a|}{1 - |a||f(s)|} \le \frac{|s| + |a|}{1 - |as|}.$$

On the other hand, we have

$$M'(z) = \frac{(1 - |a|^2)e^{i\theta}}{(1 - \overline{a}z)^2}$$

hence

$$\varphi'(s) = \frac{(1 - |a|^2)e^{i\theta} f'(s)}{(1 - \overline{a}f(s))^2},$$

thus,

$$|\varphi'(0)| = (1 - |a|^2)|f'(0)| \le 1 - |a|^2.$$

If

$$\varphi'(0) = 1 - |a|^2,$$

then $f'(0) = 1$ and reciprocally, $f'(0) = 1$ implies $\varphi'(0) = 1 - |a|^2$. $\square$

A generalization of Schwarz lemma, [1], can be formulated as follows:

Proposition 3. If $u(z)$ is analytic for $|z| < R$ and satisfies the conditions $|u(z)| \le K$, $u(z_0) = w_0$ for $|z_0| < R$, where $|w_0| < K$, then

$$\frac{K|u(z) - w_0|}{|K^2 - \overline{w}_0 u(z)|} \le \frac{R|z - z_0|}{|R^2 - \overline{z}_0 z|}. \tag{2}$$

Let $a = re^{i\alpha}$, where $0 < r < 1$ and $\alpha \in \mathbb{R}$. With

$$u(z) = (1 - r^2)M(z + a) = \frac{(1 - r^2)z}{(1 - r^2 - re^{-i\alpha}z)},$$

we have

$$u(0) = 0, \quad u'(z) = \frac{(1 - r^2)^2}{(1 - r^2 - re^{-i\alpha}z)^2}$$

and

$$u^{'}(0) = (1 - r^2) < 1.$$

We can let $u(z)$ be defined in the disk $|z| < R$, where $R \ge \frac{1}{r}$. Then, by writing the generalized Schwarz lemma for $u(z)$ and $z_0 = w_0 = 0$, the inequality (2) becomes:

$$|u(z)| \ \le \frac{K}{R}|z|, \tag{3}$$

which in terms of the function $M(z)$ is:

$$|M(z + a)| \le \frac{K}{(1 - r^2)R}|z|. \tag{4}$$

We can find a value for K if we plug

$$z = z(t) = R(\cos t + i \sin t),$$

into

$$M(z + a) = \frac{ze^{i\theta}}{(1 - |a|^2 - \overline{a}z)}$$

and look for the maximum of

$$|M(z(t) + a)| = \frac{R|e^{i(t+\theta)}|}{|1 - r^2 - rRe^{i(t-\alpha)}|}.$$

An elementary computation shows that this value is reached for $t = \alpha$ and it is

$$K = \frac{R}{|1 - r^2 - rR|},$$

thus

$$\frac{K}{(1-r^2)R} = \frac{1}{(1-r^2)(r^2+rR-1)}$$

and we can state:

Theorem 2. For any analytic function $f(s)$ and any one of its fundamental domains Ω we have:

$$|M(f(s)+a)| \leq \frac{|f(s)|}{(1-r^2)(r^2+rR-1)}, \tag{5}$$

for $s \in \Omega$ and $|f(s)| < R$, with equality at the point s_0 for which $f(s_0) = 0$. Such a point s_0, if it exists, it is unique. For any other point s with $|f(s)| < R$, the inequality (5) is strict.

Proof: Indeed, with the value we have found for K, replacing z by $f(s)$ in inequality (4), it becomes (5). Since $M(a) = 0$, if $f(s_0) = 0$ we have 0 in both terms of (5) for $s = s_0$. The existence of s_0 is not guaranteed, as the functions e^s and $\Gamma(s)$ show. However, if it exists, due to the fact that f is injective in Ω, its uniqueness is granted. On the other hand, if s_0 exists such that $f(s_0) = 0$, we need to show that the inequality is strict, for any other point s. For any $\rho > 0$, the image by $M(z)$ of the circle

$$C_\rho : z(t) = \rho e^{it}, t \in [0, 2\pi)$$

is a circle Γ_ρ. If $\rho < R$, then, due to the univalence of $M(z)$, the circle Γ_ρ is strictly interior to the circle Γ_R. By the *maximum principle*, [1], no value of $|M(f(s)+a)|$ on Γ_ρ can be equal to K, which is the maximum value of $|M(f(s)+a)|$ on Γ_R. Hence the inequality (5) is strict for any s with $|f(s)| < R$, $s \neq s_0$, which completely proves the theorem. $\square$

Since

$$M(f(s)+a) = \frac{f(s)e^{i\theta}}{(1 - r^2 - re^{-i\alpha}f(s))}$$

the inequality (5) is equivalent to

$$\frac{|f(s)|}{|1 - r^2 - re^{-i\alpha}f(s)|} \leq \frac{|f(s)|}{(1-r^2)(r^2+rR-1)} \tag{6}$$

which is, for $s \neq s_0$:

$$|1 - r^2 - re^{-i\alpha}f(s)| > (1-r^2)(r^2+rR-1) \tag{7}$$

The inequality (5) reveals a connection between the image by f of a point $s \in \Omega$ with $|f(s)| \leq R$ and $M(f(s)+a)$. The strict inequalities in (6) and (7) are satisfied for every such s, if $s \neq s_0$.

In [4], the conformal mapping of the complex plane by a function (1) with

$$a = \frac{1}{2}(1+i) = \frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}} \text{ and } \theta = 0$$

is illustrated. We have $r = \frac{\sqrt{2}}{2}$, hence for $R = \sqrt{2}$ we obtain $K = 2\sqrt{2}$, thus, with this data the inequality (5) becomes

$$\left| M\left(f(s) + \frac{1}{2}(1+i) \right) \right| \leq 4|f(s)| \tag{8}$$

and the inequality (7) becomes

$$|1 - \sqrt{2}(1-i)f(s)| \geq 8. \tag{9}$$

For every one of the functions f studied in [4] and [5] we can indicate the pre-image by f of the disk centered at the origin of radius $\sqrt{2}$. The inequalities (8) and (9) take place in all these pre-images.

2 The Conformal Self-Mapping of the Fundamental Domains of the Exponential Function

It is known, [4] and [5] that the fundamental domains of the exponential function are strips

$$\Omega_k = \{s = \sigma + it | 2k\pi < t < 2(k+1)\pi, \sigma \in \mathbb{R}\},$$

for $k \in \mathbb{Z}$.

For every Möbius transformation M as that given in (1) with $\theta = 0$, the function

$$\chi(s) = Log_{|\Omega_k}(M(e^s + a)),$$

where $Log_{|\Omega_k}$ is the branch of the multivalued function Log corresponding to Ω_k, is a conformal self-mapping of Ω_k, as in Proposition 1. The function e^s maps conformally this strip onto the complex plane with a slit L alongside the positive real half axis. Every segment $s = \sigma_0 + it$, $2k\pi \leq t < 2(k+1)\pi$, is mapped one to one by e^s onto the circle centered at the origin and of radius e^{σ_0} and the half strip corresponding to $\sigma < 0$ is mapped conformally onto the interior of this circle while the half strip corresponding to $\sigma > 0$ is mapped conformally onto the exterior of this circle. Moreover, every line

$$s = \sigma + it_0, \sigma \in \mathbb{R}, t_0 \in [2k\pi, 2(k+1)\pi]$$

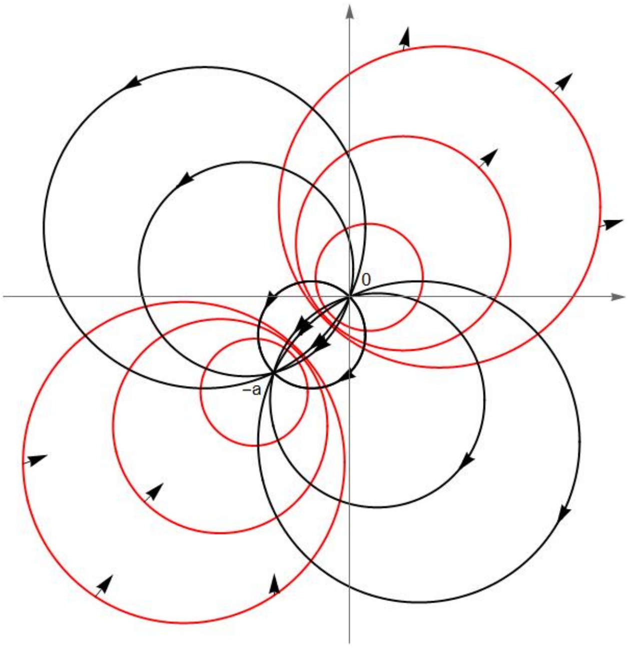

Figure 2: The Steiner net of the Möbius transformation $M(z+a)$, where M is given by formula (1) and $a = \frac{1}{2}(1+i)$

is mapped one to one by e^s onto the ray $\arg z = t_0$. When $t_0 = 2k\pi$ and $t_0 = (2k+1)\pi$, the ray is the same, namely the positive real half axis, which is the slit L.

By the inequality (5) we have

$$|M(e^s+a)| < \frac{e^\sigma}{(1-r^2)(r^2+rR-1)} < \frac{e^\sigma}{r^2(1-r^2)}, \tag{10}$$

for every $s = \sigma + it$. This is a strict inequality, since there is no s_0 such that $e^{s_0} = 0$. It is known, [4], that the fixed points of

$$M(z) = \frac{(z - re^{i\alpha})}{(1 - re^{-i\alpha}z)}$$

are $\xi_1 = e^{i\alpha}$ and $\xi_2 = -e^{i\alpha}$. To find the fixed points of

$$M(z+a) = \frac{z}{(1 - r^2 - re^{-i\alpha}z)},$$

we need to solve the equation $M(z+a) = z$. For $\alpha = \frac{\pi}{4}$, an easy computation gives $z_1 = 0$ and $z_2 = -re^{i\frac{\pi}{4}}$. The last point is on the ray through the origin making an angle $\frac{5\pi}{4}$ with the positive real half axis.

The Steiner net of the Möbius transformation $M(z + re^{i\frac{\pi}{4}})$, Fig. 2, is formed with the Apollonius circles

$$|z - c_1| = r \text{ and } |z - c_1'| = r$$

and the orthogonal circles passing through 0 and $-re^{i\frac{\pi}{4}}$, which are

$$|z - c_2| = r \text{ and } |z - c_2'| = r,$$

where

$$\begin{aligned}
c_1 &= \frac{\sqrt{2}}{4}(\sqrt{8r^2+1}-1)e^{i\frac{\pi}{4}},\\
c_1' &= -(re^{i\frac{\pi}{4}} + c_1),\\
c_2 &= \frac{1}{4}\left[(\sqrt{8r^2-1}-1) - i(\sqrt{8r^2-1}+1)\right],\\
c_2' &= -\left(\frac{\sqrt{2}}{4}e^{i\frac{5\pi}{4}} - c_2\right).
\end{aligned}$$

The pre-image by e^s of z_2 is located on the line

$$s = \sigma + i\left(\frac{\pi}{4} + \pi\right), \sigma \in \mathbb{R},$$

and it is $s_2 = \ln r + \frac{5\pi}{4}i$, while z_1 has no pre-image since it is a lacunary value for e^z. However, it is obvious that if $\sigma \to -\infty$ on any line $\sigma + it$ then $e^{\sigma+it} \to 0 = z_1$.

The Ω_k-Steiner net, Fig. 3, corresponding to this Möbius transformation and to the exponential function together with the inequality (10) allow for an accurate description of the conformal self-mapping $\chi(s) = Log_{|\Omega_k}(M(e^s + re^{i\frac{\pi}{4}}))$.

The configuration above changes drastically when instead of $M(z+a)$ we take the function

$$u(z) = (1-r^2)M(z+a) = \frac{(1-r^2)z}{(1-r^2-re^{-i\alpha}z)}.$$

Indeed, it can be easily seen that $u(z)$ is a parabolic Möbius transformation with the unique fixed point $z = 0$. Then $\chi(s) = Log_{|\Omega_k}(u(e^s))$ has no fixed point, as it can be seen in Fig. 4.

In the case of Fig. 2, the circles of the Steiner nets are given by

$$\begin{aligned}
&\left|z \pm \frac{\sqrt{2}}{4}(1+i)\right| = \frac{1}{2}, &&\left|z \pm \frac{\sqrt{2}}{4}(1-i)\right| = \frac{1}{2},\\
&\left|z \pm \frac{\sqrt{2}}{2}(1+i)\right| = 1, &&\left|z \pm \frac{\sqrt{2}}{2}(1-i)\right| = 1,\\
&\left|z \pm \frac{3\sqrt{2}}{4}(1+i)\right| = \frac{3}{2}, &&\left|z \pm \frac{3\sqrt{2}}{4}(1-i)\right| = \frac{3}{2}.
\end{aligned}$$

Although the topic of this section has been tackled in [4], here we used different Steiner nets and we treated a completely different situation where the net is parabolic.

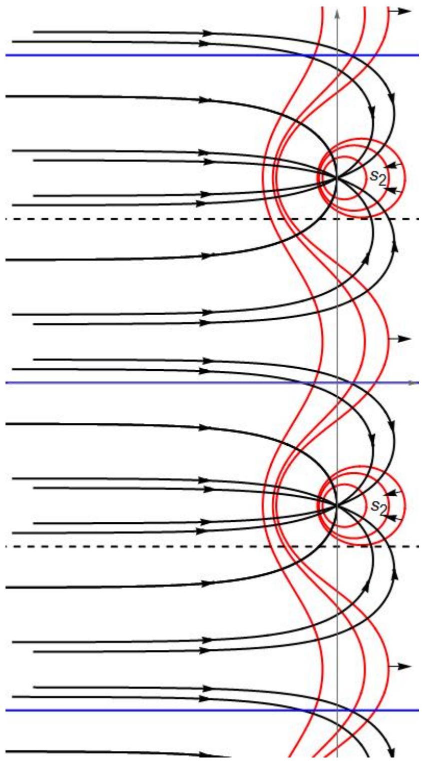

Figure 3: An illustration of the conformal self-mapping of Ω_k by the function $Log_{|\Omega_k}(M(e^s + a))$

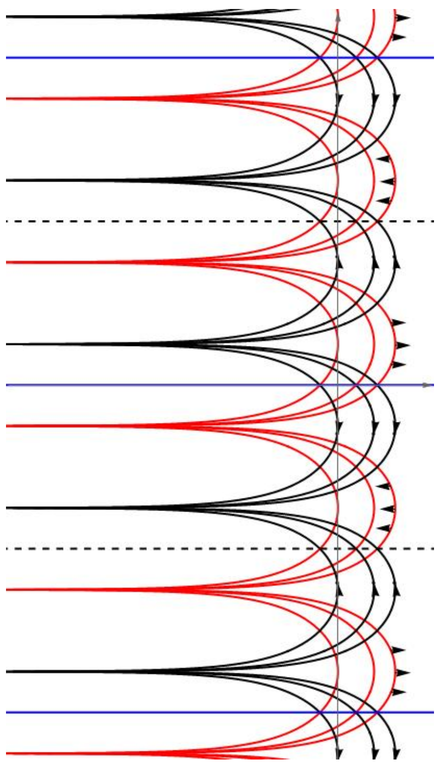

Figure 4: An illustration of the conformal self-mapping of Ω_k by the function $Log_{|\Omega_k}(u(z))$

3 Conformal Self-Mappings of the Fundamental Domains of the Functions of the Class S

An important chapter of the geometric function theory, [3], [6], [7] and [8] deals with the class S of functions f analytic and univalent in the unit disk. In the following, we will adopt a more global approach than that of geometric function theory, in the sense that f will be seen as defined in the whole complex plane. In fact we can prove:

Theorem 3. Every function f of class S which can be extended by continuity to the unit circle C and transforms the unit circle into a circle or a line Γ admits an analytic continuation $\widetilde{f}$ to $\overline{\mathbb{C}}$ with the exception of some poles located on the unit circle. When Γ is a line, the unit disk and the exterior of the unit disk are fundamental domains for $\widetilde{f}$.

Proof: We continue to denote by f the extended function to the unit circle C. For every $z \in \overline{\mathbb{C}}$, let z^* be the symmetric of z with respect to Γ, i.e., [1], $z^* = \overline{z}$ if $\Gamma = \mathbb{R}$. If the line Γ makes an angle α with the real axis, then there is a constant c such that $e^{i\alpha}z + c$ transforms $\mathbb{R}$ into Γ and symmetric points with respect to $\mathbb{R}$ into symmetric points with respect to Γ, hence, in this case, $e^{i\alpha}z + c$ is the symmetric of $e^{i\alpha}\overline{z} + c$ with respect to Γ. Then, if $w = e^{i\alpha}z + c$, we have $w^* = e^{i\alpha}\overline{z} + c = e^{i\alpha}\overline{(w - c)e^{-i\alpha}} + c = e^{2i\alpha}\overline{w} + k$, where $k = c - e^{i\alpha}\overline{c}$.

If Γ is a circle centered in a and of radius R, then the symmetric of $f(z)$ with respect to Γ is

$$\frac{R^2}{\overline{(f(1/\overline{z}))} - \overline{a}} + a$$

and we have an analytic function in this case as well.

When Γ is a line, there must be a point $e^{i\theta_0}$ on the unit circle such that

$$\lim_{\theta \nearrow \theta_0} f(e^{i\theta}) = \lim_{\theta \searrow \theta_0} f(e^{i\theta}) = \infty,$$

hence $e^{i\theta_0}$ is a pole for $\widetilde{f}$. Then an arc of the unit circle centered at $e^{i\theta_0}$ is mapped two to one onto an interval on Γ ending at ∞. Therefore $f(z)$ maps conformally the unit disk onto the complex plane with a slit and that interval is part of the slit.

Likewise, the function $\widetilde{f}(z)$ maps the exterior of the unit disk onto the complex plane with the same slit, therefore the unit disk and the exterior of the unit disk are fundamental domains of $\widetilde{f}(z)$, and the theorem is completely proved. □

Corollary. Under the conditions of Theorem 3, all the functions $\psi(z) = f(M(z))$, where $M(z)$ is given by formula (1), have as fundamental domains the unit disk and the exterior of the unit disk.

Proof: Indeed, $M(z)$ transforms the unit circle into itself, therefore, the image by $\psi(z)$ of the unit circle coincides with the image by $f(z)$ of the same circle. The only difference is that instead of the pole $e^{i\theta_0}$ of f, we will have a pole $M^{-1}(e^{i\theta_0})$ for ψ. □

Theorem 4. Let f be a function of class S which is analytic in $\overline{\mathbb{C}}$ except for some poles on the unit circle. Then $\widetilde{f}(z) = f(z)$ for $|z| \geq 1$. If

$$\widetilde{f}(z) = e^{2i\alpha}\overline{f(1/\overline{z})},$$

then

$$|f(z)| = |f(1/\overline{z})|$$

If

$$\widetilde{f}(z) = e^{2i\alpha}\overline{f(1/\overline{z})} + k$$

then

$$|f(z)| \leq |f(1/\overline{z})| + |k|.$$

Proof: We have seen that $\widetilde{f}(e^{i\theta}) = f(e^{i\theta})$ for every $\theta \in \mathbb{R}$. By the permanence of functional equations, [2], we must have $\widetilde{f}(z) = f(z)$ everywhere. If $f(z)$ carries the unit circle into a line making the angle α with the positive real half axis, then by Theorem 3,

$$\widetilde{f}(z) = e^{2i\alpha}\overline{f(1/\overline{z})} + k,$$

hence

$$f(z) = e^{2i\alpha}\overline{f(1/\overline{z})} + k,$$

and

$$|f(z)| = |f(1/\overline{z})|$$

when $k = 0$ and

$$|f(z)| \leq |f(1/\overline{z})| + |k|$$

when $k \neq 0$. We will check the affirmation of this theorem on some examples which follow. □

The functions in these examples are all defined in the whole complex plane and belong to the class S. One of the most studied such functions is Koebe's function, [9],

$$\begin{aligned} k(z) &= \frac{z}{(1-z)^2} = z(1 + z + z^2 + ...)^2 \\ &= z(1 + 2z + 3z^2 + ...) \\ &= z + 2z^2 + 3z^3 + ..., \end{aligned} \tag{11}$$

which satisfies the (now proved) Biberbach hypothesis. Moreover, by a theorem of Bieberbach if

$$f(z) = z + a_2z^2 + a_3z^3 + ...,$$

then

$$|a_2| \leq 2,$$

with equality if and only if $f(z)$ is Koebe's function.

As proven in [5], there must exist two Möbius transformations M_1 and M_2 such that

$$k(z) = M_2 \circ T \circ M_1(z) \tag{12}$$

where $T(\zeta) = \zeta^2$. Indeed, it can be easily checked, [9], that the equality (12) is true when

$$M_1(z) = \frac{1+z}{1-z}$$

and

$$M_2(z) = \frac{1}{4}(z-1),$$

hence

$$k(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right].$$

It might seem surprising that $k(z)$ is an univalent function (in the unit disk!), although it is a second degree rational function. This mystery is solved when we realize that the unit disk Ω, and the exterior of the unit disk $\overline{\mathbb{C}}\backslash\overline{\Omega}$ are fundamental domains of $k(z)$. Indeed,

$$\begin{aligned} k(e^{i\theta}) &= \frac{1}{4}\left\{\left[\frac{1+e^{i\theta}}{1-e^{i\theta}}\right]^2 - 1\right\} \\ &= \frac{1}{4}\left\{\frac{[(1+e^{i\theta})(1-e^{-i\theta})]^2}{|1-e^{i\theta}|^4} - 1\right\} \\ &= \frac{1}{4}\left\{\frac{[(2i\sin\theta)]^2}{|1-e^{i\theta}|^4} - 1\right\} \in \mathbb{R}, \end{aligned}$$

hence $k(z)$ maps the unit circle on the real axis. Moreover,

$$k\left(\frac{1}{\overline{z}}\right) = \frac{\frac{1}{\overline{z}}}{\left(1-\frac{1}{\overline{z}}\right)^2} = \frac{\overline{z}}{(1-\overline{z})^2} = k(\overline{z}),$$

hence

$$\widetilde{k}(z) = \overline{k\left(\frac{1}{\overline{z}}\right)} = \overline{k(\overline{z})} = k(z).$$

This last equality is due to the fact that the series expansion of $k(z)$ has real coefficients. We conclude that the analytic continuation of the function $k(z)$ defined in the unit disk is the function $k(z)$ defined on $\overline{\mathbb{C}}$.

The inequality $|k(z)| < |z|$ for $|z| < 1$ implies

$$\left|k\left(\frac{1}{\overline{z}}\right)\right| = |\overline{k(z)}| = |k(z)| < \left|\frac{1}{\overline{z}}\right| = \frac{1}{|z|},$$

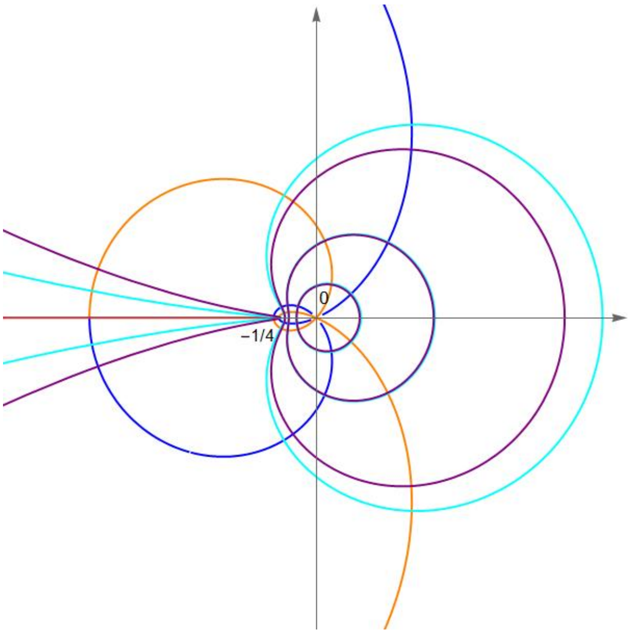

Figure 5: $k(z)$ maps conformally the unit disk and the exterior of the unit disk onto the complex plane with the slit $(-\infty, -\frac{1}{4})$

for $|z| > 1$, i.e.,

$$|k(z)| < \frac{1}{|z|},$$

for $|z| > 1$.

It can be easily checked that the function $k(z)$ maps conformally the unit disk and the exterior of the unit disk onto the whole complex plane with a slit alongside the real axis from $-\infty = k(1)$ to $-1/4 = k(-1)$.

Fig. 5 above illustrates this property by using the images by the function $k(z)$ of the net in Fig. 1. It is visible that the circles symmetric with respect to the unit circle are mapped by $k(z)$ onto the same curves. The same thing is true for the parts of the rays inside the unit circle and outside the unit circle.

Remark. Koebe One-Quarter Theorem, [10] and [11], states that the range of every function f of class S contains the disk

$$\left\{ w \mid |w| < \frac{1}{4} \right\}.$$

By the result above, this affirmation is trivial for the function $k(z)$. We can say even more, namely that the range of $k(z)$ contains the half-plane

$$\left\{ w \mid \Re w > -\frac{1}{4} \right\}.$$

On the other hand, in the proof of Koebe's Theorem it is assumed that f has an omitted value in the unit disk. By Theorem 3, such a value does not exist for the functions of class S which admit analytic continuation in the whole plane, except for some poles located on the unit circle. Therefore, an alternative proof of Koebe's Theorem is required for this class of functions.

Let us look now for the application of the theory from [5] in this case. We will use (1) with $\theta = 0$ to define

$$\chi_M(s) = k^{-1}_{|\Omega} \circ M \circ k(s).$$

As stated above, the fixed points of

$$M(z) = \frac{z - a}{1 - \overline{a}z},$$

with $a = re^{i\alpha}$, are $\xi_1 = e^{i\alpha}$ and $\xi_2 = -e^{i\alpha}$. The Steiner net corresponding to $M(z)$ is formed with circles centered at points c_1 on the second diagonal and passing through ξ_1 and ξ_2 and the orthogonal Apollonius circles centered at points c_2 on the first diagonal. If we denote by r the radii of those circles, then an easy computation gives

$$c_1 = \pm e^{3i\frac{\pi}{4}} \sqrt{r^2 - 1}$$

and

$$c_2 = \pm e^{i\frac{\pi}{4}} \sqrt{r^2 + 1},$$

hence these circles have the equations

$$|z \pm e^{3i\frac{\pi}{4}} \sqrt{r^2 - 1}| = r$$

and

$$|z \pm e^{i\frac{\pi}{4}} \sqrt{r^2 + 1}| = r.$$

An illustration of the corresponding Steiner net can be seen in Fig. 6.

On the other hand, it is known, [4], that the fixed points of $\chi_M(s)$ are those points s for which $k(s) = \xi_1$ and $k(s) = \xi_2$. These points s satisfy the equations

$$s^2 - (2 \pm e^{-i\alpha})s + 1 = 0.$$

The roots of these equations are

$$1 \pm \frac{1}{2} e^{-i\alpha} (1 \pm \sqrt{1 \pm 4e^{i\alpha}}).$$

Thus, there are two fixed points of $\chi_M(s)$ in Ω, and two fixed points in $\overline{\mathbb{C}} \backslash \Omega$, as seen in Fig. 7. The computer generated coordinates of these points are

$$\begin{aligned} z1 &= 0.379621 + 0.137809i, \\ z2 &= 0.386752 - 0.526531i, \\ z3 &= 2.32749 - 0.844916i, \\ z4 &= 0.906142 + 1.23364i. \end{aligned}$$

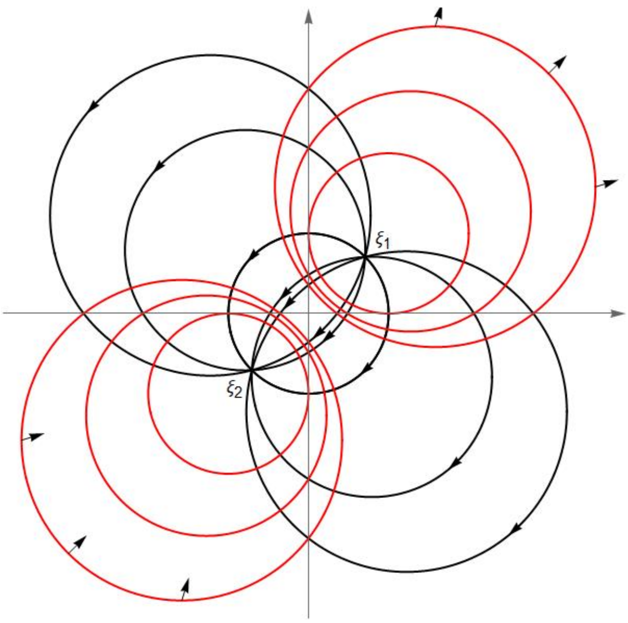

Figure 6: The Steiner net for the Möbius transformation (1)

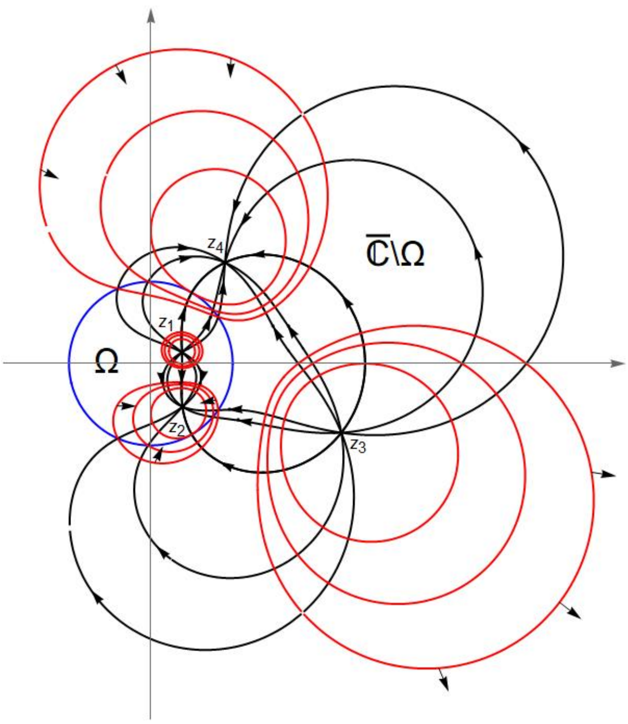

Figure 7: The conformal self-mapping of the complex plane induced by $k(z)$ and the Möbius transformation (1)

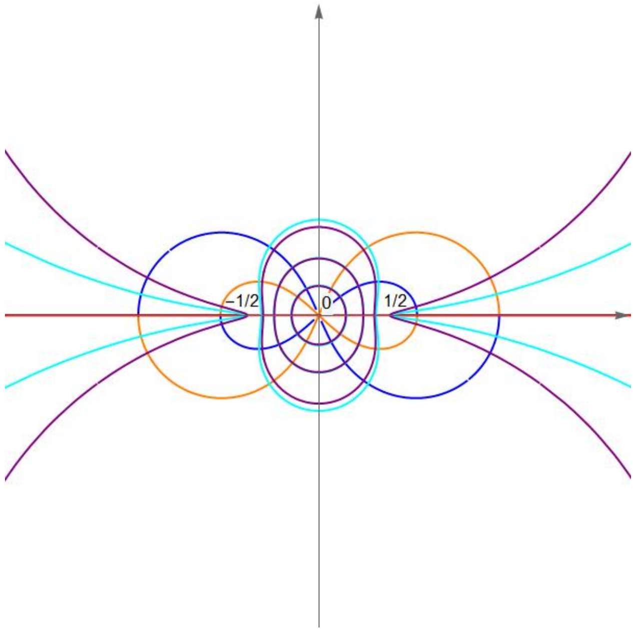

Figure 8: $\varphi(z)$ maps conformally the unit disk and the exterior of the unit disk onto the complex plane with the slit $(-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, +\infty)$

Another function of the class S is:

$$\varphi(s) = \frac{is}{1-s^2} \tag{13}$$

We have

$$\begin{aligned}\varphi(e^{i\theta}) &= \frac{ie^{i\theta}}{1-e^{2i\theta}} \\ &= \frac{i}{e^{-i\theta}-e^{i\theta}} \\ &= -\frac{\frac{1}{2}}{\frac{e^{i\theta}-e^{-i\theta}}{2i}} \\ &= -\frac{1}{2\sin\theta} \in \mathbb{R}\end{aligned}$$

hence $\varphi(z)$ maps the unit circle on the real axis with $\varphi(\pm i) = \mp\frac{1}{2}$ and $\varphi(\pm 1) = \infty$. Thus, the range of $\varphi(s)$ contains the disk centered at the origin and of radius $\frac{1}{2}$.

Consequently, $\varphi(s)$ maps conformally the unit disk and the exterior of the unit disk onto the complex plane with the slit $(-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, +\infty)$.

This property is illustrated in Fig. 8 above, obtained by the conformal representation of the net in Fig. 1.

We have

$$\varphi\left(\frac{1}{\overline{z}}\right) = \frac{\frac{i}{\overline{z}}}{(1 - \frac{1}{\overline{z}^2})} = \frac{i\overline{z}}{\overline{z}^2 - 1} = -\varphi(\overline{z}),$$

hence

$$\widetilde{\varphi}(z) = \overline{\varphi\left(\frac{1}{\overline{z}}\right)} = \overline{\frac{i\overline{z}}{\overline{z}^2 - 1}} = -\frac{iz}{z^2 - 1} = \frac{iz}{1 - z^2} = \varphi(z).$$

Thus, the analytic continuation of the function φ defined in the unit circle is, as expected the function φ defined on $\overline{\mathbb{C}}$.

The corresponding conformal self-mapping of the unit disk and of the exterior of the unit disk induced by the Möbius transformation (1) is shown in Fig. 9. We have

$$\chi_M(s) = \varphi_{|\Omega}^{-1} \circ M \circ \varphi(s),$$

where Ω is one of the two domains, and every one of these mappings has, as in the previous example, four fixed points. These are the points s for which $\varphi(s)$ are fixed points for $M(z)$, i.e.,

$$\varphi(s) = \pm e^{i\frac{\pi}{4}}.$$

Solving the equations

$$\frac{is}{1 - s^2} = \pm e^{i\frac{\pi}{4}},$$

we get:

$$1 - s^2 = \pm ie^{-i\frac{\pi}{4}}s,$$
$$s^2 \pm ie^{-i\frac{\pi}{4}}s - 1 = 0,$$
$$s = \frac{1}{2} \mp (\pm ie^{-i\frac{\pi}{4}} \pm \sqrt{4 - e^{-i\frac{\pi}{2}}}).$$

The computer generated coordinates of the fixed points of the function $\chi_M(s) = \varphi_{|\Omega}^{-1} \circ M \circ \varphi(s)$, are:

$$z1 = -1.36122 - 0.477603i,$$
$$z2 = -0.654111 + 0.229504i,$$
$$z3 = 0.654111 - 0.229504i,$$
$$z4 = 1.36122 + 0.477603i.$$

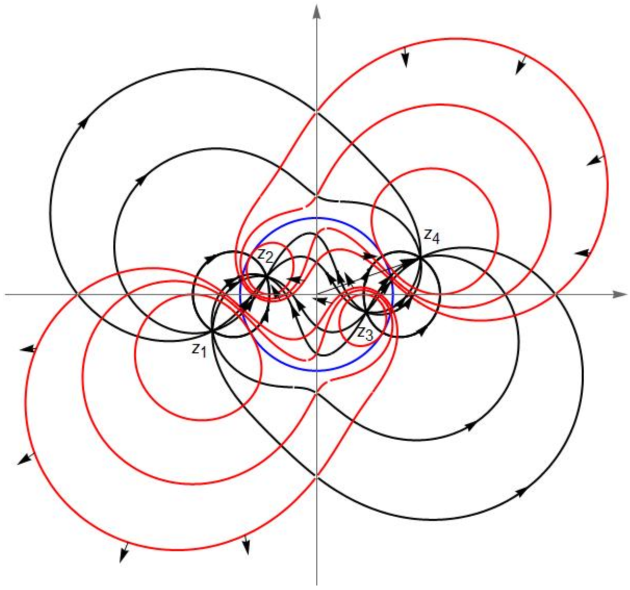

Figure 9: The conformal self-mapping of the complex plane with slits induced by the function $\varphi(z)$ and the Möbius transformation (1)

As stated in [8], the function

$$g(z) = \frac{f(M(z)) - f(a)}{(1 - |a|^2)f'(a)} \tag{14}$$

belongs to the class S if $f(s)$ is in the class S and $M(z) = (z + a)/(1 + \overline{a}z)$. Indeed, $g(z)$ is an analytic function in the unit disk, since both f and M are analytic. Moreover, $M(0) = a$, which implies $g(0) = 0$. Finally,

$$g'(z) = \frac{1}{(1 - |a|^2)f'(a)} \frac{1 - |a|^2}{(1 + \overline{a}z)^2} f'\left(\frac{z + a}{1 + \overline{a}z}\right)$$

and then $g'(0) = 1$. Therefore, $g(z)$ belongs indeed to the class S. Moreover, if $f(s)$ can be extended by continuity to the unit circle, then $g(z)$ enjoys the same property and if $f(s)$ maps the unit circle on a line, so does $g(z)$, which means that both functions admit analytic continuations to the whole complex plane and the fundamental domains of both of them are the unit disk and the exterior of the unit disk.

Fig. 10 below illustrates the conformal self-mapping of the complex plane generated by the function $k(z)$ and the Möbius transformation, $M(z)$ above.

The formula (14) allows us to generate families of functions of class S starting with a known one. The study of normality and compactness of these families of functions exceeds the purpose of this work.

The computer generated coordinates of the fixed

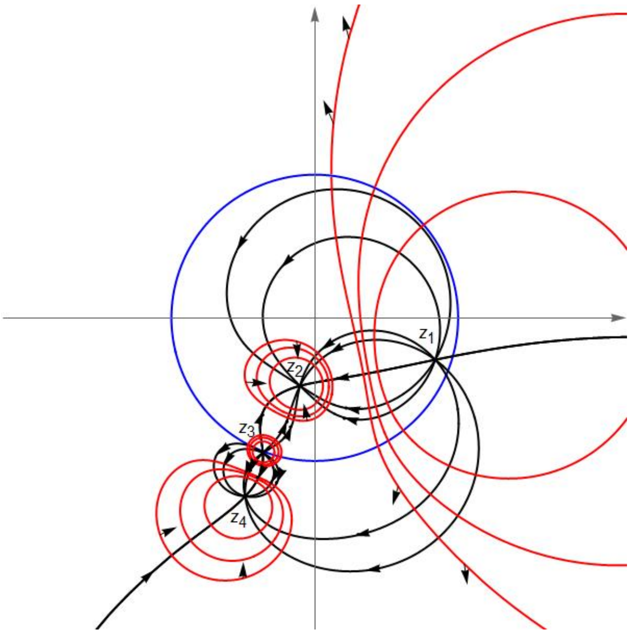

Figure 10: The conformal self-mapping of the complex plane generated by the function (14) where $f(z)$ is the Koebe function and $M(z)$ is given by formula (1)

points of the function $\chi_M(s) = g_{|\Omega}^{-1} \circ M \circ g(s)$ are

$$
\begin{aligned}
z1 &= 0.840256 - 0.292871i, \\
z2 &= -0.103106 - 0.475103i, \\
z3 &= -0.362298 - 0.944422i, \\
z4 &= -0.48444 - 1.24651i.
\end{aligned}
$$

Theorem 5. If $f(w)$ is a function of class S which admits analytic continuation to the whole complex plane, with the exception of some poles on the unit circle, then the function $g(z)$ given by the formula (14) where $w = M(z)$, admits also analytic continuation to the whole complex plane, with the exception of some poles on the unit circle.

Proof: Let $\widetilde{f}(w)$ be the the analytic continuation of $f(w)$ to the whole complex plane, except for some poles w_k on the unit circle and let

$$\widetilde{g}(z) = \frac{\widetilde{f}(M(z)) - f(a)}{(1 - |a|^2) f'(a)}$$

for $|z| \geq 1$.

Obviously, $\widetilde{g}(z)$ is an analytic function in the complex plane, except for $z_k = M^{-1}(w_k)$, which are poles for $\widetilde{g}(z)$. Since $|w| = 1$ if and only if $|z| = 1$ and $\widetilde{f}(w) = f(w)$ for $|w| = 1$, we have $\widetilde{g}(z) = g(z)$ for $|z| = 1$. By the permanence of functional equations, $\widetilde{g}(z) = g(z)$ everywhere. $\square$

4 Growth and Distortion Theorems Outside the Unit Disk

The growth and distortion theorems provide different inequalities valid in the unit disk, [9], for univalent functions of class S. If those univalent functions admit analytic continuations in the whole complex plane, except for some poles, then, those inequalities have analogue formulations outside the unit disk. Following [9], we can state:

Proposition 4. For any function $f(z)$ of class S we have:

$$\left| \frac{z f''(z)}{f'(z)} - \frac{2r^2}{1 - r^2} \right| \leq \frac{4r}{1 - r^2}, \tag{15}$$

where $|z| = r < 1$.

Let us notice that the inequality (15) implies

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{2(r+2)}{1 - r^2}. \tag{16}$$

Theorem 6. If $f(z)$ is an analytic function in the complex plane, except for some poles on the unit circle, belongs to the class S in the unit disk, and carries the unit circle onto a line, then we have:

$$\left| \frac{f''(w)}{f'(w)} + 2 \right| \leq \frac{2(2R+1)}{R^2 - 1}, \tag{17}$$

for every w with $|w| = R > 1$.

Proof: Let us use the change of variable $z = \frac{1}{w}$. Then, for $g(w) = f\left(\frac{1}{z}\right)$ we have $g(0) = f(\infty) = 0$ and since

$$f\left(\frac{1}{z}\right) = \frac{1}{z} + \frac{a_2}{z^2} + \ldots \text{ with } f'(0) = 1,$$

we have

$$g(w) = w + a_2 w^2 + \ldots \text{ with } g'(0) = 1.$$

Moreover, since $g(w)$ is a univalent function in the unit disk, it is a function of class S. Consequently,

$$\left| \frac{g''(w)}{g'(w)} \right| \leq \frac{2(r+2)}{1 - r^2},$$

where $|w| = r < 1$.

We have

$$
\begin{aligned}
g'(w) &= -\frac{1}{z^2} f'\left(\frac{1}{z}\right) = -w^2 f'(w), \\
g''(w) &= \frac{1}{z^4} f''\left(\frac{1}{z}\right) + \frac{2}{z^3} f'\left(\frac{1}{z}\right) \\
&= w^4 f''(w) + 2w^3 f'(w),
\end{aligned}
$$

hence
$$\frac{g''(w)}{g'(w)} = -w^2\frac{f''(w)}{f'(w)} - 2w,$$
thus the inequality (16) becomes inequality (17) and our theorem is proved. □

Proposition 5. (Distortion Theorem, [9]). For every function f of class S we have
$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad (18)$$
where $0 < |z| = r < 1$.

Equality occurs if and only if $f(z) = e^{i\theta}k(z)$, for $\theta \in \mathbb{R}$ and $k(z)$ is Koebe's function.

Theorem 7. If $f(z)$ is an analytic function in the complex plane except for some poles on the unit circle, belongs to the class S in the unit disk, and carries the unit circle into a line, then for every z, such that $|z| = R > 1$, we have:
$$\frac{R^4(R-1)}{(R+1)^3} \leq |f'(z)| \leq \frac{R^4(R+1)}{(R-1)^3} \qquad (19)$$

Equality occurs if and only if $f(z) = e^{i\theta}k(z)$, for $\theta \in \mathbb{R}$, where $k(z)$ is Koebe's function.

Proof: Indeed, if we make the change of variable $z(w) = \frac{1}{w}$, then for $g(w) = f(z(w))$ and for $|w| = r < 1$, the inequalities (18) become
$$\frac{1-r}{(1+r)^3} \leq \left|\frac{1}{w^2}f'(z(w))\right| \leq \frac{1+r}{(1-r)^3}. \qquad (20)$$

Replacing r by $\frac{1}{R}$ and $|z| = R$ these inequalities become (19). We have equality in (19) if and only if there is equality in (18) and this happens if and only if $f(z) = e^{i\theta}k(z)$ and the theorem is completely proved. □

Proposition 6. (Growth Theorem, [9]). For every function $f(z)$ of class S we have:
$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \qquad (21)$$
for $|z| = r < 1$.

Equality occurs if and only if $f(z)$ is a rotation of Koebe's function.

Theorem 8. If $f(z)$ is an analytic function in the complex plane with the exception of some poles on the unit circle, and carries the unit circle into a line, then for $|z| = R > 1$, we have
$$\frac{R}{(R+1)^2} \leq |f(z)| \leq \frac{R}{(R-1)^2}. \qquad (22)$$

Equality occurs if and only if $f(z)$ is a rotation of Koebe's function.

Proof: Indeed, if we make in (21) the change of variable $z(w) = \frac{1}{w}$, then for $g(w) = f(z(w))$ and for $|w| = r < 1$ the inequalities (21) become:
$$\frac{r}{(1+r)^2} \leq |f(z(w))| \leq \frac{r}{(1-r)^2}, \qquad (23)$$
for $|w| = r < 1$.

Replacing here w by $\frac{1}{z}$, where $|z| = R = \frac{1}{r} > 1$, the inequalities (23) become (22). Equality in (22) occurs if and only if there is equality in (21) and therefore $f(z)$ is a rotation of Koebe's function. Consequently, our result is proved. □

5 Conclusions

Most of the known univalent functions in the unit disk have analytic continuations in the whole complex plane with the exception of some poles on the unit circle. It was expected that these functions exhibit similar behavior outside of the unit disk. We dealt in previous works with univalent functions in the most general setting, namely in the fundamental domains of arbitrary analytic functions. The purpose of this paper was to build a bridge between the two fields.

We succeeded to prove some non trivial facts in this respect by using the theory of fundamental domains. Our findings have been illustrated by computer generated graphics.

Many open questions remain and it is expected that they will attract the attention of the researchers working in the field of univalent functions.

For example, as it is well-known, even if an analytic function $f(z)$ is not univalent in the unit disk, the disk can be partitioned into sub-domains where the function is univalent. What are the properties of the functions univalent in the unit disk, as growth and distortion etc., which hold for this type of functions? What can be said about their analytic continuation across the unit circle?

For potential applications in science, of the theory we have presented in this paper, see [12], [13] and [14].

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