Solving Nonlinear Volterra Integral Equations by Mohanad Decomposition Method

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Abstract: - In this research article, we introduce the Mohanad transform-decomposition method, which is a new analytical approach. The basic characteristics and facts of the proposed method are presented and analyzed. This new method is a simple method that combines the Mohanad transform with the decomposition method. This new approach is utilized to handle nonlinear integro-differential equations, the results obtained from this method are expressed in the form of an infinite series that converges rapidly to the exact ones. The maximum absolute error is computed for the proposed examples, and some figures are presented to show the accuracy of the obtained results. All the numerical results and computations in this study are gained by using Mathematica software.

Key-Words: - Integral transform; Mohanad transform; Adomian Decomposition method; Error analysis; Convolution theory; Integral equations; Nonlinear problems.

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1 Introduction

One of the most important tools for solving problems in science and engineering is the integral equation. Both Volterra and Fredholm integral equations are widely used, and offer useful solutions for a variety of initial and boundary value issues. Integral equations have advanced greatly as a result of advancements in potential theory, [1]. Integral equations have many applications in different fields of science and engineering, because of these applications, they got great interest from authors and specially mathematicians, they appeared in quantum mechanics astrophysics, conformal mapping, scattering, and water waves, [2], [3], [4] and [5].

The importance of studying nonlinear integral differential has been increased due to the different branches of applications that could be handled. Queuing theory and chemical kinetics, and it is expanded now in many other scientific disciplines. [6], [7], [8], [9] and [10]. So, researchers, established many methods to solve these problems, such as the homotopy analysis approach, [11], the variation method, the iteration method, [12], the least squares method, [13] and Adomian's method, [14]. Utilizing these methods, we can overcome the difficulty in the process of solving nonlinear integral equations.

By offering useful approximate analytical series solutions for such complex problems, the decomposition method has established itself as one of the most effective strategies for solving nonlinear differential and integral equations. The method of Adomian decomposition, first presented by [15] and [16], was created especially for solving integral equations. The authors in [17], [18], [19] and [20], improved this research to resolve the Volterra integral differential equation successfully. The strategy has been used later to address a variety of issues in several different domains in response to time, as stated in [21], [22], [23], [24] and [25].

Integral equations have become more than an essential tool for solving integral equations, but also crucial in deriving solutions for intricate situations. The Laplace transform is one of these transformations that has shown to be quite helpful, [26]. To further expand the scope of solving integral equations and improve the overall effectiveness of the decomposition method in addressing various scientific and engineering challenges, other transforms, such as the ARA transform, [27], a lot of scientific transforms such as formable transform, and others, [28], [29] and [30], have also been checked and verified.

Once introduced in 2013, [31], the Mohanad transform is a significant literary transformation with a huge number of mathematical issues. Given by the following integral formula is the Mohanad transform:

$$M[\varphi(w)] = u^2 \int_0^\infty e^{-uw}\varphi(w)dw, \qquad u > 0.$$

The transform can answer a wide range of problems, academics are paying close attention to it. To overcome the difficulty in nonlinear cases, we can easily merge the transform with one of the numerical methods, such as, [32], [33] and [34]. This will introduce a new hybrid method called: The Mohanad-decomposition method (MDM), it merges the two powerful techniques, the Mohanad transform and decomposition method which is the main objective of this article.

This article investigates solving nonlinear Volterra IDE of the form:

$$\varphi^{(n)}(w) = f(w) + \int_0^w R(w-v)G\big(\varphi(v)\big)dv,$$

where $R(w-v)$ is the difference kernel of the equation, $f(w)$ is piecewise continuous function, and $G(\varphi(v))$is a given analytic function of the unknown $\varphi(w)$, that could be $\cos\varphi(w)$, $\varphi^3(w)$, $\sinh\varphi(w)$ and etc.

The paper follows the following structure: In Section 2, we introduce the definition of Mohanad transform along with its basic properties, and we also explain the core concept of the Adomian decomposition method. Section 3 presents the application of the Mohanad decomposition method (MDM) for handling nonlinear Volterra integral differential equations (IDEs). To demonstrate the method's effectiveness, we solve several numerical examples of IDEs. Lastly, in Section 5, we provide the concluding remarks for this article.

2 Basic Facts

In this section, the needed properties and theorems of the Mohanad transform and the decomposition method are presented.

2.1 Mohanad Transform

In this section of the article, we introduce some definitions and properties of Mohanad integral transform.

Definition 1. Assume that $\varphi(w)$ is a continuous function with domain subset of $(0,\infty)$, then Mohanad integral transform of $\varphi(w)$ is given by the formula

$$M[\varphi(w)] = \Phi(u) = u^2 \int_0^\infty e^{-u\tau}\varphi(w)dw, \quad w > 0.$$

The Mohanad transform inverse of a function $\Phi(u)$ is defined as:

$$M^{-1}[\Phi(u)] = \varphi(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{u^2} e^{uw}\Phi(u)du, c\epsilon R.$$

Theorem 1. Assume that $\varphi(w)$ is a piecewise continuous function with domain $[0,\infty)$, and assume that the following condition holds: $|\varphi(w)| \le Me^{bw}$, for a real number $M > 0$. Then, Mohanad integral transform $M[\varphi(w)]$ is well defined for $Re(u) > b$.

Proof. The formula of Mohanad transform implies:

$$|\Phi(u)| = \left|u^2 \int_0^\infty e^{-uw}\varphi(w)dw\right| \le u^2 \int_0^\infty e^{-uw}|\varphi(w)|dw$$
$$\le u^2 \int_0^\infty e^{-uw}Me^{bw}dw = u^2 M\int_0^\infty e^{-w(u-b)}dw = \frac{Mu^2}{(u-\alpha)}, \qquad Re(u) > b > 0.$$

Thus, Mohanad transform is well defined and exists for $Re(u) > b > 0$.

We present some properties and the values of Mohanad transform to some elementary functions. Assume that $\Phi_1(u) = M[\varphi_1(w)]$ and $\Phi_2(u) = M[\varphi_2(w)]$and $a, b \in \mathbb{R}$, then

- $M[a\varphi_1(w) + b\varphi_2(w)] = a\Phi_1(u) + b\Phi_2(u)$.
- $M^{-1}[a\Phi_1(u) + b\,\Phi_2(u)] = a\varphi_1(w) + b\,\varphi_2(w)$.

Table 1, presents some quantities of Mohanad integral transform to the standard basic functions.

Table 1. Mohanad integral transform

$\varphi(w)$	$M[\varphi(w)] = \Phi(u)$
1	u
w^a	$\frac{\Gamma(a+1)}{u^{a-1}}, a > -1.$
e^{aw}	$\frac{u^2}{(u-a)}, u > a$
$\sin aw$	$\frac{a\,u^2}{(u^2+a^2)}$
$\cos aw$	$\frac{u^3}{u^2+a^2}$
$\sinh aw$	$\frac{a\,u^2}{(u^2-a^2)}$
$\cosh aw$	$\frac{u^3}{u^2-a^2}$
$\varphi'(w)$	$u\,\Phi(u) - u^2\varphi(0)$
$\varphi^{(m)}(w)$	$u^n\Phi(u) - \sum_{k=0}^{n-1} u^{m-k+1}\varphi^{(k)}(0)$
$(\varphi * \psi)(w)$	$\frac{1}{u^2} M[\varphi(w)] M[\psi(w)]$

2.2 Iterative Decomposition Technique

The Adomian decomposition method is well-known for its efficiency in solving several types of nonlinear differential equations, ordinary or partial. It is a commonly used technique in the domains of engineering, physics, and applied mathematics. The Adomian decomposition method's main idea is to divide the equation's nonlinear term into a number of elements. When these elements are added together, the result is represented astonishingly accurately. To use the method, we assume the series representation for the solution of the required equation, given by:

$$\varphi(w) = \sum_{n=0}^{\infty} \varphi_n(w) = \varphi_0(w) + \varphi_1(w) + \cdots.$$

The nonlinear term in the considered problem is then given in the form of a recursive formula. The series answer is then entered into the equation. We proceed to solve the resultant equation recursively to identify the series components $\varphi_n(\tau)$ after simplifying it. We may approximate the solution more precisely with each iteration, which produces results that are highly accurate in real-world applications.

3 Nonlinear Volterra IDEs

In this section, we apply the Mohanad integral transform to the required IDE, and then we operate the decomposition technique, that considered the basic part of the MDM. In addition, the given kernels in the equation are assumed to be in the difference form.

Now, let us construct the solution of the following IDE of the form:

$$\varphi^{(m)}(w) = f(w) + \int_0^w R(w-v)G(\varphi(v))dv, \quad (1)$$

associated with the initial conditions (ICs)

$$\varphi^{(i)}(0) = a_i, \qquad i = 0,1,\ldots,m-1. \quad (2)$$

To solve the problem (1) by MDM, firstly, apply Mohanad transform to it, to get:

$$M\left[\varphi^{(m)}(\tau)\right] = M[f(w)] + M\left[\int_0^w R(w-v)G(\varphi(v))dv\right].$$

Using the convolution and the differential properties from Table 1 of Mohanad transform, we can simplify the equation to:

$$u^m M[\varphi(w)] - u^{m+1}a_0 - u^m a_1 - \cdots - u^2 a_{m-1} = M[f(w)] + \frac{1}{u^2} M[k(w)]\, M\left[G\left(\varphi(w)\right)\right]. \quad (3)$$

Following that, we substituting the ICs (2) in Equation (3) to obtain:

$$M[\varphi(w)] = u\,a_0 + a_1 + \cdots + \frac{1}{u^{m-2}} a_{m-1} + \frac{1}{u^m} M[f(w)] + \frac{1}{u^{m+2}} M[k(w)]\, M\left[G\left(\varphi(w)\right)\right]. \quad (4)$$

Now, using the decomposition method to handle the analytic function $G\left(\varphi(w)\right)$, express the analytic function $\varphi(w)$ in a form of infinite series of the form:

$$\varphi(w) = \sum_{i=0}^{\infty} \varphi_i(w) = \varphi_0(w) + \varphi_1(w) + \cdots, \quad (5)$$

where $\varphi_i(w)$, $\tau = 0,1,\ldots$, can be calculated and expand the function $G\left(\varphi(\tau)\right)$ in the form:

$$G\left(\varphi(w)\right) = \sum_{i=0}^{\infty} A_i(w), \quad (6)$$

noting that $A_i(w), i = 0,1,2,\ldots$ are given by

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left(G\left[\sum_{j=0}^{i} \lambda^j \varphi_j\right]\Bigg|_{\lambda=0}\right), \quad i = 0,1,2,\cdots. \quad (7)$$

The Adomian polynomials are the components A_i's, are used to handle the nonlinear function $G(\varphi(\tau))$ as:

$$\begin{aligned} A_0 &= G(\varphi_0), \\ A_1 &= \varphi_1 G'(\varphi_0), \\ A_2 &= \varphi_2 G'(\varphi_0) + \frac{1}{2!}\varphi_1^2 G''(\varphi_0), \\ A_3 &= \varphi_3 G'(\varphi_0) + \varphi_1\varphi_2 G''(\varphi_0) \\ &\quad + \frac{1}{3!}\varphi_1^3 G'''(\varphi_0), \\ A_4 &= \varphi_4 G'(\varphi_0) + \left(\frac{1}{2!}\varphi_2^2 + \varphi_1\varphi_3\right) G''(\varphi_0) \\ &\quad + \frac{1}{2!}\varphi_1^2\varphi_2 G'''(\varphi_0) \\ &\quad + \frac{1}{4!}\varphi_1^4 G^{(4)}(\varphi_0). \end{aligned} \tag{8}$$

Thus, substituting Equation (5) and (6) into Equation (4), to get:

$$\begin{aligned} M\left[\sum_{i=0}^{\infty}\varphi_i(w)\right] &= u\,a_0 + a_1 + \cdots + \frac{1}{u^{m-2}}a_{m-1} \\ &\quad + \frac{1}{u^m}M[f(w)] \\ &\quad + \frac{1}{u^{m+2}}M[k(w)]\ M\left[\sum_{i=0}^{\infty}A_i(w)\right]. \end{aligned} \tag{9}$$

$$M\left[\sum_{i=0}^{\infty}\varphi_i(w)\right] = u\,a_0 + a_1 + \cdots + \frac{1}{u^{m-2}}a_{m-1} + \frac{1}{u^m}M[f(w)] + \frac{1}{u^{m+2}}M[k(w)]\ M[\textstyle\sum_{i=0}^{\infty}A_i(w)]. \tag{10}$$

The relation of Adomian decomposition technique gives us:

$$M[\varphi_0(w)] = u\,a_0 + a_1 + \cdots + \frac{1}{u^{m-2}}a_{m-1} + \frac{1}{u^m}M[f(w)]. \tag{11}$$

Equation (9), implies:

$$M[\varphi_{n+1}(w)] = \frac{1}{u^{m+2}}M[k(w)]\,M[A_n(w)]. \tag{12}$$

Remark 1. Equation (12) is well defined if the condition

$$\lim_{u\to\infty}\frac{1}{u^{m+2}}M[k(w)] = 0,$$

is satisfied. Applying the inverse Mohanad transform to Equations (11) and (12) respectively, we get the values of $\varphi_0(w), \varphi_1(w), \cdots$.

The solution of the required equation Volterra IDE (1) can be expressed in the form:

$$\varphi(w) = \varphi_0(w) + \varphi_1(w) + \cdots.$$

The proposed method is effective in expressing approximate numerical results of Volterra IDEs (linear and nonlinear). To show accuracy of the proposed technique, we present some numerical examples and solve them by MDM. Moreover, we calculate the absolute error, given by the formula:

$$AbsErr = \max\left|\varphi_{exact} - \varphi_{app}\right|,$$

defined on some interval.

4 Numerical Examples

This section presents some examples of integral equations: (IE)s and IDEs that are solved by MDM, the maximum absolute error is computed to each example to verify the efficiency of the proposed results.

Example 4.1
Take the nonlinear Volterra IE:

$$\varphi(w) = 2w - \frac{w^4}{12} + \frac{1}{4}\int_0^w (w-v)\varphi^2(w)dw\,.m \tag{13}$$

Solution. The accurate solution of problem (4.1) is (3.9) $\varphi(w) = 2w$. To solve Equation (13) by MDM, the Mohanad transform is operated to Equation (13) to get:

$$\begin{aligned} \Phi(u) &= M\left[2w - \frac{w^4}{12}\right] + \frac{1}{4}u\,M[\tau]\,M[\varphi^2(w)] \\ &= 2 - \frac{2}{u^3} + \frac{1}{4u^2}M[\varphi^2(w)]. \end{aligned} \tag{14}$$

The substitution of the series $\Phi(u)$ and the usage of the Adomian polynomials for $\varphi^2(w)$, give:

$$M[\varphi_0(w)] = 2 - \frac{2}{u^3},$$
$$M[\varphi_{n+1}(w)] = \frac{u}{4}M[M_n(w)], n \geq 0.$$

The nonlinear term $\varphi^2(w)$ is decomposed by Equation (7), to get the components as follows:

$$\begin{aligned} A_0 &= \varphi_0^2, \\ A_1 &= 2\varphi_0\varphi_1, \\ A_2 &= \varphi_1^2 + 2\varphi_0\varphi_2, \\ A_3 &= 2\varphi_1\varphi_2 + 2\varphi_0\varphi_3, \\ A_4 &= \varphi_2^2 + 2\varphi_1\varphi_3 + 2\varphi_0\varphi_4. \end{aligned} \tag{15}$$

Comparing the terms obtained in equation (7) and operating the inverse Mohanad transform, to get:

$$\varphi_0(\tau) = 2w - \frac{w^4}{12},$$
$$\varphi_1(w) = \frac{w^4}{12} - \frac{w^7}{126} + \frac{w^{10}}{51840},$$
$$\varphi_2(w) = \frac{w^7}{504} - \frac{w^{10}}{181440} + \frac{127w^{13}}{56609280} - \frac{w^{16}}{298598400},$$

$$\varphi_3(w) = \frac{w^4}{12} - \frac{w^7}{504} + \frac{w^{10}}{2792} - \frac{19w^{13}}{14152320} + \frac{71w^{16}}{2264371200} - \frac{7893w^{19}}{575787643000000}. \quad (16)$$

Thus, the numerical approximation series solution is given by:

$$\varphi(w) = \varphi_0(w) + \varphi_1(w) + \varphi_2(w) + \varphi_3(w) + \cdots = 2w + \frac{w^4}{12} - \frac{w^7}{126} - \frac{w^{10}}{362880} + \frac{51w^{13}}{56609280} + \cdots. \quad (17)$$

Table 2 proposes comparisons between the accurate solution and obtained numerical approximated solution to Example 4.1. To prove the accuracy of the MDM, we calculate the absolute error.

Table 2. Approximate and exact solutions of Example (4.1)

Nodes	Exact Solution	Approximate Solution	Absolute Error
0.0	0.0	0.0000000000	0.0000000000
0.1	0.2	0.2000083325	0.0000083325
0.2	0.4	0.4001332317	0.0001332317
0.3	0.6	0.6006732643	0.0006732643
0.4	0.8	0.8021203322	0.0021203322
0.5	1.0	1.0051463480	0.0051463480
0.6	1.2	1.2105779450	0.0105779450
0.7	1.4	1.4193552730	0.0193552730
0.8	1.6	1.6328034650	0.0328034650
0.9	1.8	1.8508857190	0.0508857190
1.0	2.0	1.8833526250	0.1166473750

$$M[\varphi_0(w)] = u^2 + \frac{3u^2}{2(u-1)} - \frac{u^2}{2(u-3)}, \quad M[\varphi_{n+1}(w)] = \frac{u^3}{u-1} M[A_n(w)], n \geq 0. \quad (18)$$

Figure 1 below, presents the graph of approximate and exact solutions. The absolute error is presented in Figure 2, of Example 4.1.

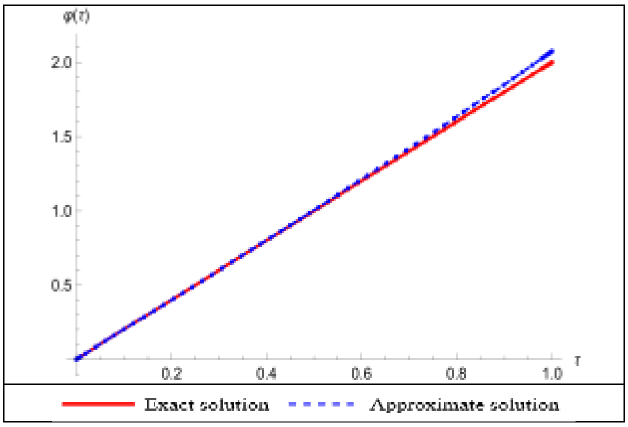

Fig. 1: The exact and approximate solutions of Example 4.1

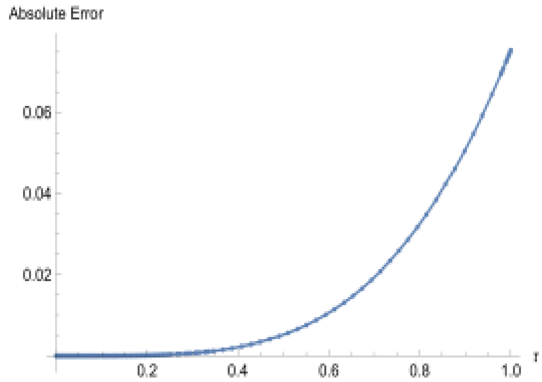

Fig. 2: The graph of absolute error of Example 4.1

Example 4.2. Take the nonlinear Volterra IE:

$$\varphi'(w) = \frac{3}{2}e^w - \frac{1}{2}e^{3w} + \int_0^w e^{w-v}\varphi^3(v)dv. \quad (19)$$
$$\varphi(0) = 1 \quad (20)$$

Solution. Operating Mohanad integral transform to Equation (19), we obtain:

$$\Phi(u) = u^2 + \frac{3u^2}{2(u-1)} - \frac{u^2}{2(u-3)} + \frac{u^3}{(u-1)} M[\varphi^3(w)],$$

which can be simplified to:

$$\Phi(u) = u + \frac{3u^2}{2(u-1)} - \frac{u^2}{2(u-3)} + \frac{u^3}{u-1} M[\varphi^3(w)].$$

Also, we get:
The Adomian components of the polynomials $A_n(w)$ of $\varphi^3(w)$, can be obtained by:

$$A_0 = \varphi_0^3,$$
$$A_1 = 3\varphi_0^2\varphi_1,$$
$$A_2 = 3\varphi_0^2\varphi_2 + 3\varphi_0\varphi_1^3,$$
$$A_3 = 3\varphi_0^2\varphi_3 + 6\varphi_0\varphi_1\varphi_2 + \varphi_1^3.$$

Applying the inverse Mohanad transform to the functions in (19) and using the recursive formula, we have:

$$\varphi_0(w) = 1 + w - \frac{1}{2}w^3 - \frac{w^4}{2} - \frac{13}{40}w^5 + \cdots,$$
$$\varphi_1(w) = \frac{1}{2}w^2 + \frac{2}{3}w^3 + \frac{5}{12}w^4 + \frac{7}{120}w^5 + \cdots,$$
$$\varphi_2(w) = \frac{1}{8}w^4 + \frac{11}{40}w^5 + \cdots$$

.

Thus, the approximate solution of Example 4.2 is:

$$\varphi(w) = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \cdots,$$

that converges directly to the exact solution $\varphi(\tau) = e^{\tau}$.

In Table 3, we introduce the accurate and approximate solutions of Example 4.2, and to prove the accuracy of the method, we calculate the absolute error.

Table 3. The exact and approximate solutions of Example 4.2, and the absolute error

Nodes	Exact Solution	Approximate Solution	Absolute Error
0.0	1	1	0
0.1	1.105170918	1.1051709181	$2.2204460493 \times 10^{-16}$
0.2	1.221402758	1.2214027582	0
0.3	1.349858807	1.3498588076	$2.2204460493 \times 10^{-16}$
0.4	1.491824697	1.4918246976	$2.2204460492 \times 10^{-16}$
0.5	1.648721270	1.6487212707	$8.8817841970 \times 10^{-16}$
0.6	1.822118800	1.8221188004	$9.5479180118 \times 10^{-15}$
0.7	2.013752707	2.0137527075	$8.1268325403 \times 10^{-14}$
0.8	2.225540928	2.2255409285	$5.3290705182 \times 10^{-13}$
0.9	2.459603111	2.4596031112	$2.7911006839 \times 10^{-12}$
1.0	2.718281828	2.7182818284	$1.228617207 \times 10^{-11}$

In Figure 3, we sketch the exact and approximate solutions. We also sketch the absolute error of the exact and approximate solutions of Example 4.2 in Figure 4.

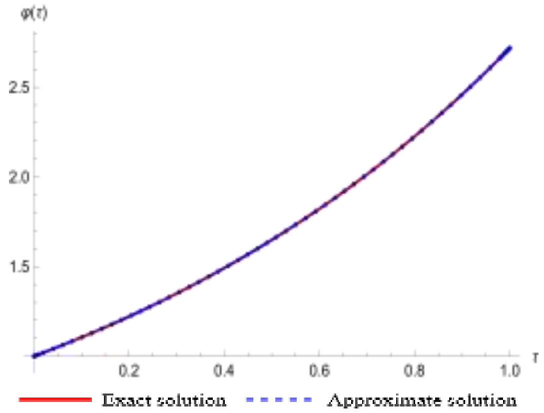

Fig. 3: The approximate and exact solutions of the Example 4.2

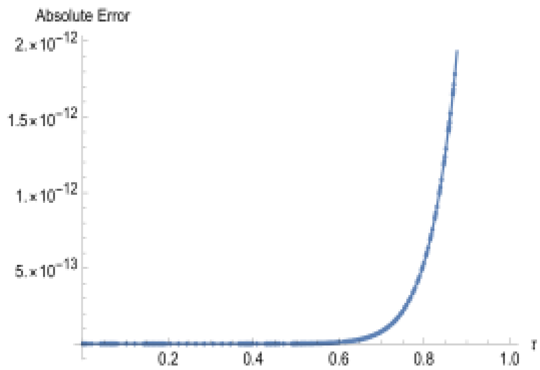

Fig. 4: The graph of absolute error of Example 4.2.

5 Conclusion

The main objective of this study was to introduce an innovative and efficient approach for solving nonlinear Volterra integral differential equations (IDEs). We achieved this by presenting approximate solutions for a family of nonlinear IDEs in the form of infinite series solutions, employing the MDM (Mohanad transform combined to Adomian's decomposition method). Several examples of Volterra IDEs were examined to validate and demonstrate the simplicity and efficiency of the proposed method. The findings of this research article indicate that MDM is a straightforward and effective method for handling nonlinear IDEs. The accuracy and efficiency in providing approximate solutions proposed in this article offer promising prospects for solving a wide range of challenging problems. In future research, we plan to further enhance and refine the method to tackle nonlinear fractional integral equations. This extension would broaden the scope of its applicability and could potentially open up new opportunities for solving more complex mathematical models.

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Conflict of Interest

The authors have no conflicts of interest to declare.