Explicit Runge-Kutta Method for Evaluating Ordinary Differential Equations of type $v^{vi} = f(u, v, v')$

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Abstract: - The initiative of this paper is to present the Runge Kutta Type technique for the development of mathematical solutions to the problems concerning to ordinary differential equation of order six of structure $v^{vi} = f(u, v, v')$ denoted as RKSD with initial conditions. The three and four stage Runge-Kutta methods with order conditions up to order seven (RKSD7) have been designed to evaluate global and local truncated errors for the ordinary differential equation of order six. The framework and evaluation of equations with their results are well established to obtain the effectiveness of RK method towards implicit function satisfying the required initial conditions and for obtaining zero-stability of RKSD7 in terms of their accuracy with maximum precision under minimal processing.

Key-Words: - Ordinary Differential Equations (ODE), Explicit Runge-Kutta Type methods, Local and Global Truncation Error, Zero Stability.

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1 Introduction

In the field of science and engineering there are many real-life situations that can be modelized mathematically, [1], into the linear and higher order differential equations, [2]. In past, researchers analyzed many numerical problems involving linear or ordinary differential equations up to fourth order with different initial conditions using Runge Kutta method, Laplace transformations or many others. In applied sciences and engineering, we observe large number of physical problems involving initial value problems concerned with higher order ordinary differential equations till fourth order. For instance, free vibration analysis of ring structures has been studied by [3]. Moreover some researchers have developed methods likewise schemes of Coupled compact for accuracy of sixth order in space was developed to obtain numerical solutions, [4], Langrages Polynomial with fictional points, [5]. The study, [6], has used neural network for solving differential equations, while [7], contributed by provided solution with B-series and coloring methodology in evolution of differential equations and fluid dynamics which are non-linear in nature and fails to discuss about the most important point, i.e., stability of the solution for enhancing the efficiency of any numerical method. Reducing higher order equations to lower one for solving them is also an approach used by many researchers, [8], [9], [10], [11]. Situations concerned to oscillatory problem. The authors in [12], have solved such problems using approach of finite differences. Also, efficiency of the methods designed by them, like, [13], solved many applied physics problems using multi step methods. Subsequently, two derivative RK method was derived by [14], [15], for solving special first order DE, [15]. In 2021, [16], [17], contributed in analysis of error for differential equations of non-linear and linear type.The authors in [18], solved oscillating systems by optimizing sixth-order RKN method which is explicit in nature.The authors in [19], solved fifth order ODE using generalized Runge- Kutta integrators. The authors in [20], had applied RKN methods which are implicit in nature. Therefore, there are lots of studies and analysis been done for providing the solutions to ordinary differential equation up to higher order. There are several symbolic algorithms for regular initial and boundary value problems for differential equations as well as differential-algebraic equations, see for example, [21], [22], [23], [24]. But, in case of ordinary differential equation of sixth order the initiative had been done but not been proved beneficial or in other word the solution had not been initiated towards minimizing error in shorter time, number of operation and using less memory space. Thus, the current paper focus on evaluating the solutions for sixth order ODE using a single-step method by Runge-Kutta method denoted by RKSD. Moreover, it proves accuracy and stability of the discussed method in minimizing the error with its derivations.

2 Runge-Kutta Type Sixth-order ODE

The initial value problem of sixth order ODE examined in the present paper are:

$$v^{vi}(u) = f(u, v, v') \tag{1}$$

with initial conditions as

$$\begin{aligned} &v(u_0) = \alpha_0, v'(u_0) = \alpha_0^{'}, v''(u_0) = \alpha_0^{''},\\ &v'''(u_0) = \alpha_0^{'''}, v^{iv}(u_0) = \alpha_0^{iv},\\ &v^{v}(u_0) = \alpha_0^{v} \end{aligned} \tag{2}$$

$$\begin{aligned} v_{(n+1)} =& v_n + hv_n^{'} + \frac{h^2}{2}v_n^{''} + \frac{h^3}{3!}v_n^{'''} + \frac{h^4}{4!}v_n^{iv}\\ &+ \frac{h^5}{5!}v_n^{v} + h^6\sum_{i=1}^{s} b_i k_i \end{aligned} \tag{3}$$

$$\begin{aligned} v_{(n+1)}^{'} =& v_n^{'} + hv_n^{''} + \frac{h^2}{2}v_n^{'''} + \frac{h^3}{3!}v_n^{iv}\\ &+ \frac{h^4}{4!}v_n^{v} + h^5\sum_{i=1}^{s} b_i^{'} k_i \end{aligned} \tag{4}$$

$$v_{(n+1)}^{''} = v_n^{''} + hv_n^{'''} + \frac{h^2}{2}v_n^{iv} + \frac{h^3}{3!}v_n^{v} + h^4\sum_{i=1}^{s} b_i^{''} k_i \tag{5}$$

$$v_{(n+1)}^{'''} = v_n^{'''} + hv_n^{iv} + \frac{h^2}{2}v_n^{v} + h^3\sum_{i=1}^{s} b_i^{'''} k_i \tag{6}$$

$$v_{(n+1)}^{iv} = v_n^{iv} + hv_n^{v} + h^2\sum_{i=1}^{s} b_i^{iv} k_i \tag{7}$$

$$v_{(n+1)}^{v} = v_n^{v} + h\sum_{i=1}^{s} b_i^{v} k_i \tag{8}$$

where

$$k_1 = f(u_n, v_n, v_n^{'}), \tag{9}$$

$$\begin{aligned} k_i =& f(u_n + c_i h, v_n + hc_i v_n^{'} + \frac{(h^2c_i^2)}{2}v_n^{''}\\ &+ \frac{(h^3c_i^3)}{3!}v_n^{'''} + \frac{(h^4c_i^4)}{4!}v_n^{iv} + \frac{(h^5c_i^5)}{5!}v_n^{v}\\ &+ h^6\sum_{j=1}^{s} a_{ij}k_i, v_n^{'} + hc_i v_n^{''} + \frac{(h^2c_i^2)}{2}v_n^{'''}\\ &+ \frac{(h^3c_i^3)}{3!}v_n^{iv} + \frac{(h^4c_i^4)}{4!}v_n^{v} + h^5\sum_{j=1}^{s} \bar{a_{ij}}k_j);\\ &for\ i = 1, 2, 3, ...s. \end{aligned} \tag{10}$$

For numerical and algebraic calculations requiring computation efforts, Mathematica software is used to evaluate values of weights, nodes and coefficients and arranged them in Butcher tableau (Table 1) form:

Table 1: The Butcher tableau RKSD Method

c	A	$\bar{A}$				
	b^T	$b^{'T}$	$b^{''T}$	$b^{'''T}$	b^{ivT}	b^{vT}

The principal motive in the construction of RKSD explicit method is for finding the least value of truncation local errors, [25], [26], [27], [28]. The method computes the value to the v_{n+1}^{p} where p is the derivative i.e. $p = 0, 1, 2, \ldots, v_{n+1}$, parameters for obtaining the approximate value to $v(u_{n+1})$, $v^{'}(u_{n+1})$, $v^{''}(u_{n+1})$, $v^{'''}(u_{n+1})$, $v^{iv}(u_{n+1})$, $v^{v}(u_{n+1})$ where v_{n+1} is the calculated solution and $v(u_{n+1})$ is taken as the analytic solution. Equation (3)-(8) be presented as

$$\begin{aligned} v_{n+1} &= v_n + h\psi, & v_{n+1}^{'} &= v_n^{'} + h\psi^{'},\\ v_{n+1}^{''} &= v_n^{''} + h\psi^{''}, & v_{n+1}^{'''} &= v_n^{'''} + h\psi^{'''},\\ v_{n+1}^{iv} &= v_n^{iv} + h\psi^{iv}, & v_{n+1}^{v} &= v_n^{v} + h\psi^{v}. \end{aligned}$$

where

$$\begin{aligned} \psi(u_n, v_n, v_n^{'}) =& v_n^{'} + \frac{h}{2}v_n^{''} + \frac{h^2}{3!}v_n^{'''} + \frac{h^3}{4!}v_n^{iv}\\ &+ \frac{h^4}{5!}v_n^{v} + h^5\sum_{i=1}^{s} b_i k_i \end{aligned} \tag{11}$$

$$\begin{aligned} \psi^{'}(u_n, v_n, v_n^{'}) =& v_n^{''} + \frac{h}{2}v_n^{'''} + \frac{h^2}{3!}v_n^{iv}\\ &+ \frac{h^3}{4!}v_n^{v} + h^4\sum_{i=1}^{s} b_i^{'} k_i \end{aligned} \tag{12}$$

(13)

$$\psi^{''}(u_n, v_n, v_n^{'}) = v_n^{'''} + \frac{h}{2}v_n^{iv} + \frac{h^2}{3!}v_n^{v} + h^3\sum_{i=1}^{s} b_i^{''}k_i \tag{14}$$

$$\psi^{'''}(u_n, v_n, v_n^{'}) = v_n^{iv} + \frac{h}{2}v_n^{v} + h^2\sum_{i=1}^{s} b_i^{'''}k_i \tag{15}$$

$$\psi^{iv}(u_n, v_n, v_n^{'}) = v_n^{v} + h\sum_{i=1}^{s} b_i^{iv}k_i \tag{16}$$

$$\psi^{v}(u_n, v_n, v_n^{'}) = \sum_{i=1}^{s} b_i^{v}k_i \tag{17}$$

Elementary differentials of the scalar equations are as follows:

$$F_1^{(6)} = v^{(vi)} = f(u, v, v_n^{'}), \tag{18}$$

$$F_1^{(7)} == f_u + f_v v^{'} + f_{v'}v_{uu}, \tag{19}$$

$$\begin{aligned} F_1^{(8)} = & f_{uu} + v_u f_{uv} + f_{uv'}v_{uu} + v_u^2 f_{vv} \\ & + f_{vv'}v_u v_{uu} + f_v v_{uu} \\ & + f_{v'v'}v_{uu}^2 + f_{v'}v_{uuu}. \end{aligned} \tag{20}$$

The local truncation error is obtained by having

$$\begin{aligned} \tau_{n+1}^{p} = & h\psi^{p}(u_n, v_n, v_n^{'}) - \Delta^{p}(u_n, v_n, v_n^{'}), \\ & \text{where } p = (0), ..., (v). \end{aligned} \tag{21}$$

Using (18)-(20), Taylor series functions of $v^p(u)$ can be represented as:

$$\begin{aligned} \Delta = & v_n^{'} + \frac{1}{2}hv_n^{''} + \frac{1}{3!}h^2v_n^{'''} + \frac{1}{4!}h^3v_n^{iv} + \frac{1}{5!}h^4v_n^{v} \\ & + \frac{1}{6!}h^5F_1^{(6)} + O(h^6) \end{aligned} \tag{22}$$

$$\begin{aligned} \Delta^{'} = & v_n^{''} + \frac{1}{2}hv_n^{'''} + \frac{1}{3!}h^2v_n^{iv} + \frac{1}{4!}h^3v_n^{v} + \frac{1}{5!}h^4F_1^{(6)} \\ & + \frac{1}{6!}h^5F_1^{(7)} + O(h^6) \end{aligned} \tag{23}$$

$$\begin{aligned} \Delta^{''} = & v_n^{'''} + \frac{1}{2}hv_n^{iv} + \frac{1}{3!}h^2v_n^{v} + \frac{1}{4!}h^3F_1^{(6)} + \frac{1}{5!}h^4F_1^{(7)} \\ & + \frac{1}{6!}h^5F_1^{(8)} + O(h^6) \end{aligned} \tag{24}$$

$$\begin{aligned} \Delta^{'''} = & v_n^{iv} + \frac{1}{2}hv_n^{v} + \frac{1}{3!}h^2F_1^{(6)} + \frac{1}{4!}h^3F_1^{(7)} \\ & + \frac{1}{5!}h^4F_1^{(8)} + \frac{1}{6!}h^5F_1^{(9)} + O(h^6) \end{aligned} \tag{25}$$

$$\begin{aligned} \Delta^{iv} = & v_n^{v} + \frac{1}{2!}hF_1^{(6)} + \frac{1}{3!}h^2F_1^{(7)} + \frac{1}{4!}h^3F_1^{(8)} \\ & + \frac{1}{5!}h^4F_1^{(9)} + \frac{1}{6!}h^5F_1^{(10)} + O(h^6) \end{aligned} \tag{26}$$

$$\begin{aligned} \Delta^{v} = & F_1^{(6)} + \frac{1}{2!}hF_1^{(7)} + \frac{1}{3!}h^2F_1^{(8)} + \frac{1}{4!}h^3F_1^{(9)} \\ & + \frac{1}{5!}h^4F_1^{(10)} + \frac{1}{6!}h^5F_1^{(10)} + O(h^6) \end{aligned} \tag{27}$$

Further on, substituting the equations (18)-(20) into equations (11)-(17), we get

$$\begin{aligned} \sum_{i=1}^{s} b_i k_i = & \sum_{i=1}^{s} b_i F_1^{(6)} + \sum_{i=1}^{s} b_i c_i h F_1^{(7)} + \frac{1}{2}\sum_{i=1}^{s} b_i c_i^2 h^2 F_1^{(8)} \\ & + \frac{1}{3!}\sum_{i=1}^{s} b_i c_i^3 h^3 F_1^{(9)} + +O(h^6). \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{i=1}^{s} b_i^p k_i = & \sum_{i=1}^{s} b_i^p F_1^{(6)} + \sum_{i=1}^{s} b_i^p c_i h F_1^{(7)} + \frac{1}{2}\sum_{i=1}^{s} b_i^p c_i^2 h^2 F_1^{(8)} \\ & + \frac{1}{3!}\sum_{i=1}^{s} b_i^p c_i^3 h^3 F_1^{(9)} + +O(h^6) \end{aligned} \tag{28}$$

where p is the derivative i.e. $p = 0, 1, 2, ...v$.

Using equations (11)-(17) and equations (22)-(27), the local truncation errors equation (21) will be represented as

$$\tau_{n+1} = h^6[\sum b_i k_i - (\frac{1}{6!}F_1^{(6)} + \frac{1}{7!}hF_1^{(7)} + \frac{1}{8!}h^2F_1^{(8)} + \frac{1}{9!}h^3F_1^{(9)} + ...)] \tag{29}$$

$$\tau_{n+1}^{'} = h^5[\sum b_i^{'} k_i - (\frac{1}{5!}F_1^{(6)} + \frac{1}{6!}hF_1^{(7)} + \frac{1}{7!}h^2F_1^{(8)} + \frac{1}{8!}h^3F_1^{(9)} + ...)] \tag{30}$$

$$\tau_{n+1}^{''} = h^4[\sum b_i^{''} k_i - (\frac{1}{4!}F_1^{(6)} + \frac{1}{5!}hF_1^{(7)} + \frac{1}{6!}h^2F_1^{(8)} + \frac{1}{7!}h^3F_1^{(9)} + ...)] \tag{31}$$

$$\tau_{n+1}^{'''} = h^3[\sum b_i^{'''} k_i - (\frac{1}{3!}F_1^{(6)} + \frac{1}{4!}hF_1^{(7)} + \frac{1}{5!}h^2F_1^{(8)} + \frac{1}{6!}h^3F_1^{(9)} + ...)] \tag{32}$$

$$\tau_{n+1}^{iv} = h^2[\sum b_i^{iv} k_i - (\frac{1}{2!}F_1^{(6)} + \frac{1}{3!}hF_1^{(7)} + \frac{1}{4!}h^2F_1^{(8)} + \frac{1}{5!}h^3F_1^{(9)} + ...)] \tag{33}$$

$$\tau_{n+1}^{v} = h[\sum b_i^{v} k_i - (F_1^{(6)} + \frac{1}{2!}hF_1^{(7)} + \frac{1}{3!}h^2F_1^{(8)} + \frac{1}{4!}h^3F_1^{(9)} + ...)] \tag{34}$$

3 RKSD7 Method Order Conditions

The order conditions of RKSD7 are:
The order terms of v:

$$Sixth\ order : \sum b_i = \frac{1}{720}, \tag{35}$$

$$Seventh\ order : \sum b_i c_i = \frac{1}{5040}. \quad (36)$$

The order terms of v' :

$$Fifth\ order : \sum b_i' = \frac{1}{120}, \quad (37)$$

$$Sixth\ order : \sum b_i' c_i = \frac{1}{720}, \quad (38)$$

$$7th\ order : \sum b_i' c_i^2 = \frac{1}{2520}, \sum b_i' \bar{a_{ij}} = \frac{1}{5040}. \quad (39)$$

The order terms of v'' :

$$Fourth\ order : \sum b_i'' = \frac{1}{24}, \quad (40)$$

$$Fifth\ order : \sum b_i'' c_i = \frac{1}{120}, \quad (41)$$

$$Sixth\ order : \sum b_i'' c_i^2 = \frac{1}{360}, \sum b_i'' \bar{a_{ij}} = \frac{1}{720}. \quad (42)$$

$$7th\ order : \sum b_i'' c_i^3 = \tfrac{1}{840}, \sum b_i'' \bar{a_{ij}} c_j = \tfrac{1}{5040}, \sum b_i'' a_{ij} = \tfrac{1}{5040}, \sum b_i'' c_i \bar{a_{ij}} = \tfrac{1}{1680} \quad (43)$$

The order terms of v''' :

$$Third\ order : \sum b_i''' = \frac{1}{6}, \quad (44)$$

$$Fourth\ order : \sum b_i''' c_i = \frac{1}{24}, \quad (45)$$

$$Fifth\ order : \sum b_i''' c_i^2 = \frac{1}{60}, \quad (46)$$

$$Sixth\ order : \sum b_i''' c_i^3 = \frac{1}{120}, \sum b_i''' \bar{a_{ij}} = \frac{1}{720}. \quad (47)$$

$$7th\ order : \sum b_i''' c_i^4 = \tfrac{1}{210}, \sum b_i''' \bar{a_{ij}} c_j = \tfrac{1}{5040}, \sum b_i''' a_{ij} = \tfrac{1}{5040}, \sum b_i''' c_i \bar{a_{ij}} = \tfrac{1}{1260} \quad (48)$$

The order terms of v^{iv} :

$$Second\ order : \sum b_i^{iv} = \frac{1}{2}, \quad (49)$$

$$Third\ order : \sum b_i^{iv} c_i = \frac{1}{6}, \quad (50)$$

$$Fourth\ order : \sum b_i^{iv} c_i^2 = \frac{1}{12}, \quad (51)$$

$$Fifth\ order : \sum b_i^{iv} c_i^3 = \frac{1}{20}, \quad (52)$$

$$Sixth\ order : \sum b_i^{iv} c_i^4 = \frac{1}{30}, \sum b_i^{iv} \bar{a_{ij}} = \frac{1}{720}. \quad (53)$$

$$7th\ order : \sum b_i^{iv} c_i^5 = \tfrac{1}{42}, \sum b_i^{iv} \bar{a_{ij}} c_j = \tfrac{1}{5040}, \sum b_i^{iv} a_{ij} = \tfrac{1}{5040}, \sum b_i^{iv} c_i \bar{a_{ij}} = \tfrac{1}{1008} \quad (54)$$

The order terms of v^v :

$$First\ order : \sum b_i^v = 1, \quad (55)$$

$$Second\ order : \sum b_i^v c_i = \frac{1}{2}, \quad (56)$$

$$Third\ order : \sum b_i^v c_i^2 = \frac{1}{3}, \quad (57)$$

$$Fourth\ order : \sum b_i^v c_i^3 = \frac{1}{4}, \quad (58)$$

$$Fifth\ order : \sum b_i^v c_i^4 = \frac{1}{5}, \sum b_i^v \bar{a_{ij}} = \frac{1}{120}. \quad (59)$$

$$6th\ order : \sum b_i^v c_i^5 = \tfrac{1}{6}, \sum b_i^{iv} \bar{a_{ij}} = \tfrac{1}{720}, \sum b_i^v \bar{a_{ij}} c_i c_j = \tfrac{1}{144}. \quad (60)$$

$$7th\ order : \sum b_i^v c_i^6 = \tfrac{1}{7}, \sum b_i^v \bar{a_{ij}} c_j = \tfrac{1}{5040}, \sum b_i^v a_{ij} = \tfrac{1}{5040}, \sum b_i^v c_i \bar{a_{ij}} = \tfrac{1}{840}, \sum b_i^v a_{ij} c_i c_j = \tfrac{1}{840}. \quad (61)$$

4 Zero-Stability of RKSD7 Method

The most important precondition for obtaining the convergence of numerical problem is evaluating zero-stability of the system, as explained by [1]. The methodology used in current research paper be written in an array representation as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{n+1} \\ hv_{n+1}' \\ h^2 v_{n+1}'' \\ h^3 v_{n+1}''' \\ h^4 v_{n+1}^{iv} \\ h^5 v_{n+1}^{v} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_n \\ hv_n' \\ h^2 v_n'' \\ h^3 v_n''' \\ h^4 v_n^{iv} \\ h^5 v_n^{v} \end{bmatrix}$$

The characteristic equation represented by $\rho(\xi)$ can be presented as:

$$\rho(\xi) = |I.\xi - A|$$

Hence, $\rho(\xi) = (\xi - 1)^6$ we we get the roots to be $\xi = 1, 1, 1, 1, 1, 1$, which is the zero-stability of the given proposed method.

5 Construction of RKSD Methods

5.1 A Third Stage Seventh Order RKSD

The motive of current section is the derivation of a third stage with 7th order RKSD method, where we use the conditions of Equations (35)-(61) respectively, as simultaneous equations for calculating the values of c_i, b_i^p for $i = 1, 2, 3$ as follows:

$$c_2 = \frac{1}{5}(\frac{1-5c_3}{1-4c_3}), b_1 = \frac{28c_2c_3 - 4(c_2+c_3)+1}{20160c_2c_3},$$
$$b_2 = \frac{4c_3-1}{20160c_2(c_3-c_2)}, b_3 = \frac{1-4c_2}{20160c_3(c_3-c_2)},$$
$$b_1^{'} = \frac{378c_2c_3 - 63(c_2+c_3)+18}{45360c_2c_3},$$
$$b_2^{'} = \frac{63c_3-18}{45360c_2(c_3-c_2)},$$
$$b_3^{'} = \frac{18-63c_2}{45360c_3(c_3-c_2)},$$
$$b_1^{''} = \frac{15c_2c_3 - 3(c_2+c_3)+1}{360c_2c_3},$$
$$b_3^{''} = \frac{1-3c_2}{360c_3(c_3-c_2)},$$

$$b_1^{'''} = \frac{20c_2c_3 - 5(c_2+c_3)+2}{120c_2c_3},$$
$$b_2^{'''} = \frac{5c_3-2}{120c_2(c_3-c_2)},$$
$$b_3^{'''} = \frac{2-5c_2}{120c_3(c_3-c_2)},$$
$$b_1^{iv} = \frac{6c_2c_3 - 2(c_2+c_3)+1}{12c_2c_3},$$
$$b_2^{iv} = \frac{2c_3-1}{12c_2(c_3-c_2)},$$
$$b_3^{iv} = \frac{1-2c_2}{12c_3(c_3-c_2)},$$
$$b_1^{v} = \frac{6c_2c_3 - 3(c_2+c_3)+2}{6c_2c_3},$$
$$b_2^{iv} = \frac{3c_3-2}{6c_2(c_3-c_2)},$$
$$b_3^{v} = \frac{2-3c_2}{6c_3(c_3-c_2)},$$

The errors norms of $v(u)$, $v^{'}(u)$, $v^{''}(u)$, $v^{'''}(u)$, $v^{iv}(u)$, $v^{v}(u)$ are as

$$||\ \tau^{(7)}\ ||_2 = \frac{1}{5040}\sqrt{\frac{(2-15c_3-5040(b_1+b_3-5b_1c_3-10b_3c_3+20b_3c_3^2))^2}{(5-20c_3)^2}}, \tag{62}$$

$$||\ \tau^{'(7)}\ ||_2 = \frac{1}{5040}\sqrt{\frac{(-3+5c_3-5040b_3^{'}c_3(1-10c_3+20c_3^2))^2}{(5-20c_3)^2}}, \tag{63}$$

$$||\ \tau^{''(7)}\ ||_2 = \frac{1}{2520}\sqrt{\frac{(-8+25c_3-2520b_3^{''}c_3^2(1-10c_3+20c_3^2))^2}{(5-20c_3)^2}}, \tag{64}$$

$$||\ \tau^{'''(7)}\ ||_2 = \frac{1}{840}\sqrt{\frac{(-13+45c_3-840b_3^{'''}c_3^3(1-10c_3+20c_3^2))^2}{(5-20c_3)^2}}, \tag{65}$$

$$||\ \tau^{iv(7)}\ ||_2 = \frac{1}{210}\sqrt{\frac{(-18+65c_3-210b_3^{iv}c_3^4(1-10c_3+20c_3^2))^2}{(5-20c_3)^2}}, \tag{66}$$

$$||\ \tau^{v(7)}\ ||_2 = \frac{1}{42}\sqrt{\frac{(-23+85c_3-42b_3^{v}c_3^5(1-10c_3+20c_3^2))^2}{(5-20c_3)^2}}, \tag{67}$$

For evaluating the minimal value to the error norms of 7th order Equations (5.1)-(5.6) we find the value of parameters c_i, b_i, $b_i^{'}$, $b_i^{''}$, $b_i^{'''}$, b_i^{iv}, b_i^{v} for $i = 1, 2, 3$ and arranged in mnemonic device known as Butcher tableau. Hence, the result values of error norms are $||\tau^{(7)}||_2 = 0$, $||\tau^{'(7)}||_2 = -3.70074 *$ 10^{-19}, $||\tau^{''(7)}||_2 = -1.5873 * 10^{-4}$, $||\tau^{'''(7)}||_2 = -4.42177 * 10^{-4}$, $||\tau^{iv(7)}||_2 = -4.198251 * 10^{-3}$ and $||\tau^{v(7)}||_2 = -4.2635 * 10^{-2}$.

The global error of three stage seventh order is calculate as follows:

$$\left|\left| \ \tau_g^{(7)} \ \right|\right|_2 = 4.284379 * 10^{-2} \tag{68}$$

$$
\begin{array}{c|ccc|ccc|cccccccccccc}
0 & 0 & & & 0 & & \\
\frac{3}{5} & \frac{7}{487} & 0 & & \frac{-3}{800} & 0 & \\
\frac{2}{7} & \frac{26}{735} & \frac{-11}{588} & 0 & \frac{13}{170} & \frac{-55}{504} & 0 \\
\hline
 & \frac{1}{1531} & \frac{1}{133056} & \frac{1}{1293} & \frac{1}{288} & 0 & \frac{4}{823} & \frac{2}{135} & \frac{1}{475} & \frac{22}{889} & \frac{7}{144} & \frac{5}{198} & \frac{49}{528} & \frac{1}{8} & \frac{25}{132} & \frac{49}{264} & \frac{13}{36} & \frac{100}{99} & \frac{-49}{132}
\end{array}
$$

Figure 1: Butcher Table of 3 Stage 7th Order RKSD Method

5.2 Construction of 4-stage Seventh Order RKSD Methods

Similar to section 5.1, the current section is designed for the derivation of 4-stage 7th order RKSD method, where we have used conditions of Equations (35)-(61) respectively as simultaneous equations for calculating the values of $c_i, b_i, b_i^{'}, b_i^{''}, b_i^{'''}, b_i^{iv}, b_i^{v}$ for $i = 1, 2, 3, 4$ as follows

$$c_2 = \frac{5 - 24c_3}{24 - 90c_3}, b_2 = -\frac{12c_3c_4 - 3(c_3 + c_4) + 1}{60480c_2(c_2 - c_3)(c_4 - c_2)},$$

$$b_3 = -\frac{12c_2c_4 - 3(c_2 + c_4) + 1}{60480c_3(c_2 - c_3)(c_3 - c_4)},$$
$$b_4 = -\frac{12c_2c_3 - 3(c_2 + c_3) + 1}{60480c_4(c_3 - c_4)(c_4 - c_2)},$$

$$b_2^{'} = -\frac{28c_3c_4 - 8(c_3 + c_4) + 3}{20160c_2(c_2 - c_3)(c_4 - c_2)},$$
$$b_3^{'} = -\frac{28c_2c_4 - 8(c_2 + c_4) + 3}{20160c_3(c_2 - c_3)(c_3 - c_4)},$$

$$b_4^{'} = -\frac{28c_2c_3 - 8(c_2 + c_3) + 3}{20160c_4(c_3 - c_4)(c_4 - c_2)},$$
$$b_2^{''} = -\frac{21c_3c_4 - 7(c_3 + c_4) + 3}{2520c_2(c_2 - c_3)(c_4 - c_2)},$$

$$b_3^{''} = -\frac{21c_2c_4 - 7(c_2 + c_4) + 3}{2520c_3(c_2 - c_3)(c_3 - c_4)},$$
$$b_4^{''} = -\frac{21c_2c_3 - 7(c_2 + c_3) + 3}{2520c_4(c_3 - c_4)(c_4 - c_2)},$$

$$b_2^{'''} = -\frac{5c_3c_4 - 2(c_3 + c_4) + 1}{120c_2(c_2 - c_3)(c_4 - c_2)},$$
$$b_3^{'''} = -\frac{5c_2c_4 - 2(c_2 + c_4) + 1}{120c_3(c_2 - c_3)(c_3 - c_4)}$$

$$b_4^{'''} = -\frac{5c_2c_3 - 2(c_2 + c_3) + 1}{120c_4(c_3 - c_4)(c_4 - c_2)},$$
$$b_2^{iv} = -\frac{10c_3c_4 - 5(c_3 + c_4) + 3}{60c_2(c_2 - c_3)(c_4 - c_2)},$$

$$b_3^{iv} = -\frac{10c_2c_4 - 5(c_2 + c_4) + 3}{60c_3(c_2 - c_3)(c_3 - c_4)},$$
$$b_4^{iv} = -\frac{10c_2c_3 - 5(c_2 + c_3) + 3}{60c_4(c_3 - c_4)(c_4 - c_2)},$$

$$b_2^{v} = -\frac{6c_3c_4 - 4(c_3 + c_4) + 3}{12c_2(c_2 - c_3)(c_4 - c_2)},$$
$$b_3^{v} = -\frac{6c_2c_4 - 4(c_2 + c_4) + 3}{12c_3(c_2 - c_3)(c_3 - c_4)},$$

$$b_4^{v} = -\frac{6c_2c_3 - 4(c_2 + c_3) + 3}{12c_4(c_3 - c_4)(c_4 - c_2)},$$
$$c_2 = \frac{1}{6}\left(\frac{1 - 6(c_2 + 6c_3) + 30c_2c_3}{-5(c_2 + 6c_3) + 20c_2c_3}\right).$$

The errors norms of $v^p(u)$ with $p = 0, i, ..., v$ as an derivative.

$$|| \ \tau^{(7)} \ ||_2 = \frac{1}{5040}\sqrt{\frac{(11 - 73c_3 - 5040(55b_1 + 55b_3 + 31b_4 - 24b_1c_3 - 48b_3c_3 + 66b_4c_3 + 90b_3c_3^2))^2}{(24 - 90c_3)^2}}, \tag{69}$$

$$|| \ \tau^{'(7)} \ ||_2 = \frac{1}{5040}\sqrt{\frac{(-13 + 12c_3 - 5040[b_3^{'}c_3(5 - 48c_3 + 90c_3^2) - b_4^{'}(19 - 66c_3)])^2}{(24 - 90c_3)^2}}, \tag{70}$$

$$|| \ \tau^{''(7)} \ ||_2 = \frac{1}{2520}\sqrt{\frac{(-37 + 102c_3 - 2520[b_3^{''}c_3^2(5 - 48c_3 + 90c_3^2) - b_4^{''}(19 - 66c_3)])^2}{(24 - 90c_3)^2}}, \tag{71}$$

$$|| \ \tau^{'''(7)} \ ||_2 = \frac{1}{840}\sqrt{\frac{(-61 + 192c_3 - 840[b_3^{'''}c_3^3(5 - 48c_3 + 90c_3^2) - b_4^{'''}(19 - 66c_3)])^2}{(24 - 90c_3)^2}}, \tag{72}$$

$$\left|\left| \ \tau^{iv(7)} \ \right|\right|_2 = \frac{1}{210}\sqrt{\frac{(-85+282c_3-210[b_3^{iv}c_3^4(5-48c_3+90c_3^2)-b_4^{iv}(19-66c_3)])^2}{(24-90c_3)^2}}, \tag{73}$$

$$\left|\left| \ \tau^{v(7)} \ \right|\right|_2 = \frac{1}{42}\sqrt{\frac{(-109+372c_3-42[b_3^{v}c_3^5(5-48c_3+90c_3^2)-b_4^{v}(19-66c_3)])^2}{(24-90c_3)^2}}, \tag{74}$$

For the least value of error norms of 7th order Equations above we find value of the parameters c_i, b_i, $b_i^{'}$, $b_i^{''}$, $b_i^{'''}$, b_i^{iv}, b_i^{v} for $i = 1,2,3,4$ as shown in the Butcher Figure 1. Hence the result values of error norms are $\left|\left|\tau^{(7)}\right|\right|_2 = 3.250539 * 10^{-2}$, $\left|\left|\tau^{'(7)}\right|\right|_2 = -2.97306 * 10^{-19}$, $\left|\left|\tau^{''(7)}\right|\right|_2 = -1.33788 * 10^{-18}$, $\left|\left|\tau^{'''(7)}\right|\right|_2 = 2.65617 * 10^{-4}$, $\left|\left|\tau^{iv(7)}\right|\right|_2 = -2.23367 * 10^{-3}$ and $\left|\left|\tau^{v(7)}\right|\right|_2 = -1.166532 * 10^{-2}$.

The global error of four stage seventh order is calculated in Figure 2 as:

$$\left|\left| \ \tau_g^{(7)} \ \right|\right|_2 = 3.460838 * 10^{-2} \tag{75}$$

$$\begin{array}{c|cccc|cccc|ccccccccccccccccc}
0 & 0 & & & & 0 & & & & & & & & & & & & & & & & & & & & \\
\frac{6}{7} & \frac{351}{490} & 0 & & & \frac{-11}{8} & 0 & & & & & & & & & & & & & & & & & & & \\
\frac{109}{372} & \frac{-98}{643} & \frac{14}{605} & 0 & & \frac{110}{403} & \frac{-4}{183} & 0 & & & & & & & & & & & & & & & & & & \\
1 & \frac{125}{36} & \frac{80}{491} & \frac{-53}{133} & 0 & \frac{-310}{39} & \frac{-10}{17} & \frac{861}{667} & 0 & & & & & & & & & & & & & & & & & \\
\hline
 & \frac{1}{156} & \frac{1}{655} & \frac{1}{123} & \frac{1}{108} & \frac{1}{295} & \frac{1}{161} & \frac{2}{135} & \frac{1}{194} & \frac{1}{2450} & \frac{12}{877} & \frac{1}{170} & \frac{16}{589} & \frac{1}{114} & \frac{9}{206} & \frac{2}{137} & \frac{79}{705} & \frac{-1}{272} & \frac{77}{912} & \frac{111}{860} & \frac{133}{408} & \frac{-3}{76} & \frac{63}{919} & \frac{426}{853} & \frac{109}{214} & \frac{-61}{789}
\end{array}$$

Figure 2: Butcher Table of Four Stage Seventh Order RKSD Method

6 Numerical Example

The results of the given methods discussed in the Section 5.1 and Section 5.2 are tested with the help of example of sixth order. The result were also tested shown in Figure 3 to compare it with existing implicit RK methods of the same order and with the direct method of solving the sixth order differential equation with constant coefficient.

Example 1. *Consider the homogeneous linear equation given as:*

$$v^{vi}(u) + v^{'}(u) = 0$$

with initial conditions as

$$\begin{aligned} v(u_0) =& 0, v'(u_0) = 1, v''(u_0) = 1, v'''(u_0) = 0, \\ v^{iv}(u_0) =& 1, v^{v}(u_0) = 2. \end{aligned}$$

The exact solution is

$$\begin{aligned} v(u) =& c_1 + c_2e^{-u} + e^{(\frac{1+\sqrt{5}}{4})u}[c_3cos(\frac{\sqrt{10-2\sqrt{5}}}{4})u \\ &+ c_4sin(\frac{\sqrt{10-2\sqrt{5}}}{4})u] + e^{(\frac{1-\sqrt{5}}{4})u} \\ &[c_5cos(\frac{\sqrt{10+2\sqrt{5}}}{4})u + c_6sin(\frac{\sqrt{10+2\sqrt{5}}}{4})u] \end{aligned}$$

where

$$\begin{aligned} c_1 =& \frac{5242880+5242880\sqrt{5}}{16(163840+163840\sqrt{5})}, \\ c_2 =& \frac{1048576\sqrt{5}}{(40-8\sqrt{5})(163840+163840\sqrt{5})} \\ c_3 =& \frac{256(-256-512\sqrt{5})}{163840+163840\sqrt{5}}, \\ c_4 =& \frac{4\sqrt{10}(204800+40960\sqrt{5})}{5(163840+163840\sqrt{5})(\sqrt{5-\sqrt{5}})} \end{aligned}$$

$$\begin{aligned} c_5 =& \frac{32\sqrt{5}(-11796480-1310720\sqrt{5})}{5(163840+163840\sqrt{5})(640-128\sqrt{5})} \\ c_6 =& \frac{(83886080\sqrt{10})}{(163840+163840\sqrt{5})(640-128\sqrt{5})(\sqrt{5-\sqrt{5}})} \end{aligned}$$

7 Conclusion

This paper gives Runge-Kutta technique for solving and assessing the local truncated error for sixth order ODE of form $v^{vi} = f(u, v, v^{'})$ possessing initial conditions. The initiative of introducing the three and four stage seventh order RKSD7 to the sixth order ODE is proved to be beneficial for evaluation of values of norms of zero stability and error of RKSD7 with efficient values of the weights b_i and the nodes c_i and arranged them in the form of Butcher tableau. The objective of providing accurate solution by minimizing error in shorter time, number of operations, use less memory space for ODE of sixth order has been attained in numerical form. Hence, the current paper findings proved be beneficial for analyzing many problems related to engineering, science, medical areas with more accuracy by minimizing the errors. From numerical results, the best outcome been received is that the number of function evaluations of both RKSD7 methods are less than number of function evaluations for other existing RK methods.

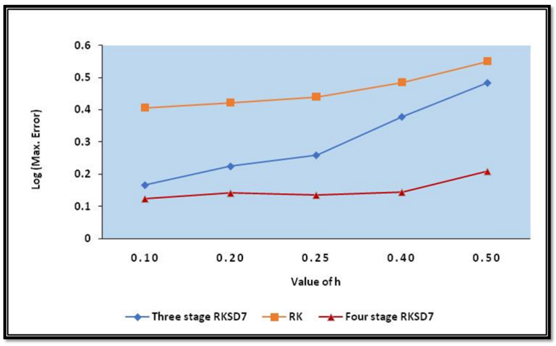

Figure 3: Efficiency curves for RK, three and four stage RKTF7 with step size $h = 0.1, 0.2, 0.25, 0.4, 0.5$

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