

On Solving the Regular Problem of Two-Phase Filtration Taking Into Account The Energy and one Model Problem of Filtration in Potentials by Monte Carlo Methods

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Abstract: - It should be noted that in some practical tasks, it is impossible not to take into account the temperature change. In this case, the energy equation should be added to the filtration equations. The algorithms of «random walk by spheres» and «random walk along boundaries» of Monte Carlo methods are used to solve regular degenerate filtration problems of two immiscible inhomogeneous incompressible liquids in a porous medium. The derivatives of the solution are evaluated using Monte Carlo methods. A model problem of filtration of a two-phase incompressible liquid with capillary forces is considered.

Key-Words: - Monte Carlo method, Dirichlet problem, Markov chains, Levy function, unbiased estimation of the solution, integral operator.

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1 Introduction

In the previous work, two approximate methods for solving two-phase filtration problems - regular and degenerate were proposed. In each of these methods, a linear differential problem is solved at an intermediate stage, which can be approximated by known difference schemes. In both methods, a linear elliptic problem is solved first concerning a given saturation value, [1]. If the temperature change is taken into account, then the energy equations are added to the filtration equations, [2], [3].

2 Formulation of a Regular Filtration Problem Taking Into Account the Energy

It is known if the conditions are met:

$$\begin{aligned} \operatorname{div}_x F &= 0, \quad (x, t) \in Q, \\ F \cdot \vec{n} &= 0, \quad (x, t) \in S, \end{aligned} \quad (1)$$

then the filtering task is called regular. It follows from (1) that almost everywhere in

$$\Omega \quad 0 \leq \delta_0 = \min_{s, (x, t)} s(x, t) \leq s(x, t) \leq \max_{s, (x, t)} s(x, t) = 1 - \delta_1 \leq 1.$$

and for a stationary task $0 \leq \delta_0 \leq s(x) \leq 1 - \delta_1 \leq 1, \quad x \in \Omega$. On regular solutions $K_0(\delta_0, \delta_1) = \text{const.} > 0$. Also, in the regular case, the function $a(x, s) = C_\varepsilon = \text{const.}$, because $\bar{p}_\varepsilon(x) = \text{const.}$ is in a homogeneous environment. So, we got to determine s again the Dirichlet problem for the Poisson equation. We denote $C_\varepsilon = C_\varepsilon \cdot C_\varepsilon^{-n} (1/C_\varepsilon) \cdot \operatorname{div} \vec{W}_\varepsilon = W_{1,\varepsilon}$. Then

$$\begin{cases} \Delta s = -W_{1,\varepsilon}, & \text{in } \Omega \\ s = s_0(x), & \text{on } \partial\Omega \end{cases} \quad (2)$$

$$s = s_0(x), \text{ on } \partial\Omega \quad (3)$$

The problem (2) – (3) is solved using the «random walk by spheres» algorithm. In [2], (see paragraph 11. Unsolved problems, p. 5, p. 303) it is noted that in practical problems it is impossible to neglect temperature changes. That is, to the filtration equations

$$\begin{cases} m \frac{\partial s}{\partial t} = \operatorname{div}(K_0 a \nabla s + K_1 \nabla p + \vec{f}_0) \equiv -\operatorname{div} \vec{V}_1(s, p) \\ \operatorname{div}(K \nabla p + \vec{f}) \equiv -\operatorname{div} \vec{V}(s, p) = 0, \end{cases} \quad (4)$$

$$\operatorname{div}(K \nabla p + \vec{f}) \equiv -\operatorname{div} \vec{V}(s, p) = 0, \quad (5)$$

the energy equation should be added

$$c\rho \frac{\partial \theta}{\partial t} = \operatorname{div}(\chi \nabla \theta) - \rho \vec{U} \nabla \theta, \quad (6)$$

where θ – is the temperature of the mixture, C и χ – heat capacity and thermal conductivity of the mixture, ρ – average density of the mixture, \vec{U} – average speed.

We consider in a homogeneous isotropic medium Ω with a boundary $\partial\Omega$ stationary filtration problem taking into account the temperature

$$\left\{ \begin{array}{l} \Delta p(x) = -h_1(x) \text{ in } \Omega, \quad (7) \\ p(x) = p_0(x) \text{ on } \partial\Omega, \quad (8) \\ \vec{V}(x) = c_3 \nabla p(x) + \vec{f}(x) \text{ in } \Omega, \quad (9) \\ \vec{W}(x) = -b\vec{V}(x) + \vec{F}(x) \text{ in } \Omega, \quad (10) \\ \operatorname{div}(c_2 a \nabla s + \vec{W}) = 0 \text{ in } \Omega, \quad (11) \\ s = s_0(x) \text{ on } \partial\Omega, \quad (12) \\ \operatorname{div}(\chi \nabla \theta) = p\vec{U} \cdot \nabla \theta \text{ in } \Omega, \quad (13) \\ \theta = \theta_0 \text{ on } \partial\Omega \quad (14) \end{array} \right.$$

We construct an algorithm for solving the stationary problem of two-phase filtration taking into account temperature in a homogeneous and isotropic medium Ω with a boundary $\partial\Omega$. For doing so, first, by solving the problem using the «random walk by spheres» algorithm, [4]

$$\left\{ \begin{array}{l} \Delta p(x) = -h_1(x) \text{ in } \Omega, \quad (15) \\ p(x) = p_0(x) \text{ on } \partial\Omega \quad (16) \end{array} \right.$$

we evaluate $p_\varepsilon(x)$, $\nabla p_\varepsilon(x)$ и $\Delta p_\varepsilon(x)$, (here and further index ε shows on ε bias of estimates), [5].

We determine:

$$\begin{aligned} \vec{V}_\varepsilon(x) &= c_3 \nabla p_\varepsilon(x) + \vec{f}(x), \\ \operatorname{div} \vec{V}_\varepsilon(x) &= c_3 \nabla p_\varepsilon(x) + \operatorname{div} \vec{f}(x) \end{aligned}$$

and

$$\begin{aligned} W_\varepsilon(x) &= -b\vec{V}_\varepsilon(x) + \vec{F}(x), \\ \operatorname{div} \vec{W}_\varepsilon(x) &= -\operatorname{div} \vec{V}_\varepsilon(x) + \operatorname{div} \vec{F}(x) \end{aligned}$$

in the area of Ω .

We assume that the filtering task is regular. Then to determine the saturation s_ε we get the task

$$\begin{cases} \Delta s = -W_{1\varepsilon} \text{ in } \Omega, & (17) \\ s = s_0(x) \text{ on } \partial\Omega, & (18) \end{cases}$$

Where $W_{1\varepsilon}(x) = (1/c_6) \cdot \operatorname{div} \vec{W}_\varepsilon$. Solution of the problem (17), (18) $s_\varepsilon(x)$ we also evaluate it by «random walk by spheres». Now we consider the stationary energy equation in the domain Ω

$$\operatorname{div}(\chi \nabla \theta) - \rho \vec{U} \nabla \theta = 0 \text{ in } \Omega, \quad (19)$$

We suppose that on the boundary $\partial\Omega$ areas Ω $\theta = \theta_0$ and χ, ρ и \vec{U} – constant values.

In this case, they get the task

$$\begin{cases} \Delta \theta - c_7 \cdot \sum_{i=1}^n \frac{\partial \theta}{\partial x_i} = 0 \text{ in } \Omega, & (20) \\ \theta = \theta_0 \text{ on } \partial\Omega, & (21) \end{cases}$$

Where $c_7 = \frac{\rho}{\Xi} = \text{const.}$, $u_i (i = 1, 2, 3)$ – vecto

r components \vec{U} . Modeling of the Markov chain, on the trajectories of which estimates of the solution of the problem (20), (21) are constructed, is based on the von Neumann selection method, [6].

We consider a more general case. Let be given an elliptic operator

$$\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j} + \tilde{c}(x), \quad (22)$$

where are the coefficients $a_{ij}(x)$, $b_j(x)$ and $\tilde{c}(x)$ – real

measurable functions defined in Ω . Suppose that

the matrix of higher coefficients $A(x) = \{a_{ij}\}_{i,j=1}^n$ is

symmetrical. Through $D(x) \subset \Omega$ let's denote a

collection of regions where the Green's function for a given operator is known. For example, a collection

of balls of maximum radius centered in x . $D(x) = \{y : r \leq R(x)\}$. Let

$\lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_n(x)$ – matrix

eigenvalues $A(x)$, r – distance between

points x and y , $r = |x - y|$. Then by virtue of the ellipticity condition $\lambda_1(x) \geq \nu$, and from the conditions on the coefficients $a_{ij}(x)$ it follows that $\lambda_n(x) \leq \nu_1 = \text{const} < +\infty$.

We determine the function $\sigma(x, y)$ equality

$$\sigma(x, y) = \left(\sum_{i,j=1}^n a_{ij}(y)(x_i - y_i)(x_j - y_j) \right)^{1/2},$$

$$R_1 = \max_{y \in \Omega} \sigma(x, y)$$

For an arbitrary summable on $[0, R_1]$ we determine the functions $p(\sigma)$

$$q(R) = \int_0^R p(\sigma) d\sigma$$

and $\mu(x) = \left(q(R) \sigma_n (n-2) |A(x)|^{1/2} \right)^{-1}$, where σ_n –

the surface area of a sphere of radius 1 in R^n .

Now for the case $\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \equiv \Delta$, where Δ – Laplace

operator, we describe the process of modeling a Markov chain. To do this, we use the following density

(majorant) $p_2(x, y) = 2 \frac{\nu_1}{\nu} \mu(x) (n-2) p(\sigma) \sigma^{1-n}$. In

general, as the majorant density, by choosing the density $p_2(x, y)$, it is easy to construct a Markov chain using the Neumann method, since the modeling $p_2(x, y)$ does not cause difficulties. For modeling the main Markov chain with density $p_1(x, y)$ we use the

inequality $\alpha \cdot p_1(x, y) \leq p_2(x, y)$, where

$\alpha \in [0, 1]$ – evenly distributed on $[0, 1]$ random variable. Modeling efficiency $p_1(x, y)$ with majorant $p_2(x, y)$ equal to 1/2. Hence, the average number of samples to obtain one implementation is 2, [7].

3 Solutions of One Model Filtration Problem in Potentials

Problem statement. We consider the model problem of filtration of a two-phase incompressible liquid taking into account capillary forces. In the cylinder $\{\Omega \times (0, T)\}$ with a boundary $\partial Q = \partial \Omega \times (0, T)$ we look for a solution to the problem

$$\text{div}(N \text{grad} R) + \text{div}(M \text{grad} P) + f_1 + f_2 = 0, \text{ in } Q \quad (23)$$

$$\begin{aligned} \frac{\partial R}{\partial t} &= \text{div}(M \text{grad} R) + \text{div}(N \text{grad} P) + \\ &+ f_1 - f_2 = 0, \text{ in } Q \end{aligned} \quad (24)$$

$$2R(x, 0) = \varphi(x), \quad (25)$$

$$\frac{\partial}{\partial n} (P + R) = 0, \text{ on } \partial \Omega \quad (26)$$

$$P = R, \text{ on } \partial \Omega \quad (27)$$

where $M = k_1 + k_2 > 0$, $N = k_1 - k_2$.

Here the desired functions are P and R

$$2R = U_1 - U_2, \quad 2P = U_1 + U_2.$$

We discretize (23)-(27) only by the time variable and we will assume that the moment of time $t = n\tau$ we know P^n and R^n . Then to determine P^{n+1} on time layers from (23), (27) we will have a Dirichlet problem for an elliptic equation

$$\text{div}(N \text{grad} P^{n+1}) + f^n, \text{ in } \Omega \quad (28)$$

$$P^{n+1} = R^n, \text{ on } \partial \Omega \quad (29)$$

where $f^n = \text{div}(N \text{grad} R^n) + f_1^n + f_2^n$.

Using (24), (25) and (26) to determine R^{n+1} on time layers, we obtain the Neumann problem for the elliptic equation, [8],

$$R^{n+1} - \tau \text{div}(M \text{grad} R^{n+1}) = g^{n+1} \text{ in } \Omega, \quad (30)$$

$$\frac{\partial R^{n+1}}{\partial n} = -\frac{\partial P^{n+1}}{\partial n} \equiv g^{n+1} \text{ on } \partial\Omega, \quad (31)$$

Where $g^{n+1} = \tau(\operatorname{div}(N \operatorname{grad} P^{n+1}) + f_1^{n+1} - f_2^{n+1}) + R^n$.

We solve the problem (28), (29) using either «random walk by spheres» or «random walk along boundaries» algorithms, [9], [10], [11], [12], [13], [14]. The derivatives of the solution are evaluated using Monte Carlo methods $\operatorname{grad} P^{n+1}$ and $\operatorname{div}(N \operatorname{grad} P^{n+1})$.

We will assume that the area Ω is bounded convex, and the boundary $\partial\Omega$ is smooth enough that Green's formula is valid. Also we assume $M = c = \operatorname{const} > 0$.

Then $\operatorname{div}(M \operatorname{grad} P^{n+1}) = M \Delta P^{n+1}$ and in these assumptions using the Levy function for the operator $-\Delta + a(x)$, $a(x) \geq 0$ it is possible to determine an integral equation for solving the problem (30), (31) with the norm of the integral operator acting in $C(\bar{\Omega})$, a smaller unit.

Equation (30) is written as:

$$-\Delta R^{n+1} + a(x)R^{n+1} = \tilde{g}^{n+1}, \quad x \in \Omega. \quad (32)$$

where $a(x) = 1/\tau = \operatorname{const} > 0$, $\tilde{g}^{n+1} = g/a$.

Let $a(x) \leq c_1^2$, $c_1 = \operatorname{const} > 0$.

Then the function

$v(x, y) = \exp(-c_1 r)(\sigma_m(m-2)r^{m-2})^{-1}$, where $r = |x - y|$ is a Levy function for the operator $-\Delta + a(x)$, in doing so, this operator is formally self-adjointed. The integral equation for the problem (32), (31) will have the form, [9]

$$R^{n+1}(x) = \int_{\bar{\Omega}} (1 - q(x, y)) \hat{p}(x, y) R^{n+1}(y) d\mu(y) + F, \quad x \in \Omega$$

where

$$\hat{p}(x, y) = \frac{1 + I_{\partial\Omega}(x)}{\sigma_m} \cdot \begin{cases} k_1(r) \frac{\cos \varphi_{xy}}{r^{m-1}}, & y \in \partial\Omega, \\ k_2(r)/r^{m-1}, & y \in \Omega, \end{cases}$$

$$q(x, y) = \begin{cases} 0, & y \in \partial\Omega \\ \frac{a(y)r \exp(-c_1 r)}{k_2(r)(m-2)}, & y \in \Omega \end{cases}$$

$$F(x) = \frac{1 + I_{\partial\Omega}(x)}{\sigma_m} (F_1(x) + F_2(x)).$$

Here $I_{\partial\Omega}(x)$ is the boundary indicator, measure μ determined on σ - algebra of Borel subsets Ω equality $\mu(A) = \lambda(A) + S(\Delta \cap \partial\Omega)$, λ - Lebesgue measure in \mathbb{R}^m , S -surface area,

$$k_1(r) = (1 + \frac{c_1 r}{m-2}) \exp(-c_1 r),$$

$$k_2(r) = \frac{c_1^2 + c_1(m-3)}{m-2} \exp(-c_1 r),$$

$$F_1(x) = \int_{\Omega} \frac{\exp(-c_1 r)}{r^{m-2}} g^{-n+1}(y) dy,$$

$$F_2(x) = \int_{\partial\Omega} \frac{\exp(-c_1 r)}{r^{m-2}} h^{n+1}(y) dy.$$

Since the area Ω is convex, then $p(x, y) \geq 0$.

If $a(x) \equiv 0$, $h^{n+1}(x) \equiv 0$, $g^{-n+1}(x) \equiv 0$,

then $R^{n+1} \equiv 1$ is a solution to the Neumann problem. It is clear that $0 \leq q(x, y) \leq 1$, because $a(x) \leq c_1^2$.

If $a(x) \neq 0$, then the norm of the integral operator in equation (10) acting in $C(\bar{\Omega})$ less than one. Consequently, the Neumann-Ulam scheme is applicable to the integral equation. Now it is possible to construct unbiased estimates of the solution of the problem (10), (9), and with finite variances, [15], [16], [17], [18]. To solve the problem (8), (9), we can apply a scheme generalizing the Neumann-Ulam scheme – a method for isolating the main part of the operator. All processes of the «random walk by spheres» type with allocation \mathcal{E} -the neighborhood of the boundary fits into the scheme of allocation of the main part of the operator with a small norm of the integral operator.

In the work [14], an algorithm of random walk along the boundary for the external Neumann problem for a self-adjointed elliptic equation of the

second order of the general form is proposed. Having solved the tasks (6), (7) and (8), (9) for $(n + 1)$ -th step in time, we move on to the next time layer, etc. When numerically solving problems of filtration of a two-phase incompressible liquid, some features of these problems should be taken into account. For example, it is necessary to take into account when constructing difference schemes a feature associated with highly varying discontinuous coefficients in sub domains. This difficulty associated with the choice of a step is easily eliminated when Monte Carlo algorithms associated with the modeling of Markov chains are used to evaluate the solution. In this case, the transition to those points where the coefficients suffer a gap is not carried out. That is, the trajectories of the chain should «be able» to start at those points, and at the transition $x \rightarrow y$ get to those points in the area where the coefficients do not have break points.

4 Conclusion

The Masket–Leverett model describing the process of liquid filtration in a porous medium is a system of equations taking into account saturation and pressure obtained using nonlinear laws, solvable Monte Carlo methods and the method of differences in probability. We were able to apply the algorithms of «random walk by spheres» and «random walk along boundaries» Monte Carlo methods to solve regular, degenerate, stationary and non-stationary filtration problems of two immiscible inhomogeneous incompressible liquids in a porous medium, [19], as well as using Monte Carlo algorithms, we solved the filtration problem taking into account temperature (i.e., an energy equation is added to the filtration equations) and one model filtration problem in potentials.

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