The Complementary Join of a Graph

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Abstract: The complementary join of a graph G is introduced in this paper as the join $G+\overline{G}$ of G and its complement considering them as vertex-disjoint graphs. The aim of this paper is to study some properties and some graph invariants of the complementary join of a graph. We find the diameter, the radius and the domination number of $G+\overline{G}$ and determine when $G+\overline{G}$ is self-centered. We obtain a characterization of the Eulerian complementary joins, and show that the complementary join of a nontrivial graph is Hamiltonian. We give the clique and independence numbers of $G+\overline{G}$ in terms of the clique and independence numbers of G. We conclude this paper by determining the chromatic number, the $L(2,1)$-labeling number, the locating chromatic number and the partition dimension of the complementary join of a star.

Key-Words: Complementary join, Eulerian, $L(2,1)$-labeling number, locating chromatic number, partition dimension.

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1 Introduction

All graphs considered in this paper are finite with no loops and no multiple edges. For standard undefined notions the reader is referred to, [1].

The *join* H_1+H_2 of two vertex-disjoint graphs H_1 and H_2 is the graph obtained from the union $H_1 \cup H_2$ by adding all edges that have one end vertex in H_1 and the other in H_2, [1]. If G and $\overline{G}$ are considered as vertex-disjoint graphs, then the *complementary prism* $G\overline{G}$ of G is the graph obtained from the union of G and $\overline{G}$ by adding the perfect matching between corresponding vertices of G and $\overline{G}$, [2]. Comparing the complementary prism $G\overline{G}$ and its complement $\overline{G\overline{G}}$ it is obvious that each of them consists of a copy of G and a copy of $\overline{G}$ together with a set of edges (say E_1 and E_2, respectively) joining these copies. Notice that the join $G+\overline{G}$ also consists of a copy of G and a copy of $\overline{G}$ together with the set of edges E joining these copies where E is just the union of the two disjoint sets E_1 and E_2.

The complementary prism gained the attention of many authors, see for example, [3],[4],[5],[6]. Also the complement of the complementary prism has been studied, some of its properties were investigated in [7]. The aim of this paper is to start studying some properties and some graph invariants of the complementary join $G+\overline{G}$ of a graph G. Notice that the complementary join $G+\overline{G}$ can be viewed as a supergraph of each of the complementary prism $G\overline{G}$ and the complement $\overline{G\overline{G}}$ of the complementary prism. Indeed each of $G\overline{G}$ and $\overline{G\overline{G}}$ is isomorphic to a spanning subgraph of $G+\overline{G}$.

In this paper, we show that the complementary join of a nontrivial graph is Hamiltonian, and obtain a characterization of those complementary joins that are Eulerian. We determine the diameter and the radius of the complementary join. We express the clique and independence numbers of $G+\overline{G}$ in terms of the clique and independence numbers of G. In particular, we determine four graph invariants (the chromatic number, the $L(2,1)$-labeling number, the locating chromatic number and the partition dimension) for the complementary join of a star.

We give a formal definition in which the adapted labeling of the vertices of $G+\overline{G}$ will be used throughout this paper.

Definition 1 *Let G be a graph of order n. The complementary join $G+\overline{G}$ of G is the graph whose vertex set is the union of the two disjoint sets $V(G)=\{a_1,a_2,\cdots,a_n\}$, $V(\overline{G})=\{b_1,b_2,\cdots,b_n\}$ where b_i is the corresponding vertex of a_i, and whose edge set is the union of the three mutually disjoint sets $E(G)$, $E(\overline{G})$, and $E=\{a_ib_j : 1\le i\le n, 1\le j\le n\}$.*

For example, the complement of the complementary prism $C_3\overline{C_3}$ is the 3-sun, while the complementary join $C_3+\overline{C_3}$ of C_3 is the multipartite graph $K_{1,1,1,3}$.

The problem of assigning radio frequencies to transmitters at different locations without causing interference is called the frequency assignment problem. It was formulated as a vertex coloring problem in [8]. A variation of this problem (which is known as the $L(2,1)$-labeling or radio 2-coloring) where closed transmitters receive different frequencies while very closed transmitters receive frequencies that differ by at least 2 was introduced in [9]. Let H be a connected graph of diameter d. Let k be an integer with $1 \leq k \leq d$. The *distance* $d(u,v)$ between the two vertices u and v of H is the number of edges in a shortest u,v-path in H. A *radio k-coloring* of H is a function f from the vertex set of H to the set of positive integers such that $d(u,v)+|f(u)-f(v)| \geq 1+k$ for any two distinct vertices u and v of H, [10]. It is obvious that the radio 1-coloring is just the standard vertex coloring. It is worth to mention that the codomain of the radio k-coloring function f is assumed by some authors to be the set of positive integers, [10], [11], while by many others it is assumed to be the set of nonnegative integers, [12],[13],[14],[15],[16],[17]. In this paper, we will follow the later assumption. Thus for clarity, we restate explicitly the following definition of the span of an $L(2,1)$-labeling and the λ-number of a graph.

Definition 2 *An $L(2,1)$-labeling of a graph H is a function f from $V(H)$ to the set of nonnegative integers (called colors) such that $|f(u)-f(v)| \geq 1$ if $d(u,v) = 2$ and $|f(u)-f(v)| \geq 2$ if u and v are adjacent. The span of f is the difference between the largest and the smallest colors in $f(V(H))$. The $L(2,1)$-labeling number $\lambda(H)$ (also called the λ-number of H) is the minimum span over all $L(2,1)$-labelings of H.*

Notice that the span of an $L(2,1)$-labeling is defined in [11], to be the difference between the largest and the smallest colors plus 1.

For a subset S of the vertex set of a connected graph H and a vertex u of H, the distance between u and S is $d(u,S) = \min\{d(u,x) : x \in S\}$. A *$k$-coloring* f of a connected graph H is an onto function from the set of vertices of H to the set of colors $\{1,2,\cdots,k\}$ such that adjacent vertices have different colors. The coloring f induces an *ordered partition* $\pi = \{R_1,R_2,\cdots,R_k\}$ of the vertex set of H, where for $1 \leq i \leq k$, the *color class* R_i is the set of vertices of H receiving the color i. The *color code* of a vertex u is the k-tuple $f_\pi(u) = (d(u,R_1),d(u,R_2),\cdots,d(u,R_k))$. A *locating coloring* of H is a coloring of H in which every two distinct vertices have different color codes. The *locating chromatic number* $\chi_L(H)$ is the smallest k such that H has a locating k-coloring. The concept of locating chromatic number was introduced in [18]. The locating chromatic number of some classes of graphs was determined by several authors, [18], [19], [20], [21], [22].

Locating chromatic number is related to both coloring and partition dimension of a graph. For an ordered k-partition $\pi = \{R_1,R_2,\cdots,R_k\}$ of the vertex set of a connected graph H, the *representation* of a vertex u of H with respect to the partition π is $r(u \mid \pi) = (d(u,R_1),d(u,R_2),\cdots,d(u,R_k))$. The partition π is a *resolving partition* if distinct vertices have different representations. The *partition dimension* $pd(H)$ of the graph H is the minimum k such that H has a resolving k-partition. Studying partition dimension of graphs starts in [23]. Many authors were interested in determining the partition dimension of some classes of graphs, [23], [24], [25], [26], [27], [28]. The concept of partition dimension was extended also for disconnected graphs, [29],[30].

The following result was obtained in [25], it will be referred to in the proofs of the last two theorems in section 5 of this paper.

Lemma 3 *Let π be a resolving partition of the vertex set of a connected graph H. If u and v are distinct vertices of H such that $d(u,x) = d(v,x)$ for all $x \in V(H)-\{u,v\}$, then u and v belong to different partition classes of π.*

2 Diameter, Radius and Domination Number

The complementary join $K_1 + \overline{K_1}$ is isomorphic to K_2, which is connected. So assume that $n \geq 2$ and let $i,j \in \{1,2,\cdots,n\}$. The two vertices a_i and b_j are adjacent in $G+\overline{G}$. On the other hand, when $i \neq j$ we have: The two vertices a_i and a_j are joint by the path $a_ib_1a_j$, and the two vertices b_i and b_j are joint by the path $b_ia_1b_j$. This implies that $G+\overline{G}$ is connected.

The diameter of $G+\overline{G}$ is determined in the following result.

Proposition 4 *For any graph G of order n, we have*

$$diam(G+\overline{G}) = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n\geq 2 \end{cases}.$$

Proof. Obviously, $diam(K_1+\overline{K_1}) = 1$. So assume that G is a nontrivial graph and let x and y be two distinct vertices of $G+\overline{G}$. If one of x,y belongs to $\{a_1,a_2,...,a_n\}$ while the other belongs to $\{b_1,b_2,\cdots,b_n\}$, then x and y are adjacent in $G+\overline{G}$. Thus assume that both x and y belong to the same set $\{a_1,a_2,...,a_n\}$ or $\{b_1,b_2,\cdots,b_n\}$. Then xwy is an x,y-path in $G+\overline{G}$ where $w = b_1$ or $w = a_1$ according to whether x and y belong to $\{a_1,a_2,...,a_n\}$ or $\{b_1,b_2,\cdots,b_n\}$, respectively. Thus $d_{G+\overline{G}}(x,y) \leq 2$. But since G is not the trivial graph, we have at least one of G and $\overline{G}$ is not complete. Thus $G+\overline{G}$ has two nonadjacent vertices x_0,y_0

with either $x_0, y_0 \in \{a_1, a_2, \cdots, a_n\}$ or $x_0, y_0 \in \{b_1, b_2, \cdots, b_n\}$. This implies that $d_{G+\overline{G}}(x_0, y_0) = 2$ and therefore $diam(G + \overline{G}) = 2$. ■

For any vertex x of a graph G of order n, we have $0 \leq \deg_G x \leq n - 1$, we will say that 0 and $n - 1$ are the extreme degrees for G. Obviously, extreme degrees for G need not be attained.

Definition 5 *A vertex x of a graph G of order n is said to be of extreme degree if $\deg_G x \in \{0, n-1\}$. Moreover, we will say that G has an extreme degree whenever it has a vertex of extreme degree.*

For example, every vertex of the complete graph K_n is of extreme degree, while P_4 has no vertex of extreme degree.

Theorem 6 *Let G be a graph of order n. Then*

$$rad(G + \overline{G}) = \begin{cases} 1 & \text{if } G \text{ has an extreme degree} \\ 2 & \text{otherwise} \end{cases}.$$

Proof. By Proposition 4, we have $rad(G + \overline{G}) \leq 2$. Clearly $rad(G + \overline{G}) = 1$ if and only if there exists a vertex x that is adjacent to all other vertices of $G+\overline{G}$. Now, if x is of the type a_i, then $\deg_G x = n-1$, while if x is of the type b_i, then the corresponding vertex a_i satisfies $\deg_G a_i = 0$. ■

A self-centered graph is a graph whose radius and diameter are equal, [31]. Using Proposition 4 and Theorem 6 we have the following result.

Corollary 7 *Let G be a graph. Then $G + \overline{G}$ is self-centered if and only if either G is the trivial graph or G has no vertex of extreme degree.*

The domination number γ of $G + \overline{G}$ can be computed in view of Theorem 6.

Corollary 8 *Let G be a graph of order n. Then*

$$\gamma(G + \overline{G}) = \begin{cases} 1 & \text{if } G \text{ has an extreme degree} \\ 2 & \text{otherwise} \end{cases}.$$

Proof. Obviously, $\gamma(G + \overline{G}) = 1$ if and only if $rad(G + \overline{G}) = 1$. Thus by Theorem 6 we have $\gamma(G + \overline{G}) = 1$ if and only if G has a vertex of extreme degree. So assume that G has no vertex of extreme degree. Then $\gamma(G + \overline{G}) > 1$. But $\{a_1, b_1\}$ is a dominating set of $G+\overline{G}$, therefore $\gamma(G+\overline{G}) = 2$. ■

3 When Hamiltonian? And When Eulerian?

The following two results determine precisely when $G+\overline{G}$ is Hamiltonian and when it is Eulerian. Recall that a graph H of order $m \geq 3$ in which every vertex has degree greater than or equal to $\frac{m}{2}$ is Hamiltonian, [1].

Proposition 9 *For any nontrivial graph G, the complementary join $G + \overline{G}$ is Hamiltonian.*

Proof. Let G be a graph of order $n > 1$. Then $G + \overline{G}$ has order $2n \geq 4$ and for any vertex x in $G + \overline{G}$ we have $\deg_{G+\overline{G}} x \geq n$ because every vertex of the type a_i is adjacent to every vertex of the type b_i. Therefore $G + \overline{G}$ is Hamiltonian. ■

It is well known that a nontrivial connected graph is Eulerian if and only if all of its vertices have even degrees.

Theorem 10 *Let G be a nontrivial graph of order n. Then $G + \overline{G}$ is Eulerian if and only if n is odd and every vertex of G has odd degree.*

Proof. Assume that n is odd and every vertex of G has odd degree. Then for every $i \in \{1, 2, \cdots, n\}$ we have $\deg_{G+\overline{G}} a_i = \deg_G a_i + n$ which is even, and we have $\deg_{G+\overline{G}} b_i = \deg_{\overline{G}} b_i + n = (n-1-\deg_G a_i)+n$ which is also even because $\deg_G a_i$ is odd. Therefore $G + \overline{G}$ is Eulerian.

Conversely, assume that $G + \overline{G}$ is Eulerian and let $i \in \{1, 2, \cdots, n\}$. Let $\deg_G a_i = m$. Then $\deg_{G+\overline{G}} b_i = (n - 1 - m) + n$ is even since $G + \overline{G}$ is Eulerian. Thus m must be odd. So every vertex of G has odd degree. But $\deg_{G+\overline{G}} a_i = m + n$ is also even since $G + \overline{G}$ is Eulerian. This implies that n must be odd. ■

4 Clique and Independence Numbers

A complete subgraph of a graph H is a *clique* in H. The *clique number* ω of H is the order of a largest clique in H. An *independent set* S of H is a subset of the vertex set of H such that any two elements of S are not adjacent in H. The cardinality of a maximum independent set of H is the *independence number* β of H. It is well known that the clique number of a graph equals the independence number of its complement. This means that for any graph H, we have $\omega(H) = \beta(\overline{H})$ and $\beta(H) = \omega(\overline{H})$. The next result determines the clique and independence numbers of the complementary join $G + \overline{G}$ in terms of the clique and independence numbers of G.

Theorem 11 *For any graph G, we have:*

$$\omega(G + \overline{G}) = \omega(G) + \beta(G)$$
$$\text{and } \beta(G + \overline{G}) = \max\{\beta(G), \omega(G)\}.$$

Proof. We will compute ω and β of the complementary join $G + \overline{G}$ by computing β and ω of its complement $\overline{G + \overline{G}}$, respectively. The vertex set of $\overline{G + \overline{G}}$ is $\{a_1, a_2, \cdots, a_n\} \cup \{b_1, b_2, \cdots, b_n\}$. But every vertex of the type a_i is not adjacent in $\overline{G+\overline{G}}$ to any vertex of the type b_i. Thus the edge set of $\overline{G+\overline{G}}$ is the union of the two sets $\{a_ia_j : i \neq j, a_ia_j \notin E(G)\}$ and $\{b_ib_j : i \neq j, b_ib_j \notin E(\overline{G})\}$. Therefore, $\overline{G+\overline{G}}$ is isomorphic to $\overline{G} \cup G$. Now it follows that:

$$\begin{aligned}\omega(G+\overline{G}) &= \beta(\overline{G+\overline{G}})\\ &= \beta(\overline{G} \cup G)\\ &= \beta(\overline{G}) + \beta(G)\\ &= \omega(G) + \beta(G)\end{aligned}$$

and

$$\begin{aligned}\beta(G+\overline{G}) &= \omega(\overline{G+\overline{G}})\\ &= \omega(\overline{G} \cup G)\\ &= \max\{\omega(\overline{G}), \omega(G)\}\\ &= \max\{\beta(G), \omega(G)\}\end{aligned}$$

since $\overline{G+\overline{G}} \cong \overline{G} \cup G$ where $\overline{G}$ and G have disjoint vertex sets. ■

5 Main Results

This section is devoted to determine the chromatic number, the $L(2,1)$-labeling number, the locating chromatic number and the partition dimension of the complementary join of a star $K_{1,m}$.

For $m \geq 2$, throughout this section we will assume that a_{m+1} is the central vertex of the star $K_{1,m}$, and we will denote it simply by a. Thus $V(K_{1,m}) = \{a\} \cup \{a_1, a_2, \cdots, a_m\}$ where $\deg_{K_{1,m}} a = m$.

Coloring the subgraph of $G+\overline{G}$ induced by $V(G)$ using $\chi(G)$ colors, and then coloring the subgraph of $G+\overline{G}$ induced by $V(\overline{G})$ using new $\chi(\overline{G})$ colors, we obtain a $(\chi(G)+\chi(\overline{G}))$-coloring of $G+\overline{G}$. But since each vertex in $V(\overline{G})$ is adjacent to every vertex in $V(G)$, we cannot have a common color used in both sets $V(G)$ and $V(\overline{G})$. So we have the following result.

Proposition 12 *For any graph G, we have $\chi(G+\overline{G}) = \chi(G)+\chi(\overline{G})$. In particular $\chi(K_{1,m}+\overline{K_{1,m}}) = 2+m$.*

Now, we compute the $L(2,1)$-labeling number of the complementary join of a star.

Theorem 13 *(a) For $m \geq 2$, we have $\lambda(K_{1,m}+\overline{K_{1,m}}) = 3m+1$.*

(b) $\lambda(K_{1,1}+\overline{K_{1,1}}) = 5$.

Proof. (a) Let $m \geq 2$ and let f be any $L(2,1)$-labeling of $K_{1,m}+\overline{K_{1,m}}$. Since $\{a, a_1\} \cup \{b_1, b_2, \cdots, b_m\}$ induces in $K_{1,m}+\overline{K_{1,m}}$ a clique B of order $m+2$, the colors assigned by f to any two distinct vertices of B must differ by at least 2. Thus $\lambda(K_{1,m}+\overline{K_{1,m}}) \geq \lambda(B) \geq 2m+2$. But for every i with $2 \leq i \leq m$, we have $d_{K_{1,m}+\overline{K_{1,m}}}(a_i, a) = d_{K_{1,m}+\overline{K_{1,m}}}(a_i, b_j) = 1$ for any j with $1 \leq j \leq m$, and $d_{K_{1,m}+\overline{K_{1,m}}}(a_i, a_1) = 2$. This implies that for every i with $2 \leq i \leq m$, $f(a_i)$ must differ than $f(a_1)$, and $f(a_i)$ must differ by at least 2 than each of $f(a)$ and $f(b_j)$ for any j with $1 \leq j \leq m$. Clearly $f(a_i) \neq f(a_j)$ for every distinct $i, j \in \{2, 3, \cdots, m\}$ because $d_{K_{1,m}+\overline{K_{1,m}}}(a_i, a_j) = 2$. Thus the values of $f(a_2), f(a_3), \cdots, f(a_m)$ are different and each of them pushes the lower bound $2m+2$ of $\lambda(K_{1,m}+\overline{K_{1,m}})$ up by 1. This implies that $\lambda(K_{1,m}+\overline{K_{1,m}}) \geq (2m+2)+(m-1) = 3m+1$.

Now define the function g on $V(K_{1,m}+\overline{K_{1,m}})$ as follows:

$$\begin{aligned}g(b_i) &= 2i-2 \text{ for } 1 \leq i \leq m,\\ g(a) &= 2m,\\ g(a_1) &= 2m+2,\\ g(a_i) &= 2m+i+1 \text{ for } 2 \leq i \leq m,\\ \text{and } g(b) &= 1.\end{aligned}$$

Notice that since $m \geq 2$, we have $g(x) - g(b) \geq 2m-1 \geq 3$ for every $x \in N(b) = \{a_i : 1 \leq i \leq m\} \cup \{a\}$. One can easily check that g is an $L(2,1)$-labeling of $K_{1,m}+\overline{K_{1,m}}$ with span $3m+1$. Therefore $\lambda(K_{1,m}+\overline{K_{1,m}}) \leq 3m+1$ and hence $\lambda(K_{1,m}+\overline{K_{1,m}}) = 3m+1$.

(b) The function g defined on $V(K_{1,1}+\overline{K_{1,1}})$ by $g(a_1) = 0$, $g(a) = 2$, $g(b_1) = 4$, and $g(b) = 5$ is clearly an $L(2,1)$-labeling of $K_{1,1}+\overline{K_{1,1}}$. Therefore $\lambda(K_{1,1}+\overline{K_{1,1}}) \leq 5$. Suppose to the contrary that f is an $L(2,1)$-labeling of $K_{1,1}+\overline{K_{1,1}}$ having span less than 5. But since $\{a, a_1, b_1\}$ induces in $K_{1,1}+\overline{K_{1,1}}$ a clique, we must have $\{f(a), f(a_1), f(b_1)\} = \{0, 2, 4\}$. Now $f(b) \in \{1, 3\}$ because $d(b, x) \leq 2$ for any $x \in \{a, a_1, b_1\}$. This contradicts the fact that both $|f(b) - f(a)|$ and $|f(b) - f(a_1)|$ must be at least 2 since b is adjacent to both a and a_1. Therefore $\lambda(K_{1,1}+\overline{K_{1,1}}) = 5$. ■

Next, we determine the locating chromatic number of the complementary join of a star.

Theorem 14 *(a) For $m \geq 2$, we have $\chi_L(K_{1,m}+\overline{K_{1,m}}) = 2m+1$.*

(b) $\chi_L(K_{1,1}+\overline{K_{1,1}}) = 4$.

Proof. (a) Let $m \geq 2$. First, we will show that the locating chromatic number of $K_{1,m}+\overline{K_{1,m}}$ is greater than or equal to $2m+1$. Let f be any locating coloring of $K_{1,m} + \overline{K_{1,m}}$. Observe that $E(K_{1,m} + \overline{K_{1,m}})$ contains all edges of the form xy where $x \in V(K_{1,m}) = \{a\} \cup \{a_1, a_2, \cdots, a_m\}$ and $y \in V(\overline{K_{1,m}}) = \{b\} \cup \{b_1, b_2, \cdots, b_m\}$. Thus the ordered partition of $V(K_{1,m} + \overline{K_{1,m}})$ induced by f must be of the form $\pi = \{R_1, R_2, \cdots, R_k, S_1, S_2, \cdots, S_p\}$ for some positive integers k and p, where $\cup_{i=1}^{k} R_i = V(K_{1,m})$ and $\cup_{i=1}^{p} S_i = V(\overline{K_{1,m}})$. Clearly $\{a\}$ is one of the color classes $R_1, R_2, \cdots, R_k$ since a is adjacent to a_i for each $1 \leq i \leq m$, say $\{a\} = R_1$. But $\{a, a_1\} \cup \{b_1, b_2, \cdots, b_m\}$ induces in $K_{1,m} + \overline{K_{1,m}}$ a clique B of order $m + 2$, so f must use different $m + 2$ colors $1, 2, \cdots, m + 2$ to color the vertices of B. Without loss of generality we can assume that $f(b_i) = i$ for $1 \leq i \leq m$, $f(a) = m + 1$ and $f(a_1) = m + 2$. Then for every i with $2 \leq i \leq m$, we have $f(a_i) > m + 1$ because a_i is adjacent to each vertex in $\{a\} \cup \{b_1, b_2, \cdots, b_m\}$. But π is a resolving partition since f is a locating coloring of $K_{1,m} + \overline{K_{1,m}}$, thus for every distinct $i, j \in \{1, 2, \cdots, m\}$, we must have $f(a_i) \neq f(a_j)$ according to Lemma 3. Therefore f must use new $m - 1$ colors, say $m + 3, m + 4, \cdots, m + (m + 1)$, to color the vertices $a_2, a_3, \cdots, a_m$, respectively. Thus $\chi_L(K_{1,m} + \overline{K_{1,m}}) \geq (m + 2) + (m - 1) = 2m + 1$.

Second, we will show that the locating chromatic number of $K_{1,m} + \overline{K_{1,m}}$ is less than or equal to $2m+1$ by providing a locating coloring g that uses exactly $2m + 1$ colors. Define the function g on $V(K_{1,m} + \overline{K_{1,m}})$ as follows:

$$g(b_i) = i \text{ for } 1 \leq i \leq m,$$
$$g(a) = m + 1,$$
$$g(a_i) = i + m + 1 \text{ for } 1 \leq i \leq m,$$
$$\text{and } g(b) = 1.$$

Then $\{b, b_1\}$ is the only color class induced by g that contains more than one element. Observe that since $m \geq 2$, the vertices b and b_1 have different color codes because $d(b, S) = 2$ while $d(b_1, S) = 1$ where $S = \{b_2\}$ is the color class having the color 2. Thus g is a locating coloring of $K_{1,m} + \overline{K_{1,m}}$ and hence $\chi_L(K_{1,m} + \overline{K_{1,m}}) \leq 2m + 1$. Therefore $\chi_L(K_{1,m} + \overline{K_{1,m}}) = 2m + 1$.

(b) The set $\{a, a_1, b_1\}$ induces in $K_{1,1} + \overline{K_{1,1}}$ a clique B, so any locating coloring of $K_{1,1} + \overline{K_{1,1}}$ must use three colors $1, 2, 3$ to color the vertices of B. It is obvious that the color of b must be different from the colors of its neighbors a and a_1. But also b cannot be assigned the color of b_1, for otherwise b_1 and b would have the same color code. Thus b must have a new fourth color and hence $\chi_L(K_{1,1} + \overline{K_{1,1}}) = 4$. ■

Finally, we determine the partition dimension of the complementary join of a star.

Theorem 15 $pd(K_{1,m} + \overline{K_{1,m}}) = m + 2$.

Proof. Consider the partition $\delta = \{R_1, R_2, \cdots, R_{m+2}\}$ where $R_i = \{a_i, b_i\}$ for $1 \leq i \leq m$, $R_{m+1} = \{a\}$, and $R_{m+2} = \{b\}$. Then for every $i \in \{1, 2, \cdots, m\}$ we have $d_{K_{1,m}+\overline{K_{1,m}}}(a_i, R_{m+2}) = 1$ while $d_{K_{1,m}+\overline{K_{1,m}}}(b_i, R_{m+2}) = 2$. This implies that δ is a resolving partition, and hence $pd(K_{1,m} + \overline{K_{1,m}}) \leq m + 2$. On the other hand, by Lemma 3, for every distinct $i, j \in \{1, 2, \cdots, m\}$ the two vertices a_i and a_j belong to different color classes in any resolving partition. Thus we have $pd(K_{1,m} + \overline{K_{1,m}}) \geq m$. Now assume to the contrary that $pd(K_{1,m} + \overline{K_{1,m}}) \neq m + 2$ and distinguish the following two cases:

Case 1. $pd(K_{1,m} + \overline{K_{1,m}}) = m$.

Assume that $\theta = \{R_1, R_2, \cdots, R_m\}$ is a resolving partition of $K_{1,m} + \overline{K_{1,m}}$. But by Lemma 3, for every distinct $i, j \in \{1, 2, \cdots, m\}$ the two vertices b_i and b_j belong to different color classes (and the same holds for distinct a_i, a_j), thus for each i with $1 \leq i \leq m$, we have $R_i \supseteq \{a_{s_i}, b_{w_i}\}$ for some $s_i, w_i \in \{1, 2, \cdots, m\}$. Now $a \in R_k$ for some $k \in \{1, 2, \cdots, m\}$, which implies that a_{s_k} and a have the same representation $(h_1, h_2, \cdots, h_m)$ with $h_k = 0$ while $h_i = 1$ for any $i \in \{1, 2, \cdots, m\} - \{k\}$. This contradicts the assumption that θ is a resolving partition.

Case 2. $pd(K_{1,m} + \overline{K_{1,m}}) = m + 1$.

Assume that $\theta = \{R_1, R_2, \cdots, R_{m+1}\}$ is a resolving partition of $K_{1,m} + \overline{K_{1,m}}$. Again by applying Lemma 3 on distinct vertices in $\{a_1, a_2, \cdots, a_m\}$ and on distinct vertices in $\{b_1, b_2, \cdots, b_m\}$, we must have at least $m-1$ color classes of θ containing simultaneously a vertex from $\{a_1, a_2, \cdots, a_m\}$ and a vertex from $\{b_1, b_2, \cdots, b_m\}$. But by the symmetry between any two vertices in $\{a_1, a_2, \cdots, a_m\}$ and also between any two vertices in $\{b_1, b_2, \cdots, b_m\}$ with respect to distances to other vertices, we can assume without loss of generality that $R_i \supseteq \{a_i, b_i\}$ for $1 \leq i \leq m - 1$. We distinguish two subcases.

Subcase 2.1. Both a_m and b_m belong to the same color class, say R_m.

Then the vertex a cannot belong to any R_i for $1 \leq i \leq m$, because otherwise if $a \in R_k$ for some $k \in \{1, 2, \cdots, m\}$, then $R_{m+1} = \{b\}$ and the two vertices a and a_k would have the same representation $(h_1, h_2, \cdots, h_{m+1})$ with $h_k = 0$ while $h_i = 1$ for any $i \in \{1, 2, \cdots, m + 1\} - \{k\}$, a contradiction. Thus we have $R_{m+1} \supseteq \{a\}$. But now, the two vertices a_1 and b_1 have the same representation $(0, 1, \cdots, 1)$, a contradiction.

Subcase 2.2. The two vertices a_m and b_m belong to different color classes, say that $R_m \supseteq \{a_m\}$ and

$R_{m+1} \supseteq \{b_m\}$. But now we have either $a \in R_m$, $a \in R_{m+1}$ or $a \in R_k$ for some $k \leq m-1$. This implies that either $r(a \mid \theta) = r(a_m \mid \theta) = (1, 1, \cdots, 1, 0, 1)$, $r(a \mid \theta) = r(b_m \mid \theta) = (1, 1, \cdots, 1, 0)$ or $r(a \mid \theta) = r(b_k \mid \theta) = (h_1, h_2, \cdots, h_{m+1})$ with $h_k = 0$ while $h_i = 1$ for any $i \in \{1, 2, \cdots, m+1\} - \{k\}$, respectively. A contradiction in any case.

Therefore $pd(K_{1,m} + \overline{K_{1,m}}) = m + 2$. ■

6 Conclusion

This paper introduces the concept of the complementary join $G + \overline{G}$ of a graph G and investigates some of its properties. Two related previously studied concepts are the complementary prism $G\overline{G}$ and its complement $\overline{G\overline{G}}$. These three graphs $H_1 = G\overline{G}$, $H_2 = \overline{G\overline{G}}$ and $H = G + \overline{G}$ have in common that each of them consists of a copy of G and a disjoint copy of $\overline{G}$ together with a set of edges joining G and $\overline{G}$. The three sets of edges are $E_1 = \{a_i b_i : 1 \leq i \leq n\}$ for H_1, $E_2 = \{a_i b_j : 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$ for H_2 and $E = \{a_i b_j : 1 \leq i \leq n, 1 \leq j \leq n\} = E_1 \cup E_2$ for H. One can consider the more general case H_g in which the edge set E_g is taken to be any specific subset of E. Notice that E_1 is a matching in the complementary prism H_1 consisting of those edges joining a vertex from G with its copy in $\overline{G}$. It seems to be interesting to study the special case H_{skew} (let us call it *the skewed complementary prism*) of H_g that generalizes the complementary prism H_1 in which E_{skew} is taken to be an arbitrary perfect matching whose elements join the vertices of G with the vertices of $\overline{G}$. Clearly, E_{skew} corresponds to a permutation of $V(G)$ while E_1 of the complementary prism corresponds to the identity permutation of $V(G)$.

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