Penalized Likelihood Parameter Estimation in the Quasi Lindley and Nadarajah-Haghighi Distributions

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Abstract: - The issues of performing inference on the parameters of quasi-Lindley (QL) distribution and Nadarajah-Haghighi exponential distribution (N-H) is addressed. It is shown graphically that the likelihood function of the quasi-Lindley distribution and Nadarajah-Haghighi exponential distribution is configured with a flat monotone shape. Which makes it very difficult to pick the values of the parameters that maximize the likelihood function. A penalization scheme is used to improve maximum likelihood point estimation. A penalization scheme based on the Jeffreys prior.

Key-Words: - Jeffreys Prior, Maximum Likelihood Estimation, Monotone Likelihood, Penalized Likelihood Estimation, Quasi Lindely Distribution, Nadarajah-Haghighi Exponential Distribution.

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1 Introduction

In statistical inference, the maximum likelihood method is used to estimate the parameters of a probability distribution. In most cases, the likelihood function is peaked and this searches for the peak reasonably simple. A problem occurs when the likelihood function is configured with a flat monotone shape, causing difficulties in the search for the point of maximization. A way around this problem suggested in the literature (known as Firth correction) is an adaptation of a method created to reduce the bias of maximum likelihood estimates. The method leads to finite estimates using maximization of the penalized likelihood procedure. In this case, the penalty might be interpreted as a Jeffreys type prior widely used in the Bayesian context, [1]. This problem appeared in some distributions as shown by [2]. They presented the maximization of penalized likelihood estimation MPLEs in the modified extended Weibull distribution.

In this article, we study quasi-Lindley distribution QL(α, θ) and discuss the properties of this distribution in section 2. We present the maximization of penalized likelihood estimation for quasi-Lindley distribution in section 2.1. In section 2.2, real data is introduced to compare the maximum likelihood estimation with the maximization of penalized likelihood estimation. In section 2.3, a simulation study is presented for quasi-Lindley distribution QL (α,θ) by the maximum likelihood estimation and maximization of penalized likelihood estimation. In section 3, we introduce the Nadarajah-Haghighi exponential distribution N-H (α, λ) and the properties of this distribution. We present the maximization of penalized likelihood estimation for the Nadarajah-Haghighi exponential distribution in section 3.1. In section 3.2, a real data set is used to compare between the maximum likelihood estimation and maximization of penalized likelihood estimation. In section 3.3, a simulation study is presented for Nadarajah-Haghighi exponential distribution N-H (α, λ) by the maximum likelihood estimation and maximization of penalized likelihood estimation

2 Quasi Lindley Distribution

The quasi-Lindley distribution QL (α, θ) was introduced by [3]. The probability density function (pdf) of the quasi-Lindley distribution is defined by

$$f(x,\alpha,\theta)=\frac{\theta(\alpha+x\theta)}{\alpha+1}e^{-\theta x}, x>0, \theta>0, \alpha>0 \tag{1}$$

Its cumulative distribution function (cdf) is obtained as:

$$F(x)=1-\frac{1+\alpha+\theta x}{\alpha+1}e^{-\theta x}, x>0, \theta>0, \alpha>-1 \tag{2}$$

They obtained the rth moment about the origin of quasi-Lindley distribution as:

$$\mu_r^{'}=\frac{\Gamma(r+1)(\alpha+r+1)}{\theta^r(\alpha+1)}; r=1,2 \tag{3}$$

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the quasi Lindley distribution. The log-likelihood function for the vector of parameters α, θ is:

$$l(\upsilon)=l(\alpha,\theta)=n\log(\theta)-n\log(\alpha+1)+\sum_{i=1}^{n}\log(\alpha+\theta x_i)-\theta\sum_{i=1}^{n}X_i \tag{4}$$

The maximum likelihood estimator ($\hat{\alpha}$ and $\hat{\theta}$) of α and θ can be obtained by maximizing the log-likelihood function given in equation (4) concerning α and θ. This can be done by setting the score function equal to zero and solving the resulting system of equations. The components of the score function are given

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\frac{x_i}{\alpha+\theta x_i}\sum_{i=1}^{n}X_i \tag{5}$$

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha+1}+\sum_{i=1}^{n}\frac{1}{\alpha+\theta x_i} \tag{6}$$

Figure 1 indicates plots of the density function of quasi-Lindley distribution for some parameter values α, θ. Figure 2 shows plots of the hazard function of quasi-Lindley distribution for some parameter Values α,θ

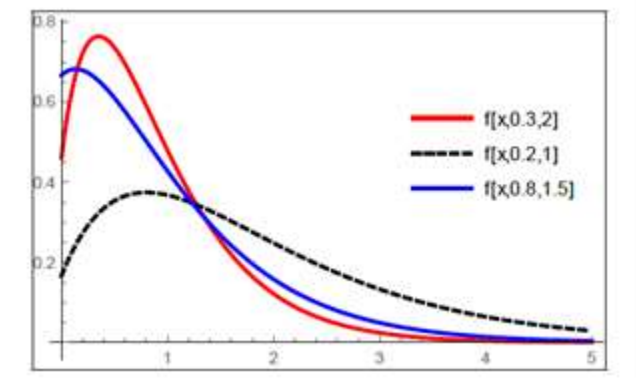


Fig. 1: Plots of the density function for some parameter values α and θ.

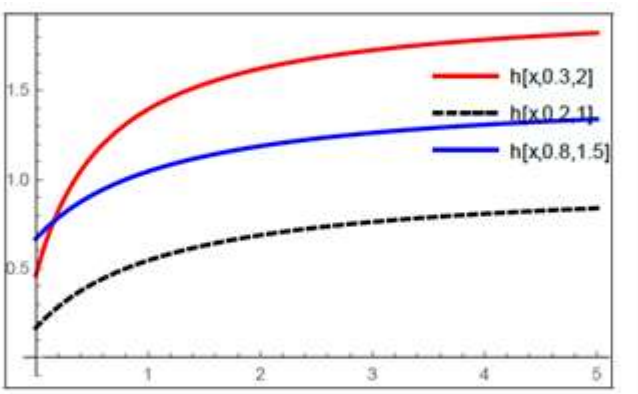


Fig. 2: Plots of the hazard function for some parameter values α and θ.

2.1 Maximization of Penalized Likelihood Estimation

In [4], the author proposed to modify the score function to reduce the bias of the MLE. His proposal is an alternative to the usual approach of subtracting from the MLE its estimated bias. The underlying idea is that since the parameter estimate may not exist, it is safer to transform the estimating equations to correct for bias prior toestimation. For the canonical parameter of the exponential family model, the rth component of the modified score equation is given by:

$$U_r^*(\nu)=U_r(\nu)+A_r(\nu)$$

Where $A_r(\nu)$ is the rth component of $(\nu) = \frac{I(\nu)B_1(\nu)}{n}, r = 1,\dots,dim(\nu)$. Here, $B_1(\nu)$ is the first-order term in the bias expansion of the MLE:

$$B(\nu) = B_1(\nu)/n + B_1(\nu)/n^2 + \dots$$

For an exponential family in canonical form,

$$A_r(\nu) = \frac{\partial}{\partial \nu_r}\left\{\frac{1}{2}\log\left|I(\nu)\right|\right\}$$

Here, the correction amounts to finding the mode of the posterior distribution obtained by using the Jeffreys invariant prior, see [2]. i.e., it amounts to finding the mode of $L^*(\nu) = L(\nu) \times |I(\nu)|^{1/2}$,where $L(\nu)$ is the likelihood function. Equivalently, estimation can be carried out by maximizing

$$l^*(\nu) = l(\nu) + \frac{1}{2}\log|I(\nu)|$$

Notice that the penalization term $|I(\nu)|^{1/2}$ is the Jeffreys invariant prior.

[2], introduced maximization of penalized likelihood estimation in the modified extended Weibull distribution and they used Jeffrey prior to find maximization of penalized likelihood estimation MPLEs. In particular, we consider the following family of penalized log likelihoods:

$$l^*(\nu) = l(\nu) + \frac{1}{2}\log|I(\nu)|, \eta \epsilon \mathbb{R} \tag{7}$$

Numerical maximization of the quasi-Lindley log-likelihood function can be problematic. A penalization scheme is used to improve maximum likelihood point estimation. A penalization scheme based on the Jeffreys invariant prior.

To estimate the parameters of quasi-Lindley distribution by maximization of penalized likelihood estimation, we use equation (7) and let $x_1, x_2, \dots, x_n$ be a random sample of size n from the quasi Lindley distribution. The elements of Fisher's (expected) information matrix are obtained as follows

$$I_{11} = E(J_{11})$$

$$I_{22} = E(J_{22})$$

$$I_{21} = I_{12} = E(J_{12})$$

Using equations (5) and (6), we get

$$J_{11} = \frac{\partial^2 l}{\partial \alpha^2} = \frac{n}{(1+\alpha)^2} + \sum_{i=1}^{n}\frac{1}{(\alpha+\theta x_i)^2} \tag{8}$$

$$J_{22} = \frac{\partial^2 l}{\partial \theta^2} = \frac{n}{\theta^2} + \sum_{i=1}^{n}\frac{x_i^{\,2}}{(\alpha+\theta x_i)^2} \tag{9}$$

$$J_{12} = \frac{\partial^2 l}{\partial \alpha \partial \theta} = \sum_{i=1}^{n}\frac{x_i}{(\alpha+\theta x_i)^2} \tag{10}$$

The information matrix is given by:

$$I(\nu) = E\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \theta} \\ \frac{\partial^2 l}{\partial \alpha \partial \theta} & \frac{\partial^2 l}{\partial \theta^2} \end{pmatrix} \tag{11}$$

The penalized likelihood function is:

$$l^*(\nu) = l(\nu) + \frac{1}{2}\log\left|I(\nu)\right|$$

where

$$I(\nu) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

Notice that the penalization term $|I(\nu)|^{1/2}$ is the Jeffreys prior.

To use the Jeffreys prior for penalizing the log-likelihood function, we need to compute the expected values of the quantities given in the equations from (8) to (10).

To illustrate the use of the proposed modified penalized likelihood function, we return to the four samples that gave rise to the log-likelihood functions presented in Figure 3. This figure contains plots of the log-likelihood function as a function of α for four different samples of size 50 obtained from a quasi Lindley distribution with α = 0.15, θ = 0.35. Notice that the likelihood function is configured with a flat monotone shape. Table 1 contains the quasi-Lindley maximum likelihood parameter estimates and the corresponding the log-likelihood values. One can notice that from Table 1, the large values of $\hat{\alpha}$ obtained using samples 3 and 4 are approximately 6882.21 and 49271.9, respectively. Table 2 contains the quasi-Lindley maximum penalized likelihood parameter estimates and one can notice that the values of $\hat{\alpha}$ obtained using samples 3 and 4 are approximately 0.9207 and 0.8715, respectively. The parameter estimates obtained using the log-likelihood function are presented in Table 1. These results are to be compared to those in Table 2. Notice that the log-likelihood functions for samples 3 and 4 contain large estimates of α.

We note that there is an improvement in the results when using maximization of penalized likelihood estimation (Table 1 and Table 2).

Notice that the values of $\hat{\alpha}$ samples 3 and 4 are reduced in Table 2 when using maximization of penalized likelihood estimation.

It is possible to compute a confidence interval for parameters of distribution in the case of the penalized likelihood estimation. That can be computed by using the bootstrap method.

The following steps were followed to obtain the confidence interval:

1. Data: $x_1,\ldots, x_n$ are drawn from a distribution F(θ) with unknown parameter θ.
2. Compute $\hat{\theta}$ that estimates θ.
3. Our bootstrap samples are drawn from $F(\hat{\theta})$
4. For each bootstrap sample $x_1^*, \ldots, x_n^*$ We compute $\hat{\theta}^*$ and the bootstrap difference $\delta^* = \hat{\theta}^* - \hat{\theta}$.
5. Use the bootstrap differences to make a bootstrap confidence interval for θ, [5].

Now, we introduce real data for quasi-Lindley distribution in the next section.

Table1. MLEs of α and θ for 4 Samples of size n = 50; from quasi Lindley (0.15,0.35) distribution.

Maximum Likelihood Estimation MLEs			
Sample	$\hat{\alpha}$	$\hat{\theta}$	log-likelihood
1	0.02524	0.38891	-127.58500
2	0.30688	0.41098	-119.95500
3	6882.21	0.18832	-133.48700
4	4 49271.9	0.21510	-126.8330

Table 2. MPLEs of α and θ for 4 Samples of size n = 50; from quasi Lindley (0.15,0.35) distribution.

Maximum Penalized Likelihood Estimation MPLEs			
Sample	$\hat{\alpha}$	$\hat{\theta}$	log-likelihood
1	0.00625	0.38856	-122.46100
2	0.25714	0.41201	-115.20600
3	0.92077	0.28230	-130.40300
4	0.87156	0.32239	-123.15000

2.2 Application for Real Data

The following data from [6], is used to compare parameter estimation in quasi-Lindley distribution between maximum likelihood estimation and penalized likelihood approach.

Data Set:

Complete data: All 50 items are put into use at t = 0 and failure times are in weeks

0.013, 0.065, 0.111, 0.111, 0.163, 0.309, 0.426, 0.535, 0.684, 0.747, 0.997, 1.284,1.304,1.647,1.829, 2.336, 2.838, 3.269, 3.977, 3.981, 4.520, 4.789, 4.849, 5.202, 5.291, 5.349, 5.911, 6.018, 6.427, 6.456, 6.572, 7.023, 7.087, 7.291, 7.787, 8.596, 9.388, 10.261, 10.713, 11.658, 13.006, 13.388, 13.842, 17.152, 17.283, 19.418, 23.471, 24.777, 32.795, 48.105

Parameters of quasi Lindley distribution are estimated by maximum likelihood estimation follows $\hat{\alpha} = 74897.9$, $\hat{\theta} = 0.127862$,

The estimate $\hat{\alpha}$ of α seems unreasonably large.

Figure 4 indicates plots of empirical and fitted cumulative distribution function (cdf) for quasi-Lindley distribution QL(α, θ) at two values of parameters F(x,74897.9,0.127862) and F(x,2.16711,0.162343), it appears that the fitted distributions both from maximum likelihood and penalized likelihood estimation fit the data reasonably well, but the extreme value of the maximum likelihood estimate seems strange and the penalized likelihood would be preferable.

Now, the Confidence interval (CI) for alpha and theta parameters of the quasi-Lindley distribution is given by:

CI for α = [59074, 149791]
CI for θ = [0.0707601, 0.155148]

Using the same real data set for the same distribution, the penalized likelihood estimation of the parameter is:

$\hat{\alpha} = 2.16711$; $\hat{\theta} = 0.162343$

The confidence intervals for parameters of the quasi Lindley distribution in the case of penalized likelihood estimation are given by:

CI for α = [1.56153, 3.12993]
CI for θ = [0.105837, 0.197349]

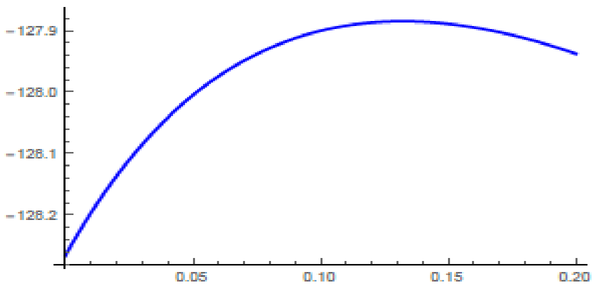
2.3 Simulation Study

We used a simulation study to assess the performance of the penalized likelihood estimation of the point estimate, of which estimates two parameters of QL(α, θ) form=1000, the sample size n is 50,75,100, and different parameter values. The following steps were followed to obtain the results:

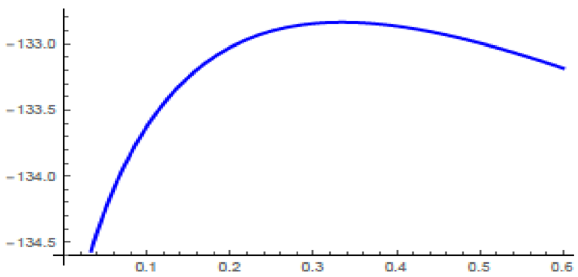
1. Select initial values for parameters α, and θ to be used for generating data.
2. Specify the sample size n to be used for the study.
3. Generate pseudo-random samples with size n from QL(α, θ) for selected value of α and θ
4. Obtain the maximum likelihood estimates and maximization of penalized likelihood estimates for α and θ for different sample sizes.
5. For each sample size, obtain the mean, bias, relative bias, variance and mean squared error (MSE) for each estimator of quasi Lindley (α, θ) for different values of η by using maximization of penalized likelihood, see equation (7).
6. Choose the value of (η) with the smallest mean square error and obtain the mean, bias and relative bias, variance and mean squared error for each estimator for different sample sizes.

Table 3 is shown the mean, bias, relative bias, and MSE for α and θ; by sample size from quasi Lindley (0.15,0.35) distribution using the maximum likelihood esttimation.

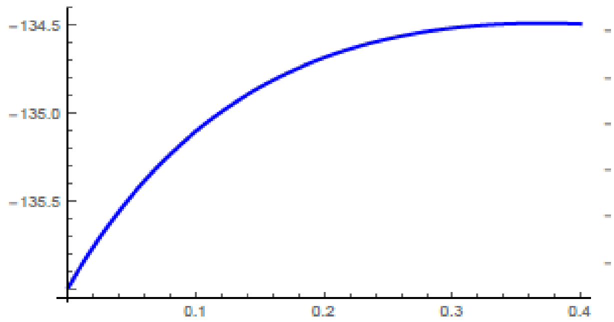
Data generation was carried out using the computational software Mathematica [10]. Simulation results are shown in Table 4, Table 5, Table 6 and Table 7 in Appendix.



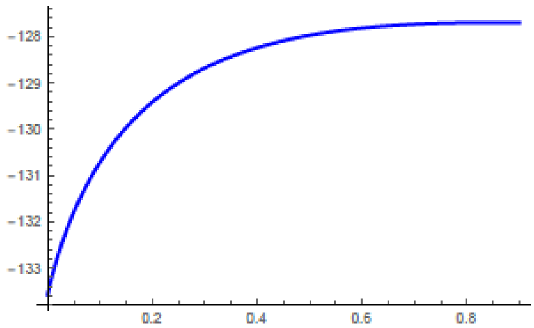
(a) sample 1



(b) sample 2



(c) sample 3



(d) sample 4

Fig. 3: Log-likelihood function for α. For each sample n = 50 from quasi Lindley distribution (0.15; 0.35)

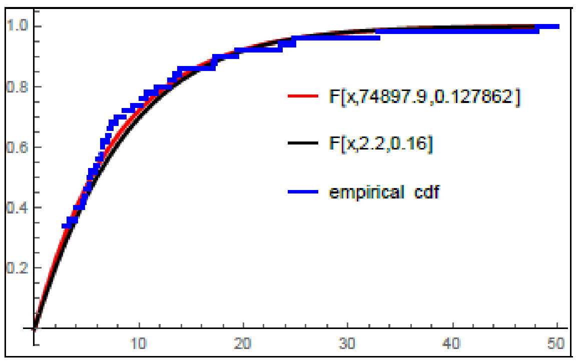


Fig. 4: Plots of empirical and fitted cumulative distribution function (cdf) for quasi-Lindley distribution QL(α, θ)

Table 3. Mean, bias, relative bias, and MSE of MLEs of α and θ; by sample size from quasi Lindley (0.15,0.35) distribution

aximum Likelihood Estimation MLEs						
n	para meter	Mean	Bias	Relative bias	Variance	MSE
25	α	911.274	-911.124	-607416	7.9515* E07	8.034* E07
	θ	0.37190	-0.02195	-6.2738	0.0032	0.003
50	α	252.221	-252.071	-168047	2.12176 E07	2.121E07
	θ	0.36579	-0.01579	-4.51319	0.0016	0.001
100	α	17.8852 0	-17.7352	-11823.5	31361	313929
	θ	0.36069	-0.01069	- 3.05490	0.0006	0.00071

For each estimator considered, the following quantities: the mean, bias, relative bias, variance, and Mean square error were computed.

Table 5 (Appendix) shows the Mean, bias, relative bias, and MSEs of α and θ using the Jeffreys prior penalization with η = 0.1,..., 0.9, 2; samples of size n = 100 drawn from quasi Lindley(0.15,0.35) distribution.

Table 8 and Table 9 in Appendix show that the smallest MSE of ˆα and ˆθ correspond to η = 2.4. Figure 5 indicates MSE of ˆα and ˆθ and the selected value η for QL distribution.

From Table 10 (Appendix), We found the mean, bias, relative bias, and variance for ˆα and ˆθ using maximization of penalized likelihood estimation with η = 2.4 and samples of size n = 50, 75, 100 from quasi Lindley distribution for different values of parameters α, θ.

The results of the simulation study show that the bias for any estimator decreases when the sample size increases. Also, the relative bias decreases when the sample size increases. The mean square error (MSE) decreases when the sample size increases.

In the next section, we introduce the maximization of penalized likelihood estimation in another distribution called the Nadarajah-Haghighi exponential distribution.

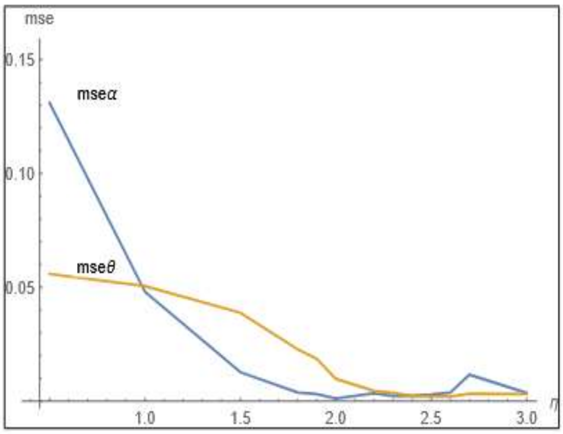


Fig. 5: MSE of ˆα and ˆθ and selected values η for QL distribution

3 Nadarajah-Haghighi Exponential Distribution

Nadarajah-Haghighi exponential distribution N-H (α, λ) was introduced in [7]. The cumulative distribution function (cdf), probability density function (pdf), and the quantile function are follows:

$$F(x) = 1 - \exp\left(1 - (1 + \lambda x)^{\alpha}\right), \tag{12}$$

$$f(x) = \alpha\lambda(1 + \lambda x)^{\alpha - 1} \exp\left(1 - (1 + \lambda x)^{\alpha}\right) \tag{13}$$

and

$$Q(p) = \frac{1}{\lambda}\left(\left(1 - \log(1 - p)\right)^{1/\alpha} - 1\right), 0 < p < 1 \tag{14}$$

Respectively.

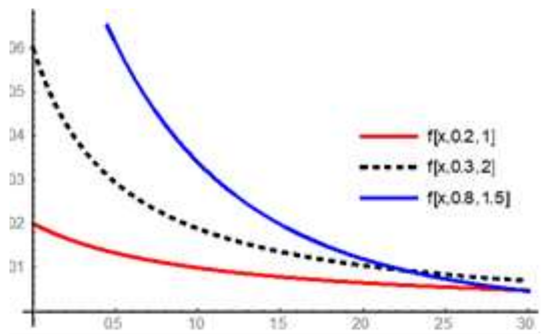


Fig. 6: Plots of the density function for some parameter values α and λ

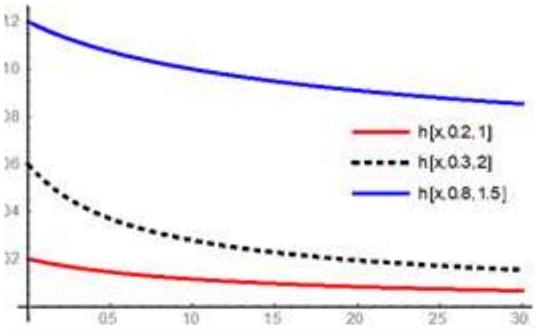


Fig. 7: Plots of the hazard function for some parameter values α and λ

Now consider estimation by the method of maximum likelihood. The log-likelihood function (LL) of the two parameters is as follows:

$$l(\alpha,\lambda)=\log L(\alpha,\lambda)=n+n\log(\alpha\lambda)+(\alpha-1)+\sum_{i=1}^{n}\log(1+\lambda x_i)^{\alpha}-\sum_{i=1}^{n}(1+\lambda x_i)^{\alpha} \tag{15}$$

It follows that the maximum-likelihood estimators (MLEs) are the simultaneous solutions of the equations:

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\log(1+\lambda x_i)-\sum_{i=1}^{n}(1+\lambda x_i)^{\alpha}\log(1+\lambda x_i)=0 \tag{16}$$

and

$$\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^{n}x_i(1+\lambda x_i)^{-1}-\alpha\sum_{i=1}^{n}x_i(1+\lambda x_i)^{\alpha-1}=0 \tag{17}$$

For interval estimation of (α, λ), one requires the Fisher information matrix. The elements of this matrix in equation (15) can be worked out as follows:

$$E\left(\frac{\partial^2\log L}{\partial\alpha^2}\right)=\frac{n}{\alpha^2}+nE\left[(1+\lambda X)^{\alpha}\left(\log(1+\lambda X)\right)^2\right] \tag{18}$$

$$E\left(\frac{\partial^2\log L}{\partial\lambda^2}\right)=\frac{n}{\lambda^2}+n(\alpha+1)E\left[X^2(1+\lambda X)^{-2}\right]+n\alpha(\alpha-1)E\left[X^2(1+\lambda X)^{\alpha-2}\right] \tag{19}$$

and

$$E\left(\frac{\partial^2\log L}{\partial\alpha\partial\lambda}\right)=nE\left[X(1+\lambda X)^{-1}\right]+n\alpha E\left[X(1+\lambda X)^{\alpha-1}\log(1+\lambda X)\right]+nE\left[X(1+\lambda X)^{\alpha-1}\right] \tag{20}$$

Where

$$E\left[(1+\lambda X)^{\alpha}\left(\log(1+\lambda X)\right)^2\right]=\alpha\lambda eJ(0,\alpha,2),$$

$$E\left[X^2(1+\lambda X)^{-2}\right]=\alpha\lambda eI(2,-2,1),$$

$$E\left[X^2(1+\lambda X)^{\alpha-2}\right]=\alpha\lambda eI(2,\alpha-2,1),$$

$$E\left[X(1+\lambda X)^{-1}\right]=\alpha\lambda eI(1,-1,1),$$

$$E\left[X(1+\lambda X)^{\alpha-1}\left(\log(1+\lambda X)\right)\right]=\alpha\lambda eJ(1,\alpha-1,1),$$

and

$$E\left[X(1+\lambda X)^{\alpha-1}\right]=\alpha\lambda eI(1,\alpha-1,1),$$

Since

$$I(a,b,c)=\frac{1}{\alpha\lambda^{\alpha+1}}\sum_{i=0}^{n}\binom{a}{i}(-1)^{a-i}c^{-(b+i)/\alpha-1}\Gamma\left(\frac{b+i}{\alpha}+1,c\right)$$

$$J(a,b,c)=\frac{1}{\alpha^{c+1}\lambda^{a+1}}\sum_{i=0}^{a}\binom{a}{i}(-1)^{a-1}\frac{\partial^c}{\partial\upsilon^c}\Gamma(\upsilon,1)\Big|_{\upsilon=(b+i)/\alpha+1}$$

the information matrix is given by:

$$I(\theta)=E\begin{pmatrix}\dfrac{\partial^2\log L}{\partial\alpha^2} & \dfrac{\partial^2\log L}{\partial\alpha\partial\lambda}\\ \dfrac{\partial^2\log L}{\partial\alpha\partial\lambda} & \dfrac{\partial^2\log L}{\partial\lambda^2}\end{pmatrix} \tag{21}$$

Figure 6 indicates plots of the density function of the Nadarajah-Haghighi exponential distribution for some parameter values α and λ. Figure 7 shows plots of the hazard function of the Nadarajah-Haghighi exponential distribution for some parameter values α and λ.

3.1 Maximization of Penalized Likelihood Estimation for N-H distribution

To calculate the penalized likelihood function, we need to calculate the Jeffreys prior, see equation (7). To use the Jeffreys prior for penalizing the log-likelihood function, we need to compute the expected values of the quantities given in equation (18) to equation (20).

To illustrate the use of the proposed modified penalized likelihood function, we shall return to the two samples that gave rise to the log-likelihood functions presented in Figure 8. This figure contains plots of log-likelihood function versus λ for two different samples of size 50 obtained from Nadarajah-Haghighi exponential distribution with α = 0.5, λ = 1 and α = 1, λ = 1. Notice that the likelihood function is configured with a flat monotone shape. Table 11 contains the Nadarajah-Haghighi exponential maximum likelihood parameter estimates and the corresponding log-likelihood values. One can notice that from Table 11 , the large values of ˆλ obtained using samples 1 and 2 are approximately 9.54432 × 109 and 1.01713 × 109, respectively. Table 12 contains the maximization of penalized likelihood estimation for Nadarajah-Haghighi exponential distribution and one can notice that the values of ˆλ obtained using samples 1 and 2 are approximately 3.67773 and 0.98556, respectively. The parameter estimates obtained using the log-likelihood function are presented in Table 11. These results are to be compared to those in Table 12. Notice that the log-likelihood functions for samples 1 and 2 contain large estimates of λ.

Table 11. MLEs of α and λ of size n = 50; the samples were drawn from the Nadarajah-Haghighi exponential distribution.

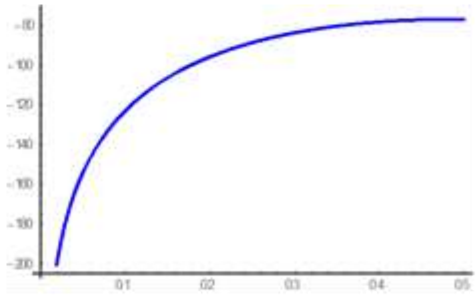
Maximum Likelihood Estimation MLEs		
Sample	$\hat{\alpha}$	$\hat{\lambda}$
1) α = 0.5,λ = 1	0.01768	9.54432 × 10^9
2)α = 1,λ = 1	0.01992	1.01713× 10^9

Table 12. MPLEs of α and λ of size n = 50; the samples were drawn from the Nadarajah-Haghighi exponential distribution

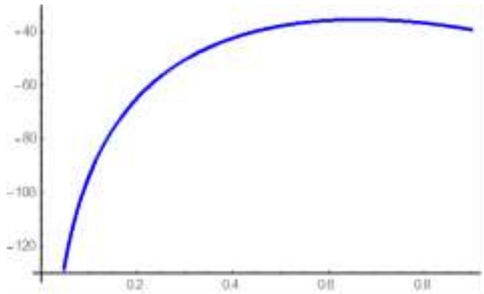
Maximum Penalized Likelihood Estimation MPLEs		
Sample	$\hat{\alpha}$	$\hat{\lambda}$
1) α = 0.5,λ = 1	0.34103	3.67773
2)α = 1,λ = 1	0.71565	0.98556

We note that there is an improvement in the results when using maximization of penalized likelihood estimation, see Table 11 and Table 12. Notice that the values of ˆλ in samples 1 and 2 are reduced in Table 12 when using maximization of penalized likelihood estimation.

We introduce real data for the Nadarajah-Haghighi exponential distribution in the next section.



(a) Sample 1



(b) Sample 2

Fig. 8: Log-likelihood function for α. For each sample n = 50 observations were obtained from Nadarajah-Haghighi exponential distribution sample1 (α = 0.5, λ = 1) and sample2 (α = 1, λ = 1)

3.2 Real Data for Nadarajah-Haghighi Exponential Distribution

The following data from [6], was used to compare parameter estimation in the Nadarajah-Haghighi exponential distribution between maximum likelihood estimation and penalized likelihood approach.

Dataset:
0.03, 0.12, 0.22, 0.35, 0.73, 0.79, 1.25, 1.41, 1.52, 1.79, 1.80, 1.94, 2.38, 2.40, 2.87, 2.99, 3.14, 3.17, 4.72, 5.09
Parameters of Nadarajah-Haghighi exponential distribution are estimated by maximum likelihood estimation as follows:

$$\hat{\alpha} = 0.0160311, \hat{\lambda} = 4.96749 \times 10^{10}$$

The estimate of ˆλ of λ seems unreasonably large.

Now, the confidence interval (CI) for α and λ parameters of the Nadarajah-Haghighi exponential distribution in the case of maximum likelihood estimation is given by

CI for

$$\alpha = [-0.0527535,\ 0.435248]$$

CI for

$$\lambda = \left[4.88508\ \times 10^{10},\ 4.96749 \times 10^{10}\right]$$

Using the same real data, see [4], parameters of the Nadarajah-Haghighi exponential distribution are estimated by penalized likelihood estimation as follows

$$\hat{\alpha} = 0.439155, \hat{\lambda} = 1.08643$$

And confidence interval for parameters of the Nadarajah-Haghighi exponential distribution in the case of penalized likelihood estimation is given by
CI for

$$\alpha = [0.397233,\ 0.682527]$$

CI for

$$\lambda = [-7.24384,\ 1.20497]$$

3.3 Simulation Study for Nadarajah-Haghighi Exponential Distribution

We used a simulation study to assess the performance of the penalized likelihood estimation of the point estimate for several cases, of which estimates two parameters of N-H (α, λ) for m=2000, the sample size n are 25, 50, and different parameter values. The following steps were followed to obtain the results:

1. Select initial values for parameters α, λ to be used for generating data.
2. Specify the sample size n to be used for generating data.
3. Generate pseudo-random samples with size n from N-H (α, λ) for selected values of α and λ.
4. Obtain the maximum likelihood estimates and the maximization of penalized likelihood estimates for α, and λ for different sample sizes.
5. For each sample size, obtain the mean, bias, relative bias, variance, and mean squared error (MSE) for each estimator for different values of (η) by using maximization of penalized likelihood, see equation (7).
6. Choose the value of (η) with the smallest mean square error and obtain the mean, bias, relative bias, variance, and mean squared error for each estimator for different sample sizes.

Data generation was carried out using the computational software Mathematica[10]. The simulation result is shown in Table 13 (Appendix).

For each estimator considered, we computed the following quantities: mean, bias, relative bias, variance, and mean square error. Table 13 (Appendix) shows that the smallest MSE of α and λ correspond to η = 1.2. From Table 14 (Appendix), we found mean, bias, relative bias, and variance for the estimation of α and λ using maximization of penalized likelihood estimation with η = 1.2; samples of size n = 25; 50 from Nadarajah-Haghighi exponential distribution for different values of parameters α, λ. In Table 15 (Appendix), we found the confidence interval for the estimator by the bootstrap method. Table 16 and Table 17 in Appendix, the results of the simulation study showing that the bias for any estimator decreases when the sample size increases. Also, the relative bias decreases when the sample size increases. Mean square error (MSE) decreases when the sample size increases.

4 Summary and Conclusion

In this article, we have shown that the maximum likelihood estimation of the parameters that index the Quasi-Lindley distribution can be problematic. The MLE estimates of parameters for the Quasi-Lindley distribution are large, so a penalized likelihood function is proposed. The penalization term is a modified version of the Jeffreys invariant prior. We used a simulation study to assess the performance of the penalized likelihood estimation of the point estimate for several cases, of which estimates two parameters of Quasi Lindley distribution QL(α, θ) for m=1000, the sample size n are 50,75,100 and different parameter values by using penalized likelihood estimation, see equation (7). We choose the value of η with the smallest mean square error. Simulation results on point estimation are found in Table 4, Table 5, Table 6 and Table 7, Table 8, Table 9 and Table 10 in Appendix. We used a simulation study to estimate parameters for several cases, of which estimates two parameters of N-H (α, λ) for m=2000, the sample size n are 25, 50, and different parameter values and different the value of η, see equation (7). We choose the value of η with the smallest mean square error (Table 13, Appemdix). A real data set from [6] was used to compare parameter estimation in the Quasi-Lindley distribution and N-H distribution between maximum likelihood estimation and maximization of the penalized likelihood approach. The results indicate that both the fitted distributions from maximum likelihood and penalized likelihood estimation reasonably well but the extreme value of the maximum likelihood estimate seems strange and the penalized likelihood would be preferable, see Figure 5. The results of the simulation study show that the bias for any estimator decreases when the sample size increases. Also, the relative bias decreases when the sample size increases. The mean square error (MSE) decreases when the sample size increases.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflicts of Interest

The authors have no conflicts of interest to declare.

APPENDIX

Table 4. Mean, bias, relative bias, and MSEs of α and θ using the Jeffreys prior penalization with η =0.1, ..., 0.9, 2.1; samples of size n = 50 drawn from quasi Lindley(0.15,0.35) distribution

Maximization of penalized likelihood estimation MPLEs						
η	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
0.1	α	0.18483	-0.03482	-23.21680	0.06308	0.06429
	θ	0.36631	-0.01630	-9.95005	0.00134	0.00161
0.3	α	0.17809	-0.02809	-18.73160	0.04392	0.04471
	θ	0.36648	-0.01648	-4.71001 8	0.00131	0.00158
0.5	α	0.16947	-0.01948	-12.98500	0.03483	0.03521
	θ	0.36734	-0.01734	-4.95467	0.00138	0.00168
0.7	α	0.16152 -	-0.01152	-7.67770	0.02776	0.02789
	θ	0.36706	-0.01706	-4.87368	0.00127	0.00155
0.9	α	0.15504	-0.00504	-3.36441	0.01924	0.01927
	θ	0.36685	-0.01685 -	-4.81674	0.00115	0.00143
1	α	0.15144	-0.00144	-0.96075	0.01732	0.01732
	θ	0.36733	-0.01733	-4.95077	0.00130	0.00160
1.1	α	0.14793	0.00207	1.38346	0.01409	0.01410
	θ	0.36607	-0.01606	-4.59021	0.00098	0.00124
1.3	α	0.13945	0.01055	7.03518	0.00983	0.00995
	θ	0.36611	-0.01611	-4.60323	0.00095	0.0012
1.5	α	0.13501	0.01498	9.98851	0.00654	0.00677
	θ	0.36478	-0.01478	-4.22347	0.00088	0.00110
1.7	α	0.13563	0.01437	9.58050	0.00358	0.00379
	θ	0.36209	-0.01209	-3.45639	0.00073	0.00088
1.9	α	0.13912	0.01088	7.25475	0.00191	0.00203
	θ	0.35856	-0.00856	-2.44487	0.00052	0.00059
2	α	0.14250	0.00749	4.99671	0.00157	0.00163
	θ	0.35630	-0.00631	-1.80224	0.00058	0.00062
2.1	α	0.14980	0.00019	0.12878	0.00001	0.00001
	θ	0.35118	-0.00118	-0.33779	0.00001	0.00001

Table 5. Mean, bias, relative bias, and MSEs of α and θ using the Jeffreys prior penalization with η = 0.1,..., 0.9, 2; samples of size n = 100 drawn from quasi Lindley(0.15,0.35) distribution

Maximization of penalized likelihood estimation MPLEs						
η	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
0.1	α	0.16601	-0.01601	-10.67440	0.01355	0.013810
	θ	0.36077	-0.01077	-4.57475	0.00055	0.00067
0.3	α	0.16480	-0.01480	-9.86835	0.01290	0.01312
	θ	0.36079	-0.01079	-3.08368	0.00055	0.00067
0.5	α	0.16277	-0.01278	-8.51867	0.01219	0.01236
	θ	0.36085	-0.01085	-3.09970	-0.00054	0.00066
0.7	α	0.15992	-0.00992	-6.61390	0.01136	0.01146
	θ	0.36101	-0.01101	-3.14560	0.00055	0.00067
0.9	α	0.15627	-0.00627	-4.18472	0.00993	0.00997
	θ	0.36113	-0.01112	-3.17966	0.00053	0.00065
1	α	0.15397	-0.00397	-2.64632	0.00926	0.00928
	θ	0.36120	-0.01120	-3.20120	0.00926	0.00065
1.1	α	0.15180	-0.00180	-1.20134	0.00820	0.00820
	θ	0.36110	-0.01110	-3.17157	0.00051	0.00063
1.3	α	0.14724	0.00276	1.84130	0.00587	0.00588
	θ	0.36042	-0.01041	-2.97690	0.00043	0.00054
1.5	α	0.14086	0.00914	6.09684	0.00445	0.00454
	θ	0.36010	-0.01010	-2.88679	0.00042	0.00052

Maximization of penalized likelihood estimation MPLEs						
1.7	α	0.13962	0.01038	6.91927	0.00243	0.00253
	θ	0.35867	-0.00867	-2.47602	0.00033	0.00041
1.9	α	0.14055	0.00945	6.29844	0.00124	0.00134
	θ	0.35659	-0.00659	-1.88260	0.00024	0.00028
2	α	0.14533	0.00467	3.11477	0.00050	0.00053
	θ	0.35356	-0.00356	-1.01647	0.00012	0.00013

Table 6. Mean, bias, relative bias, and MSEs of α and θ using the Jeffreys prior penalization with η = 1, ..., 2.4; samples of size n = 50 drawn from quasi Lindley(0.15,0.35) distribution

Maximization of penalized likelihood estimation MPLEs						
η	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
1	α	0.15950	0.00950	-6.31090	0.02020	0.02030
	θ	0.36500	0.01500	-4.29160	0.00110	0.00130
1.5	α	0.14090	0.00900	6.02230	0.00680	0.00680
	θ	0.36270	0.01270	2.58090	0.00070	0.00090
2	α	0.14490	0.00500	3.35890	0.00120	0.00120
	θ	0.35490	0.00490	-1.38710	0.00030	0.00040
2.1	α	0.14760	0.00234	1.56130	0.00059	0.00060
	θ	0.35270	0.00270	-0.77530	0.00016	0.00017
2.3	α	0.14940	0.00060	0.38790	0.00023	0.00023
	θ	0.35130	0.00130	-0.37980	0.00006	0.00006
2.4	α	0.14920	0.00084	0.56130	0.00014	0.00014
	θ	0.35160	0.00160	-0.45850	0.00005	0.00006

Table7. Mean, bias, relative bias, and MSEs of α and θ using the Jeffreys prior penalization with η = 1, ..., 2.4; samples of size n = 100 drawn from quasi Lindley(0.15,0.35) distribution

Maximization of penalized likelihood estimation MPLEs						
η	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
1	α	0.154600	-0.004560	-3.045800	0.011270	0.011290
	θ	0.360900	-0.010870	-3.106900	0.000520	0.000640
1.5	α	0.140300	0.009700	6.464300	0.005210	0.005310
	θ	0.360100	-0.010100	2.770400	0.000420	0.000520
2	α	0.143700	0.006300	4.198600	0.001090	0.001130
	θ	0.354600	-0.004600	-1.302100	0.000210	0.000230
2.1	α	0.145800	0.004130	2.755360	0.000700	0.000720
	θ	0.353200	-0.003250	-0.928070	0.000120	0.000130
2.3	α	0.149700	0.000320	0.211830	0.000020	0.000020
	θ	0.351200	-0.001190	-0.341040	0.000009	0.000011
2.4	α	0.149900	0.000099	0.066653	0.000009	0.000010
	θ	0.351100	-0.001049	-0.299887	0.000002	0.000003

Table 8. Mean, bias, relative bias, and MSEs of α and θ using the Jeffreys prior penalization with η = 0.5, ..., 3; samples of size n = 50 drawn from quasi Lindley (0.2,1) distribution

Maximization of penalized likelihood estimation MPLEs						
η	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
0.5	α	0.450304	-0.150304	-50.10140	0.108400	0.130991
	θ	2.009500	-0.009497	-7.51521	0.055801	0.055892
1	α	0.38009	-0.08009	-26.69830	0.041559	0.047975
	θ	2.02771	-0.02771	-1.73201	0.049741	0.050509
1.5	α	0.326186	-0.026185	-8.72861	0.012007	0.0126931
	θ	2.03464	-0.034640	-1.73201	0.037606	0.0388056
1.8	α	0.311974	-0.011970	-3.99145	0.003673	0.0038159
	θ	2.025710	-0.025706	-1.28530	0.022137	0.0227987
1.9	α	0.308770	-0.008778	-2.92613	0.003085	0.0031621
	θ	2.020240	-0.020241	-1.01203	0.018151	0.0185607
2	α	0.309824	-0.009824	-3.27473	0.001081	0.0011772
	θ	2.013650	-0.013651	-0.682575	0.009564	0.0097504
2.2	α	0.19608	0.00392	1.96120	0.003428	0.003444
	θ	0.98696	0.01303	1.30359	0.004404	0.004574
2.3	α	0.19926	0.00074	0.37008	0.002220	0.002221
	θ	0.98919	0.01091	1.08139	0.003566	0.003683
2.4	α	0.20058	-0.00058	-0.29079	0.002316	0.002317
	θ	0.99251	0.00749	0.74922	0.002258	0.002314
2.5	α	0.19975	0.00024	0.12271	0.002909	0.002909
	θ	0.99229	0.00771	0.77109	0.002054	0.002114
2.6	α	0.19845	0.001546	0.77301	0.003736	0.003738
	θ	0.99162	0.008382	0.83829	0.002035	0.002106
2.7	α	0.19587	0.004133	2.06671	0.011602	0.011619
	θ	0.99081	0.009188	0.91879	0.003187	0.003272
3	α	0.20026	-0.000268	-0.13400	0.003575	0.003575
	θ	0.99119	-0.008812	0.88120	0.002959	0.003036

Table 9. Mean, bias, relative bias, and MSEs of α and θ using the Jeffreys prior penalization with η = 0.5, ..., 3; samples of size n = 50 drawn from quasi Lindley (0.8,1.5) distribution

Maximization of penalized likelihood estimation MPLEs						
η	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
0.5	α	0.90498	-0.104976	-13.1220	0.36014	0.37116
	θ	1.51909	-0.019092	-6.9984	0.04841	0.04877
1	α	0.70489	0.095103	11.8879	0.11717	0.12622
	θ	1.54101	-0.041015	-2.7343	0.04637	0.04805
1.5	α	0.54894	0.251058	31.3823	0.04286	0.10589
	θ	1.54450	-0.044498	-2.9665	0.05463	0.05661
1.8	α	0.44183	0.358173	44.77170	0.03299	0.16128
	θ	1.53350	-0.033496	-2.23309	0.06908	0.07019
1.9	α	0.40854	0.391457	48.9321	0.03138	0.18462
	θ	1.51763	-0.017631	-1.17541	0.07183	0.07214
2	α	0.38089	0.419102	52.38770	0.03326	0.18462
	θ	1.50055	-0.000550	-0.03667	0.07109	0.07109
2.2	α	0.38767	0.412329	51.54110	0.03783	0.20784
	θ	1.42797	0.072031	4.80204	0.05028	0.05547
2.3	α	0.41584	0.384157	48.0196	0.03749	0.18507

Maximization of penalized likelihood estimation MPLEs						
	θ	1.38042	0.119577	7.97179	0.03891	0.05321
2.4	α	0.42318	0.376817	47.1021	0.04113	0.18312
	θ	1.33976	0.160236	10.6824	0.02986	0.05554
2.5	α	0.35983	0.44017	55.0215	0.04649	0.24025
	θ	1.29949	0.20051	13.3675	0.02914	0.06935
2.6	α	0.27353	0.52647	65.8086	0.03977	0.31694
	θ	1.23240	0.26759	17.8398	0.03891	0.11052

Table 10. Mean, bias, relative bias, Variance, and MSEs of α and θ using the Jeffreys prior penalization with η = 2.4; by sample size n drawn from QL(α, θ)

Maximization of penalized likelihood estimation MPLEs						
n	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
50	α =2	1.17230	0.82770	41.38610	0.40310	1.08830
	θ =2.5	2.32470	0.17530	7.01080	0.39680	0.42750
75	α =2	1.18497	0.81503	40.75150	0.360749	1.02502
	θ =2.5	2.35645	0.14355	5.74201	0.279884	0.30049
100	α =2	1.20750	0.79260	39.62770	0.23990	0.86810
	θ =2.5	2.38680	0.11320	4.52980	0.22120	0.23400
50	α=3	2.88895	0.11105	0.16535	3.70154	0.17768
	θ=0.5	0.50559	-0.00559	0.00179	-1.11774	0.00183
75	α=3	2.90021	0.00979	0.15757	3.32639	0.16753
	θ=0.5	0.50697	-0.00697	0.00128	-1.39373	0.00133
100	α=3	2.91664	0.08335	0.11667	2.77850	0.12362
	θ=0.5	0.50763	-0.00763	0.00092	-1.52563	0.00098
50	α=0.3	0.30982	-0.00982	-3.27270	0.00003	0.0001300
	θ=2	2.00085	-0.00085	-0.04256	0.00065	0.0006500
75	α=0.3	0.30996	-0.00996	-3.32054	0.000015	0.0001144
	θ=2	1.99976	0.00023	0.01193	0.00039	0.0003900
100	α=0.3	0.31002	-0.01001	-3.33827	0.000017	0.0001140
	θ=2	1.99975	0.00025	0.01272	0.00017	0.0001700

Table 13. Mean, bias, relative bias, and MSEs of α and λ using the Jeffreys prior penalization with η = 0.1, ..., 2.5; samples of size n = 50 drawn from N-H (1,1) distribution

Maximization of penalized likelihood estimation MPLEs						
η	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
0.1	α	1.3735	-0.3735	-37.3513	0.1098	0.2479
	λ	1.4420	-0.4420	-44.2004	0.0479	0.2433
0.3	α	1.3303	-0.3303	-33.0334	0.11707	0.2262
	λ	1.4420	-0.4473	-44.7334	0.04875	0.2489
0.5	α	1.2508	-0.2508	-25.0841	0.1118	0.1747
	λ	1.4590	-0.4590	-45.9007	0.0503	0.2609
0.7	α	1.1644	-0.1644	-16.4447	0.0910	0.1181
	λ	1.4690	-0.4690	-46.9044	0.0501	0.2701
0.9	α	1.0803	-0.0802	-08.0255	0.0728	0.0792
	λ	1.4760	-0.4760	-47.6049	0.0501	0.2767
1	α	1.0364	-0.0364	-3.6436	0.0646	0.0659
	λ	1.4797	-0.4797	-47.9681	0.0503	0.2804
1.1	α	0.9923	0.0077	0.7735	0.0584	0.0584
	λ	1.4832	-0.4832	-48.3228	0.0518	0.2853
1.2	α	0.9494	0.0506	05.0582	0.0519	0.05448
	λ	1.4858	-0.4858	-48.5765	0.0545	0.2876
1.3	α	0.9109	0.0891	8.9068	0.0471	0.0551
	λ	1.4869	-0.4869	-48.6989	0.0541	0.2912
1.5	α	0.8009	0.1990	19.9022	0.0458	0.0854
	λ	1.4964	-0.4964	-49.6403	0.0556	0.3020

Maximization of penalized likelihood estimation MPLEs						
2.1	α	0.5595	0.4405	44.0494	0.0182	0.2123
	λ	1.4502	-0.4502	-45.0206	0.0489	0.2515
2.3	α	0.5291	0.4709	47.0890	0.0219	0.2453
	λ	1.3348	-0.3347	-33.4756	0.0453	0.1574
2.5	α	0.4737	0.5263	52.6331	0.0365	0.3135
	λ	1.1460	-0.1460	-14.6047	0.0361	0.0575

Table 14. Mean, bias, relative bias, and MSEs of α and λ using the Jeffreys prior penalization with η = 1.2; samples of size n = 25, 50 drawn from N-H (0.5,1) distribution

Maximization of penalized likelihood estimation MPLEs						
n	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
25	α =0.5	0.514848	-0.014848	-2.969670	0.001464	0.001684
	λ =1	1.121760	-0.121758	-12.175800	1.645470	1.660300
50	α =0.5	0.519715	-0.019716	-3.943090	0.000097	0.000486
	λ =1	1.012780	-0.012777	-1.277740	0.320775	0.320939

Table 15. Mean, Confidence Interval CI for estimation of α and λ using the Jeffreys prior penalization with η = 1.2; by sample size drawn from N-H(0.5,1) distribution

n	parameter	Mean	Bias
25	α =0.5	0.514848	[0.513834,0.515385]
	λ =1	1.12176	[1.10871,1.14927]
50	α =0.5	0.519715	[0.509496,0.520078]
	λ =1	1.01278	[1.00012,1.02556]

Table 16. Mean, bias, relative bias, and MSEs of α and λ using the Jeffreys prior penalization with η = 1.2; by sample size drawn from N-H (0.5,0.5) distribution

Maximization of penalized likelihood estimation MPLEs						
n	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
25	α =0.5	0.477388	0.022612	4.522420	0.008701	0.009213
	λ =0.5	0.863929	-0.363929	-72.785800	2.813260	2.945700
50	α =0.5	0.490334	0.009666	1.933210	0.005776	0.005869
	λ =0.5	0.744447	-0.244447	-48.889300	2.620230	2.679980
100	α =0.5	0.507113	-0.007113	-1.422650	0.002242	0.002293
	λ =0.5	0.552156	-0.052156	-10.431100	0.442650	0.445370
150	α =0.5	0.513666	-0.013666	-2.733170	0.001005	0.001192
	λ =0.5	0.514280	-0.014280	-2.856060	0.070553	0.070757

Table 17. Mean, bias, relative bias, and MSEs of α and λ using the Jeffreys prior penalization with η = 1.2; by sample size drawn from N-H (1,1) distribution

Maximization of penalized likelihood estimation MPLEs						
n	parameter	Mean	Bias	Relative bias(%)	Variance	MSE
25	α =1	0.591546	0.408454	40.845400	0.0185191	0.185354
	λ =1	2.235650	-1.235650	-123.565000	19.977700	21.504600
50	α =1	0.625168	0.374832	37.483200	0.009941	0.150440
	λ =1	1.671450	-0.671453	-67.145300	15.553100	16.003900
100	α =1	0.646386	0.353614	35.361400	0.005107	0.130150
	λ =1	1.444810	-0.444815	-44.481500	11.893600	12.091500
150	α =1	0.658156	0.341844	34.184400	0.003197	0.120054
	λ =1	1.201460	-0.201456	-20.145600	9.629080	9.669660