Aggregation of Fuzzy Metric Spaces: A Fixed Point Theorem

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Abstract: In the last years, fuzzy (quasi-) metric spaces have been used as an important mathematical tool to measure the similarities between the two points with respect to a real parameter. For the reason of the importance of these structures, different kinds of methods have been investigated for use in the applied sciences. So generating new fuzzy (quasi-) metrics from the old ones with aggregation functions has been a research topic. In this paper, we provide a general fixed point theorem using residuum operators for contractions obtained through aggregation functions. We show that there are some necessary conditions and also we provide some examples to show that these conditions cannot be omitted.

Key–Words: - Aggregation function, contraction, fuzzy metric, fixed point theorem, residuum operator, t-norm.

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1 Introduction

In applied sciences, aggregation functions have become important tools in merging a collection of data (information) into a single one. The necessity of merging the information usually presented through numerical values has led to growing attention to studying numerical functions. Especially, the need for this merge is imposed in fields such as machine learning, data mining, image processing, and decision-making pattern recognition. Many methods to merge the numerical information are based on aggregation functions. Another different use of the aggregation function is for aggregating the metric spaces. In this way, aggregation functions allow us to generate new metric spaces from the existing ones. We observe that this problem has been mainly studied in [1], [2], [3]. Then, the authors obtained a characterization of those functions that aggregate a collection of quasi-metrics into a single one by means as in the papers, [4], [5], [6]. It should be pointed out that a function that aggregates quasi-metrics (on products) also aggregates metrics (on products) but the converse is not true in general.

One of the other application areas where the aggregation functions are used successfully is used for fuzzy binary relations which are provided by fuzzy databases. In light of this fact, the aggregation of fuzzy relations has been studied in the papers, [7], [8], [9], [10]. Also, the authors have studied the aggregation of fuzzy equivalence relations (or indistinguishability operators) in [11]. Some important works for aggregation of indistinguishability operators have been studied in [12], [13], [14].

In cases where the similarity degree between the elements must be measured concerning a positive real parameter, indistinguishability operators are not a suitable tool. To handle these situations, the notion of fuzzy (pseudo-) metric was initiated in the literature, [15], [16]. This new type of similarity measures can be interpreted in a similar way as fuzzy equivalence relations but in this case, the obtained measures are always relative to the parameter value. The detailed studies related to structures of fuzzy (pseudo-) metrics can be found in [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]. Nowadays, researchers have given their attention to the question of whether the aggregation function must satisfy which conditions to guarantee that merging the fuzzy quasi-metrics generates a new one. The authors have studied in detail which functions allow us to merge a collection of fuzzy (quasi-) metrics into a single one in [27]. They also presented a characterization of such functions in terms of triangular triplets, isotonicity, and supmultiplicativity.

In this paper, we provide a general fixed point theorem using residuum operators for contractions obtained through aggregation functions. We show that there are some necessary conditions and also we provide some examples to show that these conditions cannot be omitted.

2 Preliminaries

This section is devoted to compiling definitions and results useful throughout the paper.

Definition 1, [28], A triangular norm (briefly, t-norm) is a binary operation $*$ on $[0,1]$ such that, for all $x, y, z \in [0,1]$, the following axioms are satisfied:
(T1) $x * y = y * x$; $(Commutativity)$
(T2) $x * (y * z) = (x * y) * z$; $(Associativity)$
(T3) $x * y \geq x * z \;\; if \;\; y \geq z$; $(Monotonicity)$
(T4) $x * 1 = x$ $(Boundary\ Condition)$

If, in addition, $*$ is continuous (with respect to the usual topology) as a function defined on $[0,1] \times [0,1]$, we will say that $*$ is a continuous t-norm.

Proposition 2, [28], Let $*$ be a continuous t-norm, then $*$-residuum operator $\rightarrow_*$ of $*$ is uniquely determined by the formula

$$x \rightarrow_* y = \begin{cases} 1, & if\ x \leq y \\ \sup\{z \in [0,1]: x * z = y\}. & if\ x > y \end{cases}$$
(1)

According to the above equation, the residuum operator of the product t-norm ($*_p$), the minimum t-norm ($*_\wedge$) and the Lukasievicz t-norm ($*_\mathcal{L}$) are as follows:

$$x \rightarrow_{*_p} y = \begin{cases} 1, & if\ x \leq y \\ \dfrac{y}{x} & if\ x > y \end{cases},$$

$$x \rightarrow_{*_\wedge} y = \begin{cases} 1, & if\ x \leq y \\ y & if\ x > y \end{cases},$$

$$x \rightarrow_{*_\mathcal{L}} y = \begin{cases} 1, & if\ x \leq y \\ 1 + y - x & if\ x > y \end{cases}.$$

To find a deeper treatment on the residuum operator we refer the reader to [28]. Moreover, following the results were provided in [4], on the residuum operator, we can prove the following proposition, which will be useful later.

Proposition 3 [28], Let $*$ be a continuous t-norm then, $x \wedge y = x * (x \rightarrow_* y)$ for all $x, y \in [0,1]$.

An immediate corollary of the previous propositions is the following one.

Corollary 4, [28], Let $*$ be a continuous t-norm then, $x \rightarrow_* y \geq y$ for all $x, y \in [0,1]$.

Now, we are able to present the notion of fuzzy metric space due to [16]. It is worth mentioning that nowadays, such a concept is commonly used in the literature following its reformulation given in [18], which is given as follows. We obtain the following results from Remark 2 and Remark 4:

Definition 5, [18], A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$, satisfying the following conditions, for all $x, y, z \in X$ and $s, t \in (0, \infty)$
(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;
(KM3) $M(x, y, t) = M(y, x, t)$;
(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(KM5) The function $M_{x,y}: (0, \infty) \rightarrow [0, 1]$ is left-continuous, where $M_{x,y}(t) = M(x, y, t)$ for each $t \in (0, \infty)$.

Below, we can find the modification of the preceding definition given in [15].

Definition 6 [15], A GV-fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying, for all $x, y, z \in X$ and $s, t \in (0, \infty)$, conditions (KM3), (KM4) and the following ones:
(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ iff $x = y$;
(GV5) The function $M_{x,y}: (0, \infty) \rightarrow [0, 1]$ is continuous.

As usual, if $(X, M, *)$ is a (GV-)fuzzy metric space, we say that $(M, *)$, or simply M, is a (GV-)fuzzy metric on X.

It should be noted that a GV-fuzzy metric M can be regarded as a fuzzy metric defining $M(x, y, 0) = 0$ for each $x, y \in X$. So, GV-fuzzy metric spaces can be considered a particular case of fuzzy metric.

Definition 7 [27], A function $\Phi: [0,1]^n \rightarrow [0,1]$ is called a fuzzy metric aggregation function on products if whenever $*$ is a t-norm and $(X_i, M_i, *)_{n \in \mathbb{N}}$ is a family of fuzzy metric spaces then $(\Phi \circ M, *)$ is a fuzzy metric on $\prod X_i$ where $M: \prod X_i \times \prod X_i \times [0, \infty) \rightarrow [0,1]^n$ is given by $\big(M(x, y, t)\big)_i = M_i(x_i, y_i, t)$ for all $x, y \in \prod X_i$ and $t \geq 0$.

We refer to the paper [27], for the properties, characterizations and examples of fuzzy metric aggregation function on products.

An extension of Banach contraction principle for fuzzy metric spaces using the residuum operator was given as follows:

Theorem 8, [29], Let $(X, M,*)$ be a complete fuzzy metric space and $T: X \to X$ be a mapping. If there exists a $k \in (0,1)$ such that

$$M(T(x), T(y), t) \geq k \to_* M(x, y, t)$$

for all $x, y \in X$ and $t > 0$, and $*$ is Archimedean, then T has a unique fixed point in X.

3 Fixed Point Theorems under Aggregation Functions

Throughout this section, M_Φ denotes the fuzzy metric
induced by aggregation of the family of fuzzy metric spaces $(X_i, M_i,*)$ through Φ.

Definition 9 Let $\{(X_i, M_i,*)\}_{i=1}^{n}$ be a family of fuzzy metric spaces, $X = \prod_{i=1}^{n} X_i$ and let $\Phi: [0,1]^n \to [0,1]$ be a fuzzy metric aggregation function on products. A
mapping $F: (X, M_\Phi,*) \to (X, M_\Phi,*)$ is said to be a projective Φ-contraction if there exist contractive constants $c_1, c_2, \ldots, c_n \in (0,1)$ such that

$$M_i(F_i(x), F_i(y), t) \geq c_i \to_* \Phi\big(M_1(x_1, y_1, t), M_2(x_2, y_2, t), \ldots, M_n(x_n, y_n, t)\big)$$

for all $x, y \in X$, $t > 0$ and $i = 1, .., n$.

A function $\Phi: [0,1]^n \to [0,1]$ is said to belong to the class FMA if it fulfills the following conditions:
(FMA1) $x \geq \Phi(1,1, \ldots, x, \ldots, 1)$ for all $x \in [0,1]$.
(FMA2) $\Phi(x, x, \ldots, x) \geq x^n$ for all $x \in [0,1]$ where $x^n = x * x * \ldots * x$.
The following lemma guarantees the completeness for the fixed point theorem.

Lemma 10 Let $\Phi: [0,1]^n \to [0,1]$ be a fuzzy metric aggregation function on products such that $\Phi \in FMA$. Let $(X_i, M_i,*)$ be a family of arbitrary fuzzy metric spaces and $X = \prod_{i=1}^{n} X_i$. Assume that the fuzzy metric space $(X_i, M_i,*)$ is complete for all $i = 1, .., n$. The fuzzy metric space $(X, M_\Phi,*)$ is complete.

Proof. Let $\left(x^k\right)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(X, M_\Phi,*)$. Then for each $\varepsilon \in (0,1)$ there exists $k_0 \in \mathbb{N}$ such that $M_\Phi(x^k, x^m, t) > 1 - \varepsilon$ for each $k, m \geq k_0$ and $t > 0$. Fix $i \in \{1, \ldots, n\}$. We will show that the sequence $\left(x_i^k\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in $(X_i, M_i,*)$. Let $\varepsilon \in (0,1)$. We obtain the next inequality:

$$M_i\left(x_i^k, x_i^m, t\right) \geq \Phi(1,1, \ldots, M_i(x_i^k, x_i^m, t), .., 1)$$
$$\geq \Phi(M_1(x_1^k, x_1^m, t), M_2(x_2^k, x_2^m, t), \ldots,$$
$$M_n(x_n^k, x_n^m, t)) = M_\Phi(x^k, x^m, t) > 1 - \varepsilon$$

since Φ is isotone. So, (x_i^k) is a Cauchy sequence in $(X_i, M_i,*)$. Also, since i is arbitrary, we deduce that (x_i^k) is a Cauchy sequence in $(X_i, M_i,*)$ for all $i = 1, \ldots, n$. On the other hand, since $(X_i, M_i,*)$ is complete for all $i = 1, \ldots, n$, there exists $x_i \in X_i$ such that $lim_{n\to\infty} M_i\left(x_i^k, x_i, t\right) = 1$ for all $t > 0$. Then we have that for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $M_i\left(x_i^k, x_i, t\right) > 1 - \varepsilon$ for all i. Let $\varepsilon \in (0,1)$ and $t > 0$. Since $\backslash ast$ is a continuous t-norm, there exists $\varepsilon^* \in (0,1)$ such that $(1 - \varepsilon^*) * \ldots * (1 - \varepsilon^*) > 1 - \varepsilon$. Also, it is satisfied that $M_i\left(x_i^k, x_i, t\right) > 1 - \varepsilon^*$ for all $t > 0$ when $k \geq k_0$. Thus, we obtain:

$$M_\Phi\left(x^k, x, t\right) = \Phi\left(M_{1(x_1,{}^k, x_1, t)}, .., M_n\left(x_n^k, x_n, t\right)\right)$$
$$\geq \Phi\left((1 - \varepsilon^*), \ldots, (1 - \varepsilon^*)\right)$$
$$\geq (1 - \varepsilon^*) * \ldots * (1 - \varepsilon^*) > 1 - \varepsilon.$$

Hence, (x^k) converges to x in $(X, M_\Phi,*)$ which means that $(X, M_\Phi,*)$ is complete.

Proposition 11 Let $\{(X_i, M_i,*)\}_{i=1}^{n}$ be a family of fuzzy metric spaces, $X = \prod_{i=1}^{n} X_i$, $f_i: X_i \to X_i$ a contraction for all $i = 1, .., n$ and let $\Phi: [0,1]^n \to [0,1]$ be a fuzzy metric aggregation function on products. Then the mapping $F: (X, M_\Phi,*) \to (X, M_\Phi,*)$ defined by $F(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$ is a projective Φ-contraction.

Proof. Since $f_i: X_i \to X_i$ is a contraction there exists a $c_i \in (0,1)$ such that
$M_i(f_i(x_i), f_i(y_i), t) \geq c_i \to_* M_i(x_i, y_i, t)$
For all $x_i, y_i \in X_i$ and $t > 0$. Then, we obtain
$M_i(f_i(x_i), f_i(y_i), t) \geq c_i \to_* M_i(x_i, y_i, t)$
$\geq c_i \to_* \Phi(1, \ldots, M_i(x_i, y_i, t), \ldots, 1)$
$\geq c_i \to_* \Phi(M_1(x_1, y_1, t), M_2(x_2, y_2, t),$
$\ldots, M_n(x_n, y_n, t))$
which means that $F(x) = \left(f_{1(x_1)}, f_{2(x_2)}, \ldots, f_{n(x_n)}\right)$ is a projective Φ-contraction.

However, the converse of the above proposition may not be true in general as shown in the following example:

Example 12 Let $(\mathbb{R}, M_e, *_p)$ be the fuzzy metric space where $M_e(x,y,t) = \frac{t}{t+|x-y|}$ for all $x, y \in \mathbb{R}$ and $t > 0$. Consider the family of fuzzy metric spaces $\left(\mathbb{R}, M_i, *_p\right)_{i=1}^{2}$ such that $M_1 = M_2 = M_e$. Define the function $\Phi: [0,1]^2 \to [0,1]$ by $\Phi(x) = x_1.x_2$. Then Φ is a $*_p$-fuzzy metric aggregation function on products.

Consider the mapping $F: \mathbb{R}^2 \to \mathbb{R}$^2 defined by $F(x) = (x_1 + 1, x_2 + 1)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Now, we show that F is a projective Φ-contraction. For $i = 1$, we obtain

$$M_1(F_1(x), F_1(y), t) = M_1(x_1 + 1, y_1 + 1, t)$$
$$= \frac{t}{t+|x_1 - y_1|}$$
and
$$\Phi\left(M_{1(x_1,y_1,t)}, M_2(x_2, y_2, t)\right)$$
$$= \Phi(\frac{t}{t+|x_1-y_1|}, \frac{t}{t+|x_2-y_2|}\},$$
$$= \frac{t}{t+|x_1-y_1|}.\frac{t}{t+|x_2-y_2|}$$
which means that
$$M_1\left(F_{1(x)}, F_{1(y)}, t\right)$$
$$\geq c_1 \to_{*_p} \Phi\left(M_1(x_1,y_1,t), M_2(x_2,y_2,t)\right)$$
where $c_1 \geq \frac{t}{t+|x_1-y_1|}$ for all $x \neq y$ and $t > 0$. We have a similar result for $i = 2$. So, this implies that F is a projective Φ-contraction. But, $F_1: X_1 \to X_1$ and $F_2: X_2 \to X_2$ are not contractions. If we take $x = 1, y = 2,\ t = 1$ and assume that there is a $c_1 \in (0,1)$ such that
$$M_1(F_1(1), F_1(2), 1) \geq c_1 \to_{*_p} M_1(1,2,1)$$
then we obtain
$$\frac{1}{2} \geq c_1 \to_{*_p} \frac{1}{2} = \begin{cases} 1, & c_1 \leq \frac{1}{2} \\ \frac{1}{2c}, & c_1 > \frac{1}{2} \end{cases}$$
and this follows that $\frac{1}{2} \geq 1$ or $c \geq 1$ which are contradictions. So, we deduce that F_1 is not a contraction. One can easily observe that F_2 is not a contraction with a similar process.

Proposition 13 Let $\{(X_i, M_i, *)\}_{i=1}^{n}$ be a family of fuzzy metric spaces, $X = \prod_{i=1}^{n} X_i$ and $\Phi: [0,1]^n \to [0,1]$ a fuzzy metric aggregation function on products satisfying $\Phi(1,1,\ldots,x_i,\ldots,1) \geq x_i$ for all $x_i \in [0,1]$. If $F: (X, M_\Phi, *) \to (X, M_\Phi, *), F(x) = \left(f_1(x), \ldots, f_n(x)\right)$, is a projective Φ-contraction and $f_i: X_i \to X_i$ is a mapping satisfying $f_i(x) < x$ for all $x \in X$ and i, then f_i is a contraction with some constant $c_i \in (0,1)$.

Poof. Let $F: (X, M_\Phi, *) \to (X, M_\Phi, *)$, $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, be a projective Φ-contraction. Then there exists $c_i \in (0,1)$ such that
$$M_i(F_i(x), F_i(y), t)$$
$$\geq c_i \to_* \Phi(M_1(x_1,y_1,t), \ldots, M_n(x_n,y_n,t))$$
for all $x, y \in X, t > 0$ and $i = 1, .., n$. If we take $x = (x_1, x_2, \ldots, x_i, \ldots, x_n)$ and $y = (x_1, x_2, \ldots, y_i, \ldots, x_n)$, then we have
$$M_i(f_i(x), f_i(y), t)$$
$$\geq c_i \to_* \Phi(M_1(x_1, x_1, t), \ldots, M_i(x_i, y_i, t), \ldots, M_n(x_n, y_n, t))$$
$$= c_i \to_* \Phi(1,1,\ldots, M_i(x_i,y_i,t), \ldots, 1)$$
$$\geq c_i \to_* M_i(x_i, y_i, t)$$
which means that f_i is contraction for all i.

We note here that Example 12 shows that the condition "$f_i(x) < x, \forall\, x \in X\ and\ i$" cannot be deleted in the above theorem.

Theorem 14 Let $\{(X_i, M_i, *)\}_{i=1}^{n}$ be a family of fuzzy metric spaces and $X = \prod_{i=1}^{n} X_i$. If $\Phi: [0,1]^n \to [0,1]$ is a fuzzy metric aggregation function on products satisfying $FMA2$ and $F: (X, M_\Phi, *) \to (X, M_\Phi, *)$ is a projective Φ-contraction with constants $c_1, c_2, \ldots, c_n \in (0,1)$, then F is a contraction from $(X, M_\Phi, *)$ into itself.

Proof. Let $x, y \in X$ and $t > 0$. Since Φ is isotone and F is projective Φ-contraction, there exists $c_1, c_2, \ldots, c_n \in (0,1)$ such that
$$M_\Phi(F(x), F(y), t)$$
$$= \Phi\left(M_1(F_1(x), F_1(y), t), \ldots, M_n(F_n(x), F_n(y), t)\right)$$
$$\geq \Phi(c_1 \to_* \Phi\left(M_1(x_1,y_1,t), \ldots, M_n(x_n,y_n,t)\right),$$
$$c_2 \to_* \Phi\left(M_1(x_1,y_1,t), \ldots, M_n(x_n,y_n,t)\right),$$
$$\ldots, c_n \to_* \Phi(M_1(x_1,y_1,t), \ldots, M_n(x_n,y_n,t)))\}$$
Take $c = max(c_1, c_2, \ldots, c_n)$. Then we have
$$M_\Phi(F(x), F(y), t) \geq$$
$$\Phi\begin{pmatrix} c \to_* \Phi\left(M_1(x_1,y_1,t), \ldots, M_n(x_n,y_n,t)\right), \\ c \to_* \Phi\left(M_1(x_1,y_1,t), \ldots, M_n(x_n,y_n,t)\right), \\ \ldots, c \to_* \Phi\left(M_1(x_1,y_1,t), \ldots, M_n(x_n,y_n,t)\right), \end{pmatrix}$$
since $\to_*$ is antitone with respect to the first argument. Then we obtain

$M_\Phi(F(x),F(y),t)$
$\geq \left(c \rightarrow_* \Phi\big(M_1(x_1,y_1,t),\ldots,M_n(x_n,y_n,t)\big)\right)^{n-1}$
$= c^{n-1} \rightarrow_* \Phi\big(M_1(x_1,y_1,t),\ldots,M_n(x_n,y_n,t)\big)$
$= c^{n-1} \rightarrow_* M_\Phi(x,y,t)$
and hence F is a contraction since $c^{n-1} \in (0,1)$.

The following example shows that the condition "$\Phi\, satisfies\, FMA2$" of Theorem 14 cannot be omitted.

Example 15, Let $(\mathbb{R}, M_e, *_\mathcal{L})$ be the stationary fuzzy metric space where $M_e(x,y,t) = \frac{1}{1+|x-y|}$ for all $x,y \in \mathbb{R}$ and $t > 0$. Consider the family of fuzzy metric spaces $\{(\mathbb{R}, M_i, *_\mathcal{L})\}_{i=1}^2$ such that $M_1 = M_2 = M_e$. Define the function $\Phi: [0,1]^2 \rightarrow [0,1]$ by
$\Phi(x) = \begin{cases} 1, & x_1 = 1\; and\; x_2 = 1 \\ 0, & otherwise \end{cases}$. Then Φ is a $\rightarrow_{*_\mathcal{L}}$-stationary fuzzy metric aggregation function on products. However Φ does not satisfy $FMA2$ since $\Phi(x,x) = 0 < x \rightarrow_{*_\mathcal{L}} x$ for all $x \in (0,1)$.
Consider the mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $F(x) = (x_2, x_1)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$. It is clear that F is a projective Φ-contraction since there exists $c_1, c_2 \in (0,1)$ such that:
$M_1(F_1(x),F_1(y),t) = M_1(x_2,y_2,t)$
$= \dfrac{1}{1+|x_2-y_2|}$
$\geq 1 - c_1 = c_1 \rightarrow_{*_\mathcal{L}} 0$
$= c_1 \rightarrow_{*_\mathcal{L}} \Phi\big(M_1(x_1,y_1,t), M_2(x_2,y_2,t)\big)$,
$M_2(F_2(x),F_2(y),t) = M_2(x_1,y_1,t)$
$= \dfrac{1}{1+|x_1-y_1|}$
$\geq 1 - c_2 = c_2 \rightarrow_{*_\mathcal{L}} 0$
$= c_2 \rightarrow_{*_\mathcal{L}} \Phi\big(M_1(x_1,y_1,t), M_2(x_2,y_2,t)\big)$
for all $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2 (x \neq y)$ and $t > 0$.
However, F is not a contraction from $(\mathbb{R}^2, M_\Phi, *_\mathcal{L})$ from itself. Take $x = (0,0)$ and $y = (1,0)$. Then we have
$M_\Phi(F(0,0),F(1,0),t) = M_\Phi((0,0),(0,1),t)$
$= \Phi\big(M_1(0,0,t), M_2(0,1,t)\big) = \Phi\left(\dfrac{1}{2}\right) = 0$
$M_\Phi((0,0),(1,0),t) = \Phi(M_1(0,1,t), M_2(0,0,t))$
$= \Phi(\frac{1}{2}) = 0$
Assume that that there exists a $c \in (0,1)$ such that $M_\Phi(F(0,0),F(1,0),t) \geq$
$c \rightarrow_{*_\mathcal{L}} M_\Phi\big((0,0),(1,0),t\big)$.
Then we obtain $0 \geq c \rightarrow_{*_\mathcal{L}} 0 = 1 - c$ which means that $c \geq 1$. So we have a contradiction and hence F is not a contraction from $(\mathbb{R}^2, M_\Phi, *_\mathcal{L})$ into itself.

Now we give the fixed point theorem by using Archimedean t-norm and aggregation functions:

Theorem 16 Let $\{(X_i, M_i, *)\}_{i=1}^n$ be a family of complete fuzzy metric spaces, $*$ be an Archimedean t-norm and $X = \prod_{i=1}^n X_i$. If $\Phi: [0,1]^n \rightarrow [0,1]$ is a fuzzy metric aggregation function on products satisfying FMA and $F: (X, M_\Phi, *) \rightarrow (X, M_\Phi, *)$ is a projective Φ-contraction with constants $c_1, c_2, \ldots, c_n \in (0,1)$, then F has a unique fixed point in X.

Proof. Since Φ and F satisfies the conditions in Theorem 14, we can guarantee that F is a contraction. Also, we obtain from Lemma 10 that $(X, M_\Phi, *)$ is a complete metric space since $\{(X_i, M_i, *)\}_{i=1}^n$ is a family of complete fuzzy metric space and Φ belongs to FMA. In addition to these, $*$ is an Archimedean t-norm as stated in the hypothesis, so we conclude that from Theorem 8, F has a unique fixed point in X.

In the following, we give an example that shows there are contractions on $(X, M_\Phi, *)$ which are not projective Φ-contraction. So this fact implies that Theorem 16 is not a direct consequence of Theorem 8.

Example 17 Let $(\mathbb{R}, M_e, *_\mathcal{L})$ be the stationary fuzzy metric space where $M_e(x,y,t) = \frac{1}{1+|x-y|}$ for all $x,y \in \mathbb{R}$ and $t > 0$. Consider the family of fuzzy metric spaces $\{(\mathbb{R}, M_i, *_\mathcal{L})\}_{i=1}^2$ such that $M_1 = M_2 = M_e$. Define the function $\Phi: [0,1]^2 \rightarrow [0,1]$ by
$\Phi(x) = \begin{cases} 1, & x_1 = 1\; and\; x_2 = 1 \\ 0, & x_1 = 0\; and\; x_2 = 0 \\ \frac{1}{2}, & otherwise \end{cases}$. Then, Φ is a $*_\mathcal{L}$-stationary fuzzy metric aggregation function. Consider the mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $F(x) = \left(\frac{x_1+x_2}{2}, 0\right)$ for all $x = (x_1,x_2) \in \mathbb{R} \times \mathbb{R}$. Now, we show that F is a contraction mapping.

Let $x, y \in X$. Then, we obtain

$M_\Phi(F(x),F(y),t)$
$= \Phi\big(M_1(F_1(x),F_1(y),t), M_2(F_2(x),F_2(y),t)\big)$
$= \Phi\left(M_1\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2} t\right), 1\right) = 1$
$\geq c \rightarrow_{*_\mathcal{L}} M_\Phi(x,y,t)$

for all $c \in (0,1)$ and $t > 0$. Hence, F is a contraction on $(X, M_\Phi, *_\mathcal{L})$. Now, we will show that

F is not a projective Φ-contraction.

Let $x = (1,0), y = (1,2) \in \mathbb{R}^2$. Then, we have $M_1(F_1(x), F_1(y), t) = M_1\left(\frac{1}{2}, \frac{3}{2}, t\right) = \frac{1}{2}$

$$< 1 = c \rightarrow_{*_\mathcal{L}} 1 = c \rightarrow_{*_\mathcal{L}} \Phi\left(1, \frac{1}{3}\right)$$

$$= c \rightarrow_{*_\mathcal{L}} \Phi\big(M_1(1,1,t), M_2(0,2,t)\big)$$

$$= c \rightarrow_{*_\mathcal{L}} \Phi(M_1(x_1, y_1, t), M_2(x_2, y_2, t))$$

for all $c \in (0,1)$ and $t > 0$ which concludes that F is not a projective Φ-contraction.

4 Conclusion

Fixed point theory is an important tool when we solve certain functional equations such as differential equations, integral equations, fractional differential equations, matrix equations, etc. We can reformulate the considered problem in terms of investigating the existence and uniqueness of a fixed point of a function. Also, this theory has several applications in many different fields such as biology, physics, chemistry, economics, game theory, optimization theory and etc. For future work, we plan to investigate some applications of the obtained result to the mentioned areas.

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References:

[1] Borsik, J.; Dobos, J., On a product of metric spaces, *Math. Slovaca* 31(1) (1981), 193-205.
[2] Dobos, J., *Metric preserving functions*, Stroffek, kosice (1998).
[3] Pradera, A.; Trillas, E., A note on pseudometrics aggregation, *International Journal of General Systems,* 31(1) (2002), 41-51.
[4] Mayor, G.; Valero, O., Aggregation of asymmetric distances in computer science, *Information Sciences,* 180(6) (2010), 803-812.
[5] Miñana, J. J.; Valero, O., Characterizing quasi-metric aggregation functions, *International Journal of General Systems,* 48(8) (2019), 890-909.
[6] Massanet, S.; Valero, O, On aggregation of metric structures: the extended quasi-metric case, *International Journal of Computational Intelligence Systems*, 6(1) (2013), 115-126.
[7] Drewniak, J.; Dudziak, U., Preservation of properties of fuzzy relations during aggregation processes, *Kybernetika*, 43(2) (2007), 115-132.
[8] Dudziak, U., Preservation of t-norm and t-conorm based properties of fuzzy relations during aggregation process, *8th conference of the European Society for Fuzzy Logic and Technology*, Atlantis Press, (2013), 416-423.
[9] Fodor, J. C.; Roubens, M. R., Fuzzy preference modelling and multicriteria decision support, *Springer Science and Business Media*, 14 (1994).
[10] Saminger, S.; Mesiar, R.; Bodenhofer, U., Domination of aggregation operators and preservation of transitivity, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems,* 10(2002), 11-35.
[11] Mayor, G.; Recasens, J., Preserving T-transitivity, *Artif. Intell. Res. Dev.*, 288 (2016), 79-87.
[12] Calvo Sanchez, T.; Fuster-Parra, P.; Valero, O., On the problem of aggregation of partial T indistinguishability operators, In International Conference on Information Processing and Management of Uncertainty in *Knowledge-Based Systems, Springer, Cham*, (2018), 207-218.
[13] Fuster-Parra, P.; Martin, J.; Recasens, J.; Valero, O., T-equivalences: The metric behavior revisited, *Mathematics*, 8(4) (2020), 495.
[14] Miñana, J. J.; Valero, O., Aggregation of indistinguishability fuzzy relations revisited, *Mathematics,* 9(12) (2021), 1-12.
[15] George, A.; Veeramani P., On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64(3) (1994), 395-399.
[16] Kramosil, I.; Michalek, J., Fuzzy metrics and statistical metric spaces, *Kybernetika*, 11(5) (1975), 336-344.
[17] Camarena, J.G.; Greogri, V.; Morillas, S.; Sapena, A., Fast detection and removal of impulsive noise using peer groups and fuzzy metrics, *Journal of Visual Communication and Image Representation*, 19(1) (2008), 20-29.
[18] Grabiec, M., Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems*, 27 (1989), 385-389.
[19] Grecova, S.; Morillas, S., Perceptual similarity between color images using fuzzy metrics, *Journal of Visual Communication and Image Representation*, 34 (2016), 230-

235.
[20] Gregori, V.; Miñana, J.J., A Banach contraction principle in fuzzy metric spaces defined by means of t-conorms, *Revista de la Real Academia de Ciencias Exactas, Fasicas y Naturales. Serie A. Matematicas (RACSAM)*, 115 (2021), 129:1-11.
[21] Gregori V.; Miñana J.J.; Morillas, S., Some questions in fuzzy metric spaces, *Fuzzy Sets and Systems*, 204 (2012), 71-85.
[22] Gregori, V.; Morillas, S.; Sapena, A., Examples of fuzzy metrics and applications, *Fuzzy Sets and Systems*, 170 (2011), 95-111.
[23] Gregori, V.; Romaguera, S., Some properties in fuzzy metric spaces, *Fuzzy Sets and Systems*, 115 (2000), 485-489.
[24] Miñana, J.J.; Ortiz, A.; Ortiz, E.; Valero, O., On the standard fuzzy metric: generalizations and application to model estimation, *Proceedings of the Conference of the Spanish Association for Artificial Intelligence (CAEPIA)*, (2021), pp. 311-316.
[25] Ortiz, A.; Ortiz, E.; Miñana, J.J.; Valero, O., Hypothesis scoring and model refinement strategies for Fm-based Ransac, *Advances in Artificial Intelligence. CAEPIA (2021). Lecture Notes in Computer Science*, Vol. 12882 (2021), pp. 96-105.
[26] Ortiz, A.; Ortiz, E.; Miñana, J.J.; Valero, O., On the use of fuzzy metrics for robust model estimation: a ransac-based approach, *Advances in Computational Intelligence. IWANN 2021. Lecture Notes in Computer Science*, Vol. 12861 (2021), 165-177.
[27] Pedraza, T.; Rodriguez-Lopez, J.; Valero, O., Aggregation of fuzzy quasi-metrics, *Information Sciences*, (581), (2021) 362-389.
[28] Klement, E. P.; Mesiar R.; Pap E., *Triangular Norms*, Netherlands: Springer (2000).
[29] Aygün, H.; Güner, E.; Miñana, J. J.; Valero, O., Fuzzy partial metric spaces and fixed point theorems, *Mathematics*, 10(17) (2022), 3092.

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Conflict of Interest

The authors have no conflicts of interest to declare.