Applications of Locally Compact Spaces in Polyhedra: Dimension and Limits

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Abstract: - The study of applications of locally compact spaces in polyhedra in relation to their dimensions as well as homotopy and extension problems developed in the late 1940s and early 1950s under the leadership of mathematician. Many mathematicians studied application locally compact in polyhedra. A polyhedron can be obtained by subdivision, as a simplicial metric complex; thus, re-gluings of polyhedra can also be seen as simple complexes. Thus, the topology of a simplicial metric complex X is the topology quotient of the reattachment. **The objective of this work is to shed light on the applications in polyhedra of locally compact spaces and to highlight the limits of these applications.** A continuous application f of X in P defines a finite open overlay of X, and a partition of the unit subordinate to this overlay, f is homotopic to an application f ', obtained by composing the restriction to A, of an application of X in the K_R polyhedron, and a simplistic application of a sub-polyhedron K_R' in P. The problem of extension deserves to be elucidated to understand how it is possible to get around certain conceptual difficulties.

Key-Words: - polyhedron, compact spaces, locally compact spaces, paracompact spaces, CECH COHOMOLOGY, homotopy, extension.

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1 Introduction

The study of applications of locally compact spaces in polyhedra in relation to their dimensions as well as homotopy and extension problems developed in the late 1940s and early 1950s under the leadership of mathematician Henry CARDAN, [1], [2]. In a more general framework, when we consider a polyhedron P, finite or infinite, but locally finite (thus locally compact) and supposedly simplicial abstract (we decompose it if it is not), it is defined by a simplicial abstract complex K and is identified with a subspace of the "cube" I^K (I designating the bounded closed interval [0; 1]), [3]. Thus, a point of P is a system (λ_K) of real numbers between 0 and 1, all zero except a finite number, and whose sum is equal to 1; the index k runs through the set K and the set of k such that $\lambda_K \neq 0$ is a simplex of the simplicial complex K. **Our goal in this work is to study these applications in their dimension and possibly raise problems of homotopies and extension.**

These continuous applications transform, indeed, the unit I ball into a locally compact part Consequently, a continuous application f of the X space in P is then defined by the coordinate data $\lambda_K = f_K(x)$ of the transform of the point x by f; the f_K are numerical continuous functions, with values in the segment I, such as:

I. In the (open) U_K set of points x where $f_K > 0$, the f_j is zero except for a finite number;

II. The sum Σf_K is equal to one (this sum makes sense, since at each point all terms are zero except for a finite number).

If we have a system of continuous functions f_K, with values in I, defined in a locally compact space X, then the satisfaction of the first condition leads to say that it is a finite type partition. This partition is finally a partition of the unit when, in addition to the first criterion, the second condition is satisfied.

The objective of this work is to shed light on the applications in polyhedra of locally compact spaces and to highlight the limits of these applications. The text is structured as follows: the first section presents problem statements to highlight locally compact spaces in polyhedra, the second section deals with homotopy problems and the last section is devoted to the limits in the case of extensions.

2 Materials and Methods

2.1 Continuous Applications of a Locally Compact Space X in Polyhedra

This work could have been treated without leaving the framework of the Banach spaces, [1], [2], [3], [4]. Moreover, given the results that we are trying to highlight with regard to polyhedra, we preferred to place ourselves within the framework of locally compact spaces. We use without reference the elementary results of the theory of measurement, placing ourselves exclusively in the wake of the measurements on the space of continuous applications with compact supports on a given locally compact space.

One of the most important statements is that any open overlap of X is said to be finite if each set of the overlap meets only a finite number of them, [5]. Moreover, to each (fk) partition (of finite type) let us associate the U_k open overlay defined as above: this open overlay is said to be "associated with the partition"; it is of finite type.

In [3], Given an open finite $(Vk)k\in K$ overlay, a partition of the unit $(fk)k\in K$ is said to be subordinate to the overlay if it has the same set of indices k, and if, for any $k\in K$, the set Uk of points x such that $fk(x) \neq 0$ is contained in Vk.

From this demonstration, we can say that a continuous application f of X in P defines a finite open overlay of X, and a partition of the unit subordinate to this overlay.

Conversely, let R be an open finite overlay, and (f_k) a partition subordinate to this overlay: then (f_k) defines a continuous application f of X in the polyhedron K_R (topological realization of the "nerve" of the overlay of R; this nerve is the simplicial abstract complex K_R defined as follows: its vertices are the sets of the overlay R, and a set of "vertices" is a "simplex" if the corresponding sets of the overlay have a non-empty intersection).

Note that from, [6], a polyhedron can be obtained by subdivision, as a simplicial metric complex; thus re-gluings of polyhedra can also be seen as simple complexes. Thus the topology of a simplicial metric complex X is the topology quotient of the reattachment.

In order to study the applications of locally compact spaces, we start from two problems which are the following:

Problem (2.1):
Given an open overlay of finite type R, are there partitions subordinate to this overlay? Yes, if the space is normal (in other words, two disjoint closed neighbourhoods can be separated by two disjoint open neighbourhoods). Indeed, if (U_k) is a finite open overlay of a normal space X, there is an open overlay (V_k) such as $V_k \subset U_k$; or g_k a continuous function with values in the segment [0; 1], such that $g_k(x) = 1$ for all $x \in V_k$, $g_k(x)=0$ for all $x \notin U_k$.Or g(x) equals summation of $g_k(x)$ over k, which has a meaning and a numerical continuous function with values greater than or equal to 1. Just ask: $f_k(x) = g_k(x) / g(x)$ to obtain a partition of the unit subordinate to the overlay (U_k).

From two overlaps (U_k) and (V_j) (not necessarily having the same set of indices), we say that (V_j) is finer than (U_k) if any set (V_j) is contained in at least one set U_k.

Problem (2.2):
X locally compact given, is there a finer finite open overlap than an arbitrarily given open overlap? Any compact space is paracompact (trivial) (moreover, in a compact space, any finite open overlap is finite). A

paracompact space is normal. The immediate consequence is that if X is paracompact, there is always a partition of the unit whose associated overlay is finer than an arbitrarily given open overlay. Consequently, there are arbitrarily thin unit partitions.

Note: it is easy to characterize, among the locally compact X, those that are paracompact. For this, it is necessary and sufficient that X is a meeting of disjoint open subspaces (each of which is therefore also closed), each of which is an enumerable meeting of compact subsets. An equivalent condition is: X has at least one finite overlay of relatively compact open subspaces, [6].

Any closed sub-space of a paracompact space is paracompact.

Proposal (2.3): Let R be an open covering of finished type, K_R the polyhedron it defines. Each partition of the unit whose associated overlap is finer than R defines (in several ways) a continuous application of X in K_R; all these applications (whatever partition they correspond to) are homotopic.

Let R' be the overlay associated with a partition, R' being finer than R. For each simple application $K_{R'} \longrightarrow K_R$ (which associates to a set of R' a set of R containing it), let us compose the application X $K_{R'}$ defined by the partition, with $K_{R'} \longrightarrow K_R$. By definition, we thus obtain all the applications of X in K_R defined by the partition. These applications (whatever the partition whose associated overlap R' is finer than R) each define a partition of the unit subordinate to R. To show that they are all homotopic, it is enough to use the affine structure of the space of the partitions subordinate to R.

In the next part of this section according to [6], we will discuss The Approximation of an application f of a closed part A of X, in a polyhedron P

We assume X paracompact. The application f of A in P defines an open, finite overlap of A. The joining of the sets of this overlap is complementary (in X) to a closed set B, therefore paracompact; if $\dim (X - A) \leq n$, then the dimension of B is less than or equal to n. There is an open overlap R, of the finished type of X, whose trace on A is a finer overlap than that defined by f. Let us associate with R, a partition of the unit (ϕ_k) in X space. The restrictions on A of (ϕ_k) thus define an overlap of A that is finer than that defined by f. The (ϕ_k)'s define an application ϕ of X in K_R, which applies A in a sub-polyhedron $K_{R'}$ of K_R. There is an application f ' of A in P, consisting of A in $K_{R'}$ (restriction of ϕ), and a simple application $K_{R'}$ in P. In A, the applications f and f ' are homotopic (according to proposal (2.3)).

In summary: f is homotopic to an application f ', obtained by composing the restriction to A, of an application of X in the K_R polyhedron, and a simplistic application of a sub-polyhedron $K_{R'}$ in P. Therefore, to extend f' it is sufficient to know how to extend the $K_{R'}$ application in P into an application of K_R in P (problem studied previously). If f' is extendable, then f will also be extendable, by virtue of the following lemma:

Lemma (2.4):

Let X be a paracompact, and A a closed contained in X. Any continuous application of A in a polyhedron P, which. In A, is homotopic to an application extendable to X, is itself extendable to X.

To demonstrate this assertion, it is sufficient to prove that any application of A in a polyhedron P is extendable to a neighbourhood of A (where A is a closed part of a paracompact space X).

However, it is true that if A is compact then the image of A is contained in a finished polyhedron. If A is not compact, then it is sufficient to demonstrate when X is a countless meeting of compacts, $X_1 \subset X_2 \subset X_3 \subset \ldots \subset X_j \subset \ldots$. The restriction from f to $A \cap X_1$ extends to a compact neighbourhood V_1 of $A \cap X_1$; hence f_1 on $A \cup (V_1 \cap X_1) = A_1$. The restriction from f_1 to $A \cap X_2$ extends to a compact neighbourhood V_2 of ... , etc.

It can also be proved that if f and g (applications of x in the polyhedron P) coincide on a closed part $A \subset X$, they are homotopic (in X) to two applications f' and g' which coincide on a neighbourhood of A.

3 Size of A Locally Compact Space

It is important to specify that in all that follows, only locally compact spaces X with the following property are considered:

X has arbitrarily fine open overlaps of finite dimensions. Recall that from, [3], [7], an overlap is said to be of dimension greater than or equal to n, if its "nerve" is of dimension greater than or equal to n, in other words, if each point of the space belongs at most to n + 1 sets of the overlap, any closed sub-space of such space X enjoys the same property).

Definition (3.1):
It is said that dim X $\leq$ n if there are open overlaps of dimension greater than or equal to n, arbitrarily fine. In fact, dim X = n if dim X $\leq$ n and dim X $\neq$ n – 1. If dim X $\leq$ n and if A is a closed subspace of X, then dim A $\leq$ n: it is immediate.

Theorem (3.2):
If dim X $\leq$ n, any continuous application, in the sphere Sn, of a closed part A of X is extendable to X. This is a direct consequence of the extension theorem of an application in Sn of a sub-polyhedron of a polyhedron of dimension less than or equal to n.
Notice that as a reciprocity of this theorem:
Suppose only that any compact sub-space X' $\subset$ X enjoys the property Πn(X'): any continuous application in Sn of a closed part of X' is extendable into a continuous application of X' in Sn. Then dim X $\leq$ n.
Demonstration of this reciprocity:
we first observe that the property $\Pi_n(X)$ leads to $\Pi_{n+1}(X)$.
Indeed, suppose that X verifies Π_n; that is A a closed part of X, and f a continuous application of A in S^{n+1}. Consider Sn as the equator of S^{n+1} and either B = $f^{-1}(S^n) \subset A$. There is a continuous application g of X in S^n, which coincides with f on B; on A, f and g (considered as applications in S^{n+1}) are homotopic, because f(x) and g(x) are never diametrically opposed; since g(x) is extendable to X, f is also (theorem 1).
Let R be an overlap of X, of finite dimension, arbitrarily fine, and formed of relatively compact openings. Let us choose an application f of X in K_R, in the class defined by R. For any integer k, let X_k be the reciprocal image of the k-skeleton P_k of K_R, and f_k the restriction from f to X_k. We will show that f_n extends into a continuous application g_n of x in P_n, so that g(x) belongs to the smallest closed simplex containing x; then the reciprocal image, by g_n, of the canonical overlap of P_n will be an arbitrarily fine open overlap of dimension lower or equal to n. To prove the existence of g_n, we define, by downward recurrence on k $\geq$ n, an application g_k of X in P_k in the following way: $g_k = f_k$ for k large enough (in fact: for k at least equal to the dimension of K_R); g_k coincides g_{k+1} on X_k, and is deduced from g_{k+1} using the property Π_k for the compacts contained in X.

Corollary (3.3):
For dim X $\leq$ n, it is necessary and sufficient that dim Y $\leq$ n for any compact sub-space Y$\subset$X.
As a remark, we can add that this could be used as a definition for the dimension of a locally compact space X, without any restrictive hypothesis on X).
Before moving on to theorem (5.4), it is essential to first define the notion of R-application. It is thus a continuous application f of X in a space Y, such that the reciprocal image of each point of Y is "small of order R" (R designating an open overlap of X).

Theorem (3.4):
For a space X to be of dimension less than or equal to n, it is necessary and sufficient that for any open overlap R of X, there is an R-application of X in the polyhedron of dimension less than or equal to n.
The condition is obviously necessary. To show that it is sufficient, it is sufficient to demonstrate when X is compact (according to the previous corollary). Then the reciprocal image of any "fairly small" set of the polyhedron P (which can be assumed to be finite) is still small of order R.
Let us take a subdivision P' of P, fine enough so that the overlap of X defined by the application of X in P' is finer than R. Since this overlap is of dimension less than or equal to n, X has many arbitrarily fine open overlaps of dimension less than or equal to n.

Corollary (3.5):
The dimension of the product of two spaces is at most equal to the sum of the dimensions of these spaces. On the other hand, it can be smaller than the sum).
We will notice that the dimension of a quotient space may be greater than the dimension of the space itself (Peano curve).

4 CECH COHOMOLOGY

Let R' be a finished type X overlay, thinner than a finished type R overlay. Then, all the simplistic applications of $K_{R'}$ in K_R (which to a set of R', associate a set of R containing it) are homotopic (proposal 1), so in, [4], [5], they define the same homomorphism of the cohomology groups: $H^*(K_R)$ in $H^*(K_{R'})$ (any coefficients, fixed once and for all). Transitivity (obvious) of these homomorphisms. There is therefore an inductive limit (direct limit) of the $H^*(K_R)$ groups, with canonical homomorphisms of the $H^*(K_R)$ within this inductive limit. This inductive limit is the Cech cohomology group H(X), X being locally compact and paracompact.
NB: this definition is only consistent with Cech's

definition if X is compact for X not compact, this is the correct generalisation of Cech's definition). It can be shown that the cohomology groups thus obtained are those given by the axiomatics of the beams.

Proposal (4.1.1):
If dim $X \leq n$, the cohomology groups $H^p(X)$ are zero for $p > n$, whatever the group of coefficients.
This is obvious from the definition of these groups, and from the definition of the dimension.
Or A a closed part of X. For each overlap R of X, or R_A the overlap of A induced by R; then K_{RA} identifies itself with a sub-polyhedron of K_R. We can consider the inductive limit of $H^*(K_R \text{ mod } K_{RA})$; this is, by definition, the relative cohomology group H (X mod A).

Proposal (4.1.2):
We have an exact sequence of canonical homomorphisms
$\ldots \longrightarrow H^n(X \text{ mod } A) \longrightarrow H^n(X)$ $H^n(A)$ $\longrightarrow H^{n+1}(X \text{ mod } A) \longrightarrow \ldots$ (for any system of coefficients).
Indeed, we have such an exact sequence for the cohomology of the K_R polyhedron and its sub-polyhedron K_{RA}. However, the inductive limit of a family of exact sequences is an exact sequence.
Remark: if dim $(X - A) \leq n$, then $H^p(X \text{ mod } A) = 0$ for $p > n$.

Let us from, [6], recall without demonstration the well known fact: by compact X, the group H (X mod A) depends only on the space X - A, and is identified with the cohomology group (of Cech) of "second species", or "with compact supports" of the locally compact space X - A.

4.1 Effect of Continuous Application

Let f be a continuous application of X in X', which transforms a closed part $A \subset X$ into a closed part $A' \subset X'$. The reciprocal image of any finite open overlap of X' is a finite open overlap of X.

We deduce, after passing the inductive limit, a homomorphism of the cohomology groups: $H^n(X' \text{ mod } A')$ in $H^n(X \text{ mod } A)$.

4.2 Transitivity of these Homeomorphisms

Let us now have two continuous applications f and g of X in Y, whose restrictions to A are identical. We deduce a homomorphism (f, g) * of H(Y) in H (X mod A) (group of arbitrary coefficients). This homomorphism is defined by crossing the inductive limit. The homomorphism (f, g) * relating to Cech cohomology groups has properties similar to those indicated for singular cohomology.

4.3 Some important Definitions

Definition (4.4.1), [4], Let Y be an aspherical polyhedron in dimension less than n. We have already defined the fundamental class of Y as an element of $H^n(Y, H_n(Y))$ where for Y the cohomology of Cech is identified with the singular cohomology.
We then deduce that the characteristic class of an application f of X (paracompact) in Y is an element $\gamma(f)$ of $H^n(X, H_n(Y))$, image of the fundamental class by the homomorphism $H^n(Y, H_n(Y))$ defined by f.

Definition (4.4.2), [5], The deviation of a pair (f, g) of applications of X in Y, which coincide on a closed part A of X: it is an element γ (f, g) of $H^n(X \text{ mod } A, H_n(Y))$, image of the fundamental class by the homomorphism (f, g) * of $H^n(Y, H_n(Y))$ in $H^n(X \text{ mod } A, H_n(Y))$.

Definition (4.4.3):
The obstruction of an application f of A in Y (A: closed part of X paracompact): it is an element β (f) of $H^{n+1}(X \text{ mod } A, H_n(Y))$, image of the characteristic class of f (element of $H^n(A, H_n(Y))$ by the canonical homomorphism $H^n(A, H_n(Y))$ in $H^{n+1}(X \text{ mod } A, H_n(Y))$.

4.4 Extension and Homotopy Theorems

Throughout the paragraph, X denotes a locally compact space satisfying the conditions of paragraph 2; A denotes a closed part of X, and Y denotes an aspherical polyhedron of dimensions less than n (e.g. a sphere of dimension n).

The idea is to transpose to these cases theorems (4.2) and (4.4), in which X is a polyhedron and A a sub-polyhedron (it is true that then Y is not necessarily a polyhedron). Thus, we define a new theorem which is in reality only theorem (5.2) stated above.

Theorem (4.4.1):
We assume dim $X \leq n+1$. Then, for an application f of A in Y to be extendable to X, it is necessary and sufficient that the obstruction $\beta(f) \in H^{n+1}(X \text{ mod } A, H_n(Y))$ is null. This is true, strictly speaking, only for any $n \geq 2$; if $n = 1$, it is furthermore assumed that the

space Y is i-simple for any integer $i \leq n+1$, and then the conclusion remains.

Theorem (4.4.2):
It is assumed that dim $X \leq n$. Then, for two applications f and g of X in Y, which coincide on A, to be homotopic modulo A, it is necessary and sufficient that the deviation
γ (f, g) ∈ H^n (X mod A, $H_n(Y)$) is zero. Moreover, given an application f of X in Y, and an arbitrary element γ of H^n (X mod A, $H_n(Y)$), there is an application g of X in Y, equal to f on A, and such that the deviation γ (f, g) is precisely γ. In particular, (A is assumed to be empty): the classes of applications of X in Y are in one-to-one correspondence with the elements of the (Cech) cohomology group H^n (X, $H_n(Y)$).

The conclusions of this theorem are only valid, in reality, if $n \geq 2$. For $n = 1$, they are valid provided that it is also assumed that the space Y is i-simple for all $i \leq n$ (respectively $i< n$).

It is possible to apply the previous theorems especially when Y is a sphere of dimension n. But for $n = 1$, the nullity of the homotopy groups $\Pi_i(S^1)$ for $i \geq 2$ makes it possible to give the following extensions of theorems 1 and 2.

Theorem (4.4.3):
Or a closed part of X. For two applications f and g of X in S^1 to extend to X, it is necessary and sufficient that the obstruction β (f) ∈ H^2(X mod A, Z) is zero.

Theorem (4.4.4):
Or A a closed part of X. For two applications f and g of X in S^1, which coincide on A, to be homotopic modulo A, it is necessary and sufficient that the deviation γ (f, g) ∈ H^1(X mod A, Z) is zero. The classes of applications of X in S^1 correspond biunivocally to the elements of the cohomology group (of Cech) of the X space, with integer coefficients.

4.5 Application of the Extension Theorem: Cohomological Characterisation of the Dimension

Lemma (4.5.1):
If $\dim(X) \leq n+1$, and if H^{n+1}(X mod A, Z) = 0 for any closed part A of X, then any application of A in S^n is extended into an application of X in S^n.
This is a direct consequence of theorem (5.5.1).
According to the reciprocal of theorem (4.2), we see that, in the hypotheses of the lemma, the dimension of X is less than or equal to n. Reciprocally, it is clear that if $\dim(X) \leq n$, then H^{n+1}(X mod A, Z) = 0 for any closed part A of X. Consequently, we can state a new theorem.

Theorem (4.5.2):
If X is of finite dimension, the dimension of X is the largest of the integers n such that there exists a closed part A of X satisfying H^n (X mod A, Z) ≠ 0. (The integer coefficients, for cohomology, play a privileged role for the characterisation of the dimension; it can be seen that this is due to the fact that the homology group $H_n(S^n)$ is isomorphic to Z).

Remark (4.5.3):
If X is compact (a case to which we can return, since the dimension of a non-compact space is the upper limit of the dimensions of the contained compacts), the cohomology group H^n (X mod A, Z), which intervenes in the characterisation of the dimension is none other than the cohomology group with compact supports of the open subspace X - A.

As an example: the space IR^n is of dimension (topological) equal to n, because the cohomology with compact supports of an open ball is not null for dimension n.

A polyhedron of (simplistic) dimension n is of(topological) dimension n.

For a closed part of IR^n to be of dimension n, it is necessary and sufficient that it has at least one interior point. (This is sufficient according to theorem (5.6.2); it is necessary since if A has no interior point in any triangulation of IR^n, each n-simplex contains a point which does not belong to A, which allows (by central projection in each n-simplex) to find an ε-application of A in a polyhedron of dimension n - 1.

5 Conclusion

This work has highlighted the importance of locally compact space applications in the case of polyhedra, but also their limitations. A continuous application f of X in P defines a finite open overlay of X, and a partition of the unit subordinate to this overlay, f is homotopic to an application f ', obtained by composing the restriction to A, of an application of X in the K_R polyhedron, and a simplistic application of a sub-polyhedron K_R' in P. Indeed, problems may remain and are linked to homotopic applications as well as to the cohomological character of the dimension chosen to study. In future work we hope to highlight some of the problems related to homotopic applications and to study some applications of locally compact spaces in further cases other than polyhedra.

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