Growth of Solutions to Complex Linear Differential Equations with Analytic or Meromorphic Coefficients in $\overline{\mathbb{C}} - \{z_0\}$ of Finite Logarithmic Order

ABDELKADER DAHMANI, BENHARRAT BELAÏDI
Department of Mathematics, Laboratory of Pure and Applied Mathematics,
University of Mostaganem (UMAB),
B. P. 227 Mostaganem,
ALGERIA

Abstract: - In this paper, by using Nevanlinna theory near a singular point, we study the growth and the oscillation of solutions of homogeneous and non-homogeneous complex linear differential equations of the form:

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$
$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z),$$

where $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ and $F(z)$ are analytic or meromorphic functions in the extended complex plane except a finite singular point with finite logarithmic order. Under some additional conditions when an arbitrary $A_s(z)$ dominating near a singular point $z_0 \in \mathbb{C}$ the others coefficients by its logarithmic order and logarithmic type, we obtained some growth properties of solutions of the above equations. The results established in the present paper extend and improve those from other works.

Key-Words: - Linear differential equation, analytic function, meromorphic function, singular point, logarithmic order, logarithmic type.

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1 Introduction and Main Results

In this paper, we shall assume the reader is familiar with the fundamental results and standard notations of the Nevanlinna value distribution theory of meromorphic functions, [1], [2], [3], [4]. The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna theory to annuli have been made by [5], [6], [7]. Many authors have investigated the growth and oscillation of solutions of complex linear differential equations in the different domains such as the whole complex plane $\mathbb{C}$, [8], [9], [10], [11], the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, [12], [13] and more recently in the extended complex plane except a finite singular point $\overline{\mathbb{C}} - \{z_0\}$, [14], [15], [16], [17], [18], [19], [20], considering the case that at least one of the coefficients has order different to zero. In recent years, after the works, [21], [22], there has been an increasing interest in using the logarithmic order as an effective tool to measure the rate of the growth of solutions of linear differential equations and linear difference equations when all the coefficients are of order equals zero, [23], [24], [25]. In this article, we also use the logarithmic order as growth indicator for solutions of homogeneous and non-homogeneous linear differential equations, where the coefficients are analytic or meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$. For the following definitions, we use the same definitions as in [16] and [20]. Let f be a meromorphic in $\overline{\mathbb{C}} - \{z_0\}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \mathbb{C}$. The characteristic function of $f(z)$ near z_0 is defined by:

$$T_{z_0}(r, f) = m_{z_0}(r, f) + N_{z_0}(r, f)$$

where

$$m_{z_0}(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(z_0 - e^{i\phi}) \right| d\phi$$

and

$$N_{z_0}(r, f) = -\int_\infty^r \frac{n(t, f) - n(\infty, f)}{t} dt - n(\infty, f) \log r,$$

such that $n(t, f)$ counts the number of poles of $f(z)$ in $\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$, each pole according to its multiplicity.

For all $R \in (0,+\infty)$ and $p \geq 1$, we define $\exp_1 R = e^R$, $\exp_{p+1} R = \exp(\exp_p R)$, $\log_1 R = \log R$ and $\log_{p+1} R = \log(\log_p R)$. Let p and q be two integers with $p \geq q \geq 1$. The $[p,q]$-order and the $[p,q]$-type near z_0 of a meromorphic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$ are defined by:

$$\sigma_{[p,q]}(f,z_0) = \limsup_{r\to 0} \frac{\log_p^+ T_{z_0}(r,f)}{\log_q \frac{1}{r}},$$

$$\tau_{[p,q]}(f,z_0) = \limsup_{r\to 0} \frac{\log_{p-1}^+ T_{z_0}(r,f)}{\left(\log_{q-1}\frac{1}{r}\right)^{\sigma_{[p,q]}(f,z_0)}}$$

if $\sigma_{[p,q]}(f,z_0) \in (0,+\infty)$. For an analytic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$, the $[p,q]$-order and the $[p,q]$-type of $f(z)$ near z_0 are given by:

$$\sigma_{[p,q]}(f,z_0) = \limsup_{r\to 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{\log_q \frac{1}{r}},$$

$$\tau_{[p,q],M}(f,z_0) = \limsup_{r\to 0} \frac{\log_p^+ M_{z_0}(r,f)}{\left(\log_{q-1}\frac{1}{r}\right)^{\sigma_{[p,q]}(f,z_0)}}$$

if $\sigma_{[p,q]}(f,z_0) \in (0,+\infty)$, where $M_{z_0}(r,f) = \max\{|f(z)| : |z - z_0| = r\}$. The $[p,q]$ exponent of convergence of zeros and distinct zeros near z_0 of a meromorphic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$ are respectively defined by:

$$\lambda_{[p,q]}(f,z_0) = \limsup_{r\to 0} \frac{\log_p^+ N_{z_0}\left(r,\frac{1}{f}\right)}{\log_q \frac{1}{r}},$$

$$\bar{\lambda}_{[p,q]}(f,z_0) = \limsup_{r\to 0} \frac{\log_p^+ \overline{N}_{z_0}\left(r,\frac{1}{f}\right)}{\log_q \frac{1}{r}},$$

where $\overline{N}_{z_0}(r,f)$ is defined similarly as $N_{z_0}(r,f)$ but for $\bar{n}(t,f)$ which counts the number of distinct poles of $f(z)$ instead of $n(t,f)$.

Remark 1. (i) $\sigma_{[1,1]}(f,z_0) = \sigma(f,z_0)$, $\tau_{[1,1]}(f,z_0) = \tau(f,z_0)$, $\lambda_{[1,1]}(f,z_0) = \lambda(f,z_0)$ and $\bar{\lambda}_{[1,1]}(f,z_0) = \bar{\lambda}(f,z_0)$ are just the order, the type and the exponent of convergence of zeros and distinct zeros of $f(z)$ respectively, [16].
(ii) $\sigma_{[2,1]}(f,z_0) = \sigma_2(f,z_0)$, $\tau_{[2,1]}(f,z_0) = \tau_2(f,z_0)$, $\lambda_{[2,1]}(f,z_0) = \lambda_2(f,z_0)$ and $\bar{\lambda}_{[2,1]}(f,z_0) = \bar{\lambda}_2(f,z_0)$ are just the hyper order, the hyper type and the hyper exponent of convergence of zeros and distinct zeros of $f(z)$ respectively, [16].

Depending on the definitions of the logarithmic order and the logarithmic type of meromorphic functions in $\mathbb{C}$, [21], [22], we define the logarithmic order and the logarithmic type of a meromorphic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$ as follow:

$$\sigma_{\log}(f,z_0) = \limsup_{r\to 0} \frac{\log^+ T_{z_0}(r,f)}{\log\log \frac{1}{r}},$$

$$\tau_{\log}(f,z_0) = \limsup_{r\to 0} \frac{T_{z_0}(r,f)}{\left(\log\frac{1}{r}\right)^{\sigma_{\log}(f,z_0)}}$$

if $\sigma_{\log}(f,z_0) \in [1,+\infty)$. If $f(z)$ is an analytic function in $\overline{\mathbb{C}} - \{z_0\}$, then:

$$\sigma_{\log}(f,z_0) = \limsup_{r\to 0} \frac{\log^+\log^+ M_{z_0}(r,f)}{\log\log \frac{1}{r}},$$

$$\tau_{\log,M}(f,z_0) = \limsup_{r\to 0} \frac{\log^+ M_{z_0}(r,f)}{\left(\log\frac{1}{r}\right)^{\sigma_{\log}(f,z_0)}}$$

if $\sigma_{\log}(f,z_0) \in [1,+\infty)$. The logarithmic exponent of convergence of zeros and distinct zeros of a meromorphic function $f(z)$ in $\overline{\mathbb{C}} - \{z_0\}$ are respectively given by:

$$\lambda_{\log}(f,z_0) = \limsup_{r\to 0} \frac{\log^+ N_{z_0}\left(r,\frac{1}{f}\right)}{\log\log \frac{1}{r}} - 1,$$

$$\bar{\lambda}_{\log}(f,z_0) = \limsup_{r\to 0} \frac{\log^+ \overline{N}_{z_0}\left(r,\frac{1}{f}\right)}{\log\log \frac{1}{r}} - 1.$$

For $k \geq 2$, we consider the linear differential equations:

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \quad (1)$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \quad (2)$$

where $A_j(z)$ $(j = 0,1,\dots,k-1)$ and $F(z)$ are analytic or meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$. Recently in [20], the authors investigated the growth of solutions of (1) for the case that an arbitrary coefficient $A_s(z)$ dominates by its $[p,q]$-order, and they obtained the following theorem.

Theorem A ([20]). Let $A_0(z),\dots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Suppose there exists an integer $s\,(0 \leq s \leq k-1)$ such that $A_s(z)$ satisfies $\max\{\sigma_{[p,q]}(A_j, z_0) : j \neq s\} < \sigma_{[p,q]}(A_s, z_0) < +\infty$. Then, every analytic solution $f(z)\ (\not\equiv 0)$ in $\overline{\mathbb{C}} - \{z_0\}$ of (1) satisfies $\sigma_{[p+1,q]}(f, z_0) \leq \sigma_{[p,q]}(A_s, z_0) \leq \sigma_{[p,q]}(f, z_0)$.

In [14], the authors also considered (1) for the special case when the coefficients are meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$ and $A_0(z)$ is the dominant coefficient, where they obtained the following theorem on the hyper order.

Theorem B ([14]). Let $A_0(z), \ldots, A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$ satisfying $\max\{\sigma(A_j, z_0) : j \neq 0\} < \sigma(A_0, z_0)$ with

$$\liminf_{r \to 0} \frac{m_{z_0}(r, f)}{T_{z_0}(r, f)} > 0.$$

Then, every meromorphic solution $f(z)\ (\not\equiv 0)$ in $\overline{\mathbb{C}} - \{z_0\}$ of (1) satisfies $\sigma(A_0, z_0) \leq \sigma_2(f, z_0)$.

The aim of the present paper is to investigate the growth of solutions of the linear differential equations (1) and (2) considering the case that an arbitrary coefficient $A_s(z)$ dominates the other coefficients which are analytic or meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$, by its logarithmic order or its logarithmic type, where we extend the above results. It should be noted that similar results to ours were obtained for the complex plane case, [24], [25]. First, for the case when the coefficients of (1) are meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$, we obtain the following results.

Theorem 1. Let $A_0(z), \ldots, A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$ of finite logarithmic order. Suppose there exists an integer $s\,(0 \leq s \leq k-1)$ such that $A_s(z)$ satisfies

$$\limsup_{r \to 0} \frac{\sum_{j \neq s} m_{z_0}(r, A_j)}{m_{z_0}(r, A_s)} < 1$$

and

$$\liminf_{r \to 0} \frac{m_{z_0}(r, A_s)}{T_{z_0}(r, A_s)} = \delta > 0.$$

Then, every meromorphic solution $f(z)\ (\not\equiv 0)$ in $\overline{\mathbb{C}} - \{z_0\}$ of (1) satisfies $\sigma_{\log}(A_s, z_0) - 1 \leq \sigma_{\log}(f, z_0)$ and $\sigma_{\log}(A_s, z_0) \leq \sigma_{\log}(f, z_0)$ if $\sigma_{\log}(A_s, z_0) > 1$.

Theorem 2. Let $A_0(z), \ldots, A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$ of finite logarithmic order. Suppose there exists an integer $s\,(0 \leq s \leq k-1)$ such that $A_s(z)$ satisfies $\max\{\sigma_{\log}(A_j, z_0) : j \neq s\} \leq \sigma_{\log}(A_s, z_0) < +\infty$,

$$\liminf_{r \to 0} \frac{m_{z_0}(r, A_s)}{T_{z_0}(r, A_s)} = \delta > 0$$

and

$$\sum_{\sigma_{\log}(A_j, z_0) = \sigma_{\log}(A_s, z_0) \geq 1, j \neq s} \tau_{\log}(A_j, z_0) < \delta \tau_{\log}(A_s, z_0) < +\infty.$$

Then, every meromorphic solution $f(z)\ (\not\equiv 0)$ in $\overline{\mathbb{C}} - \{z_0\}$ of (1) satisfies $\sigma_{\log}(A_s, z_0) - 1 \leq \sigma_{\log}(f, z_0)$ and $\sigma_{\log}(A_s, z_0) \leq \sigma_{\log}(f, z_0)$ if $\sigma_{\log}(A_s, z_0) > 1$.

Theorem 3. Let $A_0(z), \ldots, A_{k-1}(z)$ be meromorphic functions in $\overline{\mathbb{C}} - \{z_0\}$ of finite logarithmic order. Suppose there exists an integer $s\,(0 \leq s \leq k-1)$ such that $A_s(z)$ satisfies

$$\lambda_{\log}\left(\frac{1}{A_s}, z_0\right) + 1 < \sigma_{\log}(A_s, z_0),$$

$$\max\{\sigma_{\log}(A_j, z_0) : j \neq s\} \leq \sigma_{\log}(A_s, z_0) < +\infty$$

and

$$\sum_{\sigma_{\log}(A_j, z_0) = \sigma_{\log}(A_s, z_0) \geq 1, j \neq s} \tau_{\log}(A_j, z_0) < \tau_{\log}(A_s, z_0) < +\infty.$$

Then, every meromorphic solution $f(z)\ (\not\equiv 0)$ in $\overline{\mathbb{C}} - \{z_0\}$ of (1) satisfies $\sigma_{\log}(A_s, z_0) \leq \sigma_{\log}(f, z_0)$.

Next, when the coefficients of (1) and (2) are analytic functions in $\overline{\mathbb{C}} - \{z_0\}$, we obtain the following results.

Theorem 4. Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ of finite logarithmic order. Suppose there exists an integer $s\,(0 \leq s \leq k-1)$ such that $A_s(z)$ satisfies $\max\{\sigma_{\log}(A_j, z_0) : j \neq s\} \leq \sigma_{\log}(A_s, z_0) < +\infty$ and

$$\limsup_{r \to 0} \frac{\sum_{j \neq s} m_{z_0}(r, A_j)}{m_{z_0}(r, A_s)} < 1.$$

Then, every analytic solution $f(z)\ (\not\equiv 0)$ in $\overline{\mathbb{C}} - \{z_0\}$ of (1) satisfies $\sigma_{[2,2]}(f, z_0) - 1 \leq \sigma_{\log}(A_s, z_0) - 1 \leq \sigma_{\log}(f, z_0)$. Furthermore, if $\sigma_{\log}(A_s, z_0) > 1$, then $f(z)$ satisfies $\sigma_{[2,2]}(f, z_0) \leq \sigma_{\log}(A_s, z_0) \leq \sigma_{\log}(f, z_0)$.

Theorem 5. Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ of finite logarithmic order. Suppose there exists an integer s $(0\le s\le k-1)$ such that $A_s(z)$ satisfies $\max\{\sigma_{\log}(A_j,z_0): j\neq s\}<\sigma_{\log}(A_s,z_0)<+\infty$. Then, every analytic solution $f(z)$ $(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (1) satisfies $\sigma_{[2,2]}(f,z_0)-1\le\sigma_{\log}(A_s,z_0)-1\le\sigma_{\log}(f,z_0)$. Furthermore, if $\sigma_{\log}(A_s,z_0)>1$, then $f(z)$ satisfies $\sigma_{[2,2]}(f,z_0)\le\sigma_{\log}(A_s,z_0)\le\sigma_{\log}(f,z_0)$.

Theorem 6. Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ of finite logarithmic order. Suppose there exists an integer s $(0\le s\le k-1)$ such that $A_s(z)$ satisfies $\max\{\sigma_{\log}(A_j,z_0): j\neq s\}\le\sigma_{\log}(A_s,z_0)<+\infty$ and

$$\sum_{\sigma_{\log}(A_j,z_0)=\sigma_{\log}(A_s,z_0),j\neq s}\tau_{\log}(A_j,z_0)<\tau_{\log}(A_s,z_0)<+\infty.$$

Then, every analytic solution $f(z)$ $(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (1) satisfies $\sigma_{[2,2]}(f,z_0)-1\le\sigma_{\log}(A_s,z_0)-1\le\sigma_{\log}(f,z_0)$. Furthermore, if $\sigma_{\log}(A_s,z_0)>1$, then $f(z)$ satisfies $\sigma_{[2,2]}(f,z_0)\le\sigma_{\log}(A_s,z_0)\le\sigma_{\log}(f,z_0)$.

Theorem 7. Let $A_0(z),\ldots,A_{k-1}(z)$ satisfy the hypotheses of Theorem 5 and let $F(z)$ $(\not\equiv 0)$ be an analytic function in $\overline{\mathbb{C}}-\{z_0\}$.

i) If $\sigma_{\log}(A_s,z_0)\le\sigma_{[2,2]}(F,z_0)<+\infty$, then every analytic solution $f(z)$ $(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (2) satisfies $\sigma_{[2,2]}(f,z_0)=\sigma_{[2,2]}(F,z_0)$.
ii) If $\sigma_{\log}(A_s,z_0)>\sigma_{[2,2]}(F,z_0)$, then every analytic solution $f(z)$ $(\not\equiv 0)$ in $\overline{\mathbb{C}}-\{z_0\}$ of (2) satisfies $\sigma_{[2,2]}(f,z_0)\le\sigma_{\log}(A_s,z_0)$ and $\overline{\lambda}_{[2,2]}(f,z_0)=\lambda_{[2,2]}(f,z_0)=\sigma_{[2,2]}(f,z_0)$ holds for every solution that satisfies $\sigma_{[2,2]}(f,z_0)=\sigma_{\log}(A_s,z_0)$.

Remark 2. Nevanlinna theory has a wide range of applications starting from number theory to probability and statistics and to theoretical physics, [26], [27], [28], [29] and the references cited therein.

2 Some Lemmas

The following lemmas are important to prove our results. Firstly, we denote the logarithmic measure of a set $E\subset(0,1)$ by $m_l(E)=\int_E\frac{dt}{t}$.

Lemma 1 ([20]). Let f be non-constant meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ and let $k,j\in\mathbb{N}$, such that $k\neq j$. Then:

$$m_{z_0}\left(r,\frac{f^{(k)}(z)}{f^{(j)}(z)}\right)=O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right),$$

holds for all $r\in(0,r_1]\setminus E_1$ with $m_l(E_1)<\infty$.

Lemma 2 ([30]). Let f be non-constant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\sigma_{\log}(f,z_0)=\sigma$. Then there exists a set E_2 of $(0,1)$ that has infinite logarithmic measure such that for all $|z-z_0|=r\in E_2$, we have:

$$\lim_{r\to 0}\frac{\log\log M_{z_0}(r,f)}{\log\log\frac{1}{r}}=\lim_{r\to 0}\frac{\log T_{z_0}(r,f)}{\log\log\frac{1}{r}}=\sigma$$

and for any given $\varepsilon>0$

$$M_{z_0}(r,f)>\exp\left\{\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}\right\},$$
$$T_{z_0}(r,f)>\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}.$$

Lemma 3. Let f_1,f_2 be two meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$ satisfying $\sigma_1=\sigma_{\log}(f_1,z_0)>\sigma_{\log}(f_2,z_0)=\sigma_2$. Then there exists a set $E_3\subset(0,1)$ of infinite logarithmic measure such that for all $|z-z_0|=r\in E_3$, we have:

$$\lim_{r\to 0}\frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)}=0.$$

Proof. By the definition of the logarithmic order, for any given $0<\varepsilon<\frac{\sigma_1-\sigma_2}{2}$, the exists $r_2\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_2)$, we obtain:

$$T_{z_0}(r,f_2)\le\left(\log\frac{1}{r}\right)^{\sigma_2+\varepsilon}. \qquad (3)$$

By Lemma 2, there exists a set $E_2\subset(0,1)$ of infinite logarithmic measure such that, for the above ε and for all $|z-z_0|=r\in E_3$, we have:

$$T_{z_0}(r,f_1)\le\left(\log\frac{1}{r}\right)^{\sigma_1-\varepsilon}. \qquad (4)$$

By (3) and (4), for the above ε and for all $|z-z_0|=r\in E_3=E_2\cap(0,r_2)$, we get:

$$0 \leq \frac{T_{z_0}(r, f_2)}{T_{z_0}(r, f_1)} \leq \frac{\left(\log \frac{1}{r}\right)^{\sigma_2 + \varepsilon}}{\left(\log \frac{1}{r}\right)^{\sigma_1 - \varepsilon}} = \frac{1}{\left(\log \frac{1}{r}\right)^{\sigma_1 - \sigma_2 - 2\varepsilon}} \longrightarrow 0, \quad \text{as } r \longrightarrow 0.$$

Lemma 4. Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with finite logarithmic order $1 \leq \sigma_{\log}(f, z_0) = \sigma < +\infty$ and finite logarithmic type $0 < \tau_{\log}(f, z_0) < +\infty$. Then there exists a set E_4 of $(0,1)$ that has infinite logarithmic measure such that for all $|z - z_0| = r \in E_4$, we have:

$$\lim_{r \longrightarrow 0} \frac{T_{z_0}(r, f)}{\left(\log \frac{1}{r}\right)^{\sigma}} = \tau_{\log}(f, z_0).$$

Proof. By the definition of the logarithmic type, there exists a sequence $\{r_n\}_{n=1}^{+\infty}$ tending to 0 satisfying $r_{n+1} < \frac{n}{n+1} r_n$ and

$$\lim_{n \longrightarrow +\infty} \frac{T_{z_0}(r_n, f)}{\left(\log \frac{1}{r_n}\right)^{\sigma}} = \tau_{\log}(f, z_0).$$

So, for any given $\varepsilon > 0$, there exists an integer n_0 such that for all $n \geq n_0$ and for any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we have:

$$\frac{T_{z_0}(r_n, f)}{\left(\log \frac{1}{\frac{n}{n+1} r_n}\right)^{\sigma}} \leq \frac{T_{z_0}(r, f)}{\left(\log \frac{1}{r}\right)^{\sigma}} \leq \frac{T_{z_0}\left(\frac{n}{n+1} r_n, f\right)}{\left(\log \frac{1}{r_n}\right)^{\sigma}}.$$

Since,

$$\lim_{n \longrightarrow +\infty} \frac{T_{z_0}(r_n, f)}{\left(\log \frac{1}{\frac{n}{n+1} r_n}\right)^{\sigma}} = \lim_{n \longrightarrow +\infty} \frac{T_{z_0}\left(\frac{n}{n+1} r_n, f\right)}{\left(\log \frac{1}{r_n}\right)^{\sigma}} = \tau_{\log}(f, z_0),$$

then for any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we get:

$$\lim_{r \longrightarrow 0} \frac{T_{z_0}(r, f)}{\left(\log \frac{1}{r}\right)^{\sigma}} = \tau_{\log}(f, z_0).$$

Set $E_4 = \bigcup_{n=n_0}^{+\infty} \left[\frac{n}{n+1} r_n, r_n\right]$, then $m_l(E_4) = \sum_{n=n_0}^{+\infty} \int_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_0}^{+\infty} \log\left(1 + \frac{1}{n}\right) = +\infty$.

Lemma 5 ([30]). Let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ of finite logarithmic order with $\max\{\sigma_{\log}(A_j, z_0) : j = 0, \dots, k-1\} \leq \alpha < +\infty$. Then, every analytic solution $f(z)$ ($\not\equiv 0$) in $\overline{\mathbb{C}} - \{z_0\}$ of (1) satisfies $\sigma_{[2,2]}(f, z_0) \leq \alpha$.

Lemma 6 ([16]). Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ and set $g(\omega) = f\left(z_0 - \frac{1}{\omega}\right)$. Then $g(\omega)$ is meromorphic in $\mathbb{C}$ and we have

$$T(R, f) = T_{z_0}\left(\frac{1}{R}, f\right).$$

Lemma 7 ([31]). Let f be a non-constant meromorphic function in $\mathbb{C}$ with $p \geq q \geq 1$. Then

$$\sigma_{[p,q]}(f') = \sigma_{[p,q]}(f).$$

Lemma 8. Let f be a non-constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with $p \geq q \geq 1$. Then

$$\sigma_{[p,q]}(f, z_0) = \sigma_{[p,q]}(f', z_0).$$

Proof. By Lemma 6, $g(\omega) = f\left(z_0 - \frac{1}{\omega}\right)$ is meromorphic in $\mathbb{C}$ and $\sigma_{[p,q]}(g) = \sigma_{[p,q]}(f, z_0)$. From Lemma 7 we have $\sigma_{[p,q]}(g') = \sigma_{[p,q]}(g)$, where $f'(z) = \frac{1}{\omega^2} g'(\omega)$. Setting $h(\omega) = \frac{1}{\omega^2} g'(\omega)$. It is clear that $\sigma_{[p,q]}(h) = \sigma_{[p,q]}(g')$. On the other hand by Lemma 6, we have $\sigma_{[p,q]}(h) = \sigma_{[p,q]}(f', z_0)$. Hence, $\sigma_{[p,q]}(f, z_0) = \sigma_{[p,q]}(f', z_0)$.

Lemma 9 ([30]). Let $F(z) \not\equiv 0$, $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ and let f be a non-constant analytic solution in $\overline{\mathbb{C}} - \{z_0\}$ of (2) satisfying $\max\left\{\sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(A_j, z_0) : (j = 0, \dots, k-1)\right\} < \sigma_{[2,2]}(f, z_0)$. Then, $\bar{\lambda}_{[2,2]}(f, z_0) = \lambda_{[2,2]}(f, z_0) = \sigma_{[2,2]}(f, z_0)$.

3 Proof of the Theorems

3.1 Proof of Theorem 1

Proof. Let f ($\not\equiv 0$) be a meromorphic solution of (1) in $\overline{\mathbb{C}} - \{z_0\}$. If $\sigma_{\log}(f, z_0) = \infty$, then the result is trivial. So, we suppose that $\sigma_{\log}(f, z_0) < \infty$. By (1), we have:

$$-A_s(z) = \frac{f^{(k)}(z)}{f^{(s)}(z)} + A_{k-1}(z) \frac{f^{(k-1)}(z)}{f^{(s)}(z)} + \cdots + A_{s+1}(z) \frac{f^{(s+1)}(z)}{f^{(s)}(z)} + A_{s-1}(z) \frac{f^{(s-1)}(z)}{f^{(s)}(z)}$$

$$+\cdots+A_0(z)\frac{f(z)}{f^{(s)}(z)}. \quad (5)$$

It follows that:

$$m_{z_0}\big(r,A_s(z)\big)\leq\sum_{j=0,j\neq s}^{k}m_{z_0}\left(r,\frac{f^{(j)}(z)}{f^{(s)}(z)}\right)+\sum_{j=0,j\neq s}^{k-1}m_{z_0}\left(r,A_j(z)\right)+O(1). \quad (6)$$

By Lemma 1, for a constant $r_1\in(0,1)$, there exists a set $E_1\subset(0,r_1]$ of finite logarithmic measure such that for all $|z-z_0|=r\in(0,r_1]\setminus E_1$, we have:

$$\sum_{j=0,j\neq s}^{k}m_{z_0}\left(r,\frac{f^{(j)}(z)}{f^{(s)}(z)}\right)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \quad (7)$$

Suppose that:

$$\limsup_{r\to 0}\frac{\sum_{j=0,j\neq s}^{k-1}m_{z_0}\big(r,A_j\big)}{m_{z_0}(r,A_s)}=\alpha<\beta<1.$$

Then for $r\to 0$, we get:

$$\sum_{j=0,j\neq s}^{k-1}m_{z_0}\left(r,A_j(z)\right)<\beta m_{z_0}(r,A_s). \quad (8)$$

Substituting (7) and (8) into (6), for all $|z-z_0|=r\in(0,r_1]\setminus E_1$ and $r\longrightarrow 0$, we obtain:

$$(1-\beta)m_{z_0}(r,A_s)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \quad (9)$$

By the assumption $\liminf_{r\longrightarrow 0}\frac{m_{z_0}(r,A_s)}{T_{z_0}(r,A_s)}=\delta>0$, there exists $r_3\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_3)$, we have:

$$m_{z_0}(r,A_s)\geq\frac{\delta}{2}\,T_{z_0}(r,A_s). \quad (10)$$

By Lemma 2, there exists a set $E_2\subset(0,1)$ of infinite logarithmic measure such that for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E_2$, we have:

$$T_{z_0}(r,A_s)\geq\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)-\varepsilon}. \quad (11)$$

Combining (9), (10) and (11), for any given $\varepsilon>0$ and for all $|z-z_0|=r\in E_2\cap(0,r_1]\cap(0,r_3)\setminus E_1$, we get:

$$\frac{\delta}{2}(1-\beta)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)-\varepsilon}\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \quad (12)$$

This implies that $\sigma_{\log}(A_s,z_0)-1-\varepsilon\leq\sigma_{\log}(f,z_0)$ and $\sigma_{\log}(A_s,z_0)-\varepsilon\leq\sigma_{\log}(f,z_0)$ if $\sigma_{\log}(A_s,z_0)>1$. Since $\varepsilon>0$ is arbitrary, we obtain $\sigma_{\log}(A_s,z_0)-1\leq\sigma_{\log}(f,z_0)$ and $\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$ if $\sigma_{\log}(A_s,z_0)>1$.

3.2 Proof of Theorem 2

Proof. Let $f(\not\equiv 0)$ be a meromorphic solution of (1) in $\overline{\mathbb{C}}-\{z_0\}$. First, we suppose that $\max\{\sigma_{\log}\big(A_j,z_0\big):j\neq s\}<\sigma_{\log}(A_s,z_0)=\sigma$. Then as in the proof of Theorem 1, by substituting (7) and (10) into (6), for all $|z-z_0|=r\in(0,r_1]\cap(0,r_3)\setminus E_1$, we obtain:

$$\frac{\delta}{2}T_{z_0}(r,A_s)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j=0,j\neq s}^{k-1}T_{z_0}\big(r,A_j\big). \quad (13)$$

By Lemma 3, there exists a set $E_3\subset(0,1)$ of infinite logarithmic measure such that for all $|z-z_0|=r\in E_3$, we have:

$$\max\left\{\frac{T_{z_0}\big(r,A_j\big)}{T_{z_0}(r,A_s)},j\neq s\right\}\to 0,\ \text{ as }\ r\longrightarrow 0. \quad (14)$$

Then, by (13) and (14) for all $r\in E_3\cap(0,r_1]\cap(0,r_3)\setminus E_1$ and $r\longrightarrow 0$, we get:

$$\left(\frac{\delta}{2}-o(1)\right)T_{z_0}(r,A_s)\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right). \quad (15)$$

From (15), we deduce that $\sigma_{\log}(A_s,z_0)-1\leq\sigma_{\log}(f,z_0)$ and $\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$ if $\sigma_{\log}(A_s,z_0)>1$. Now we suppose that $\max\{\sigma_{\log}\big(A_j,z_0\big):j\neq s\}=\sigma_{\log}(A_s,z_0)=\sigma$ and

$$\tau_1=\sum_{\sigma_{\log}(A_j,z_0)=\sigma_{\log}(A_s,z_0)\geq 1,j\neq s}\tau_{\log}\big(A_j,z_0\big)<\delta\tau_{\log}(A_s,z_0)=\delta\tau.$$

So, there exists a set $J_1\subseteq\{0,1,\ldots,k-1\}\setminus\{s\}$ such that for $j\in J_1$, we have $\sigma_{\log}\big(A_j,z_0\big)=\sigma_{\log}(A_s,z_0)=\sigma$ with:

$$\tau_1=\sum_{j\in J_1}\tau_{\log}\big(A_j,z_0\big)<\tau_{\log}(A_s,z_0)=\tau$$

and for $j\in J_2=\{0,1,\ldots,s-1,s+1,\ldots,k-1\}\setminus J_1$ we have $\sigma_{\log}\big(A_j,z_0\big)<\sigma_{\log}(A_s,z_0)=\sigma$. Then there exists $r_4\in(0,1)$, such that all $|z-z_0|=r\in(0,r_4)$

and for any given $\varepsilon(0<(\tau+k)\varepsilon<\delta\tau-\tau_1)$, we obtain:

$$T_{z_0}(r,A_j)\leq(\tau_{\log}(A_j,z_0)+\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_j,z_0)}$$
$$=(\tau_{\log}(A_j,z_0)+\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)},\quad j\in J_1 \quad (16)$$

and

$$T_{z_0}(r,A_j)\leq\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_j,z_0)+\varepsilon}$$
$$\leq\left(\log\frac{1}{r}\right)^{\sigma_0},\quad j\in J_2, \quad (17)$$

where $\max\{\sigma_{\log}(A_j,z_0): j\in J_2\}<\sigma_0<\sigma$. By the assumption $\liminf_{r\to 0}\frac{m_{z_0}(r,A_s)}{T_{z_0}(r,A_s)}=\delta>0$, there exists $r_5\in(0,1)$ such that for any given $\varepsilon>0$ and for all $|z-z_0|=r\in(0,r_5)$, we have:

$$m_{z_0}(r,A_s)\geq(\delta-\varepsilon)T_{z_0}(r,A_s). \quad (18)$$

By Lemma 4, there exists a set $E_4\subset(0,1)$ of infinite logarithmic measure such that for the above ε and for all $|z-z_0|=r\in E_4$, we have:

$$T_{z_0}(r,A_s)\geq(\tau-\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)}. \quad (19)$$

Combining (18) and (19), for the above ε and for all $|z-z_0|=r\in E_4\cap(0,r_5)$, we get:

$$m_{z_0}(r,A_s)\geq(\delta-\varepsilon)(\tau-\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)}$$
$$=(\delta\tau-\delta\varepsilon-\tau\varepsilon+\varepsilon^2)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)}$$
$$\geq(\delta\tau-(\tau+\delta)\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)}. \quad (20)$$

Knowing the fact that $0<\delta\leq 1$, by (20) it follows:

$$m_{z_0}(r,A_s)\geq(\delta\tau-(\tau+1)\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)}. \quad (21)$$

By substituting (7) and (16), (17) and (21) into (6), for the above ε and for all $|z-z_0|=r\in E_4\cap(0,r_1]\cap(0,r_4)\cap(0,r_5)\setminus E_1$, we obtain:

$$(\delta\tau-(\tau+1)\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_s,z_0)}$$
$$\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j=0,j\neq s}^{k-1}T_{z_0}(r,A_j)$$
$$\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j\in J_1}(\tau_{\log}(A_j,z_0)+\varepsilon)$$
$$\times\left(\log\frac{1}{r}\right)^{\sigma}+\sum_{j\in J_2}\left(\log\frac{1}{r}\right)^{\sigma_0}$$
$$\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+(\tau_1+(k-1)\varepsilon)$$
$$\times\left(\log\frac{1}{r}\right)^{\sigma}+(k-1)\left(\log\frac{1}{r}\right)^{\sigma_0}. \quad (22)$$

It follows that

$$(1-o(1))(\delta\tau-\tau_1-(\tau+k)\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma}$$
$$\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right), \quad (23)$$

which implies that, $\sigma_{\log}(A_s,z_0)-1\leq\sigma_{\log}(f,z_0)$ and $1<\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$ if $\sigma_{\log}(A_s,z_0)>1$.

3.3 Proof of Theorem 3

Proof. By (6) and (7), for all $r\in(0,r_1]\setminus E_1$, we have

$$T_{z_0}(r,A_s)=m_{z_0}(r,A_s)+N_{z_0}(r,A_s)$$
$$\leq\sum_{j=0,j\neq s}^{k}m_{z_0}\left(r,\frac{f^{(j)}(z)}{f^{(s)}(z)}\right)+\sum_{j=0,j\neq s}^{k-1}m_{z_0}(r,A_j(z))$$
$$+N_{z_0}(r,A_s)+O(1)$$
$$\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+\sum_{j=0,j\neq s}^{k-1}T_{z_0}(r,A_j)$$
$$+N_{z_0}(r,A_s). \quad (24)$$

If $\sigma_1=\max\{\sigma_{\log}(A_j,z_0): j\neq s\}<\sigma_{\log}(A_s,z_0)=\sigma$, then there exists $r_6\in(0,1)$ such that for any given $\varepsilon(0<2\varepsilon<\sigma-\sigma_1)$ and for all $|z-z_0|=r\in(0,r_6)$, we obtain:

$$T_{z_0}(r,A_j)\leq\left(\log\frac{1}{r}\right)^{\sigma_{\log}(A_j,z_0)+\varepsilon}$$
$$\leq\left(\log\frac{1}{r}\right)^{\sigma_1+\varepsilon},j=0,1,\ldots,k-1,j\neq s. \quad (25)$$

By Lemma 2, there exists a set $E_2\subset(0,1)$ of infinite logarithmic measure such that for the above ε and for all $|z-z_0|=r\in E_2$, the assumption (11) holds. By the definition of $\lambda_{\log}\left(\frac{1}{A_s},z_0\right)=\lambda$, there exists $r_7\in(0,1)$ such that for any given $\varepsilon(0<2\varepsilon<\sigma-\lambda-1)$ and for all $|z-z_0|=r\in(0,r_7)$, we get:

$$N_{z_0}(r,A_s)\leq\left(\log\frac{1}{r}\right)^{\lambda_{\log}\left(\frac{1}{A_s},z_0\right)+1+\varepsilon}. \quad (26)$$

By substituting (11), (25) and (26) into (24), for sufficiently small ε satisfying $0<2\varepsilon<\min\{\sigma-\sigma_1,\sigma-\lambda-1\}$ and for all $r\in E_2\cap(0,r_1]\cap(0,r_6)\cap(0,r_7)\setminus E_1$, we have:

$$\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)$$
$$+(k-1)\left(\log\frac{1}{r}\right)^{\sigma_1+\varepsilon}+\left(\log\frac{1}{r}\right)^{\lambda+1+\varepsilon}, \qquad (27)$$

then

$$\left(1-o(1)\right)\left(\log\frac{1}{r}\right)^{\sigma-\varepsilon}$$
$$\leq O\left(T_{z_0}\left((r,f)\right)+\log\frac{1}{r}\right). \qquad (28)$$

Thus $1<\sigma-\varepsilon\leq\sigma_{\log}(f,z_0)$. Since $\varepsilon>0$ is arbitrary, we obtain $1<\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$. Now, if $\max\{\sigma_{\log}(A_j,z_0):j\neq s\}=\sigma_{\log}(A_s,z_0)=\sigma$ and

$$\tau_1=\sum_{\sigma_{\log}(A_j,z_0)=\sigma_{\log}(A_s,z_0)\geq 1,j\neq s}\tau_{\log}(A_j,z_0)$$
$$<\tau_{\log}(A_s,z_0)=\tau,$$

then as in the proof of Theorem 2, we assume that there exists a set $J_1\subseteq\{0,1,\ldots,k-1\}\setminus\{s\}$ such that for $j\in J_1$, we have $\sigma_{\log}(A_j,z_0)=\sigma_{\log}(A_s,z_0)=\sigma$ with:

$$\tau_1=\sum_{\sigma_{\log}(A_j,z_0)=\sigma_{\log}(A_s,z_0)\geq 1,j\neq s}\tau_{\log}(A_j,z_0)$$
$$<\tau_{\log}(A_s,z_0)=\tau$$

and for $j\in J_2=\{0,1,\ldots,s-1,s+1,\ldots,k-1\}\setminus J_1$ we have $\sigma_{\log}(A_j,z_0)<\sigma_{\log}(A_s,z_0)=\sigma$. Then, there exists a $r_4\in(0,1)$, such that for any given $\varepsilon\left(0<\varepsilon<\frac{\tau-\tau_1}{k}\right)$ and for all $|z-z_0|=r\in(0,r_4)$, the assumptions (16) and (17) hold. By Lemma 4, there exists a set $E_4\subset(0,1)$ of infinite logarithmic measure such that for the above ε and for all $|z-z_0|=r\in E_4$, (19) holds. By substituting (16), (17), (19) and (26) into (24) for sufficiently small ε satisfying $0<\varepsilon<\min\left\{\frac{\sigma-\lambda-1}{2},\frac{\tau-\tau_1}{k}\right\}$ and for all $r\in E_4\cap(0,r_1]\cap(0,r_4)\cap(0.r_7)\setminus E_1$, we get:

$$(\tau-\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma}\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)$$
$$+\sum_{j\in J_1}(\tau_{\log}(A_j,z_0)+\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma}$$
$$+\sum_{j\in J_2}\left(\log\frac{1}{r}\right)^{\sigma_0}+\left(\log\frac{1}{r}\right)^{\lambda+1+\varepsilon}$$
$$\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)+(\tau_1+(k-1)\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma}$$
$$+(k-1)\left(\log\frac{1}{r}\right)^{\sigma_0}+\left(\log\frac{1}{r}\right)^{\lambda+1+\varepsilon}. \qquad (29)$$

So

$$\left(1-o(1)\right)(\tau-\tau_1-k\varepsilon)\left(\log\frac{1}{r}\right)^{\sigma}$$
$$\leq O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right), \qquad (30)$$

which implies that $1<\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$.

3.4 Proof of Theorem 4

Proof. We assume that $f(\not\equiv 0)$ is an analytic solution of (1) in $\overline{\mathbb{C}}-\{z_0\}$. By Theorem 1, we have $0\leq\sigma_{\log}(A_s,z_0)-1\leq\sigma_{\log}(f,z_0)$ and $\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$ if $\sigma_{\log}(A_s,z_0)>1$. On the other hand, by Lemma 5, we have $\sigma_{[2,2]}(f,z_0)\leq\sigma_{\log}(A_s,z_0)$. Hence $\sigma_{[2,2]}(f,z_0)-1\leq\sigma_{\log}(A_s,z_0)-1\leq\sigma_{\log}(f,z_0)$ and $\sigma_{[2,2]}(f,z_0)\leq\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$ if $\sigma_{\log}(A_s,z_0)>1$.

3.5 Proof of Theorem 5

Proof. We assume that $f(\not\equiv 0)$ is an analytic solution of (1) in $\overline{\mathbb{C}}-\{z_0\}$. By Theorem 2, we get $0\leq\sigma_{\log}(A_s,z_0)-1\leq\sigma_{\log}(f,z_0)$ and $\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$ if $\sigma_{\log}(A_s,z_0)>1$. Then, by using Lemma 5, we conclude that $\sigma_{[2,2]}(f,z_0)-1\leq\sigma_{\log}(A_s,z_0)-1\leq\sigma_{\log}(f,z_0)$ and $\sigma_{[2,2]}(f,z_0)\leq\sigma_{\log}(A_s,z_0)\leq\sigma_{\log}(f,z_0)$ if $\sigma_{\log}(A_s,z_0)>1$.

3.6 Proof of Theorem 6

Proof. Again, by Theorem 2 and Lemma 5, we get the assertions of Theorem 6.

3.7 Proof of Theorem 7

Proof. We suppose $f(z)$ is an analytic solution in $\overline{\mathbb{C}}-\{z_0\}$ of (2). Then f can be represented in the form:

$$f(z)=B_1(z)f_1(z)+B_2(z)f_2(z)$$
$$+\cdots+B_k(z)f_k(z), \qquad (31)$$

where $f_1,f_2,\ldots,f_k$ is a solution base of equation (1) corresponding to equation (2) and $B_1,B_2,\ldots,B_k$ are suitable analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ determined by the following system of equations:

$$\begin{cases}B_1'f_1+B_2'f_2+\cdots+B_k'f_k=0\\ B_1'f_1'+B_2'f_2'+\cdots+B_k'f_k'=0\\ \vdots\\ B_1'f_1^{k-1}+B_2'f_2^{k-1}+\cdots+B_k'f_k^{k-1}=F,\end{cases} \qquad (32)$$

By (32), we get

$$B_j' = F.G_j(f_1, f_2, \dots, f_k).W(f_1, f_2, \dots, f_k)^{-1}, \quad (33)$$

$j = 1, \dots, k$, where $G_j(f_1, f_2, \dots, f_k)$ is differential polynomial of $f_1, f_2, \dots, f_k$ and their derivatives with constant coefficients and $W(f_1, f_2, \dots, f_k)$ is the Wronksian of $f_1, f_2, \dots, f_k$. By (33) and Lemma 8, for $j = 1, \dots, k$, we have :

$$\sigma_{[2,2]}(B_j, z_0) = \sigma_{[2,2]}(B_j', z_0)$$
$$\leq \max\{\sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(G_j(f_1, f_2, \dots, f_k), z_0), \sigma_{[2,2]}(W(f_1, f_2, \dots, f_k), z_0)\}. \quad (34)$$

Since $G_j(f_1, f_2, \dots, f_k)$ and $W(f_1, f_2, \dots, f_k)$ are both differential polynomial of $f_1, f_2, \dots, f_k$ and their derivatives with constant coefficients, then they satisfy:

$$\max\{\sigma_{[2,2]}(G_j(f_1, f_2, \dots, f_k), z_0), \sigma_{[2,2]}(W(f_1, f_2, \dots, f_k), z_0)\} \leq \sigma_{[2,2]}(f_j, z_0). \quad (35)$$

By Theorem 5, if $\max\{\sigma_{\log}(A_j, z_0) : j \neq s\} < \sigma_{\log}(A_s, z_0) < +\infty$, then

$$\sigma_{[2,2]}(f_j, z_0) \leq \sigma_{\log}(A_s, z_0), \ j = 1, \dots, k. \quad (36)$$

By (31), (34), (35) and (36) for $j = 1, \dots, k$, we get

$$\sigma_{[2,2]}(f, z_0) \leq \max\{\sigma_{[2,2]}(f_j, z_0), \sigma_{[2,2]}(B_j, z_0)\}$$
$$\leq \max\{\sigma_{[2,2]}(F, z_0), \sigma_{\log}(A_s, z_0)\}. \quad (37)$$

i) If $\sigma_{[2,2]}(F, z_0) \geq \sigma_{\log}(A_s, z_0)$, then by (2) and (37), we deduce that $\sigma_{[2,2]}(f, z_0) = \sigma_{[2,2]}(F, z_0)$.

ii) If $\sigma_{[2,2]}(F, z_0) < \sigma_{\log}(A_s, z_0)$, then by (37), we obtain $\sigma_{[2,2]}(f, z_0) \leq \sigma_{\log}(A_s, z_0)$. Further, assume that a solution f of (2) satisfies $\sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_s, z_0)$. Then, there holds

$$\max\{\sigma_{[2,2]}(F, z_0), \sigma_{[2,2]}(A_j, z_0) : (j = 0, \dots k-1)\} < \sigma_{[2,2]}(f, z_0).$$

By Lemma 9, we conclude that $\bar{\lambda}_{[2,2]}(f, z_0) = \lambda_{[2,2]}(f, z_0) = \sigma_{[2,2]}(f, z_0) = \sigma_{\log}(A_s, z_0)$.

4 Conclusion

Throughout this article, by using the concepts of logarithmic order and logarithmic type, we have studied the growth of solutions of the linear differential equations (1) and (2) considering the case of an arbitrary coefficient $A_s(z)$ dominating the others coefficients which are analytic or meromorphic functions in $\bar{\mathbb{C}} - \{z_0\}$. We improve and extend some precedent results obtained in the papers, [14] and [20].

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