New Modified Proximal Point Algorithm for Solving Minimization and Common Fixed Point Problem over CAT(κ) Spaces

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Abstract: -In this paper, we present a newly proximal point algorithm for solving minimization and common fixed point problems in CAT(1) spaces. Under some mild conditions, we prove strong and Δ-convergence theorems. Additionally, a convex minimization application and a common fixed point problems in CAT(κ) spaces with the bounded $\kappa \in (0, \infty)$ are provided. Our findings complement and advance the pertinent recent findings in the literature.

Key-Words: Geodesic metric space, Convex function, Iteration process; Fixed point problem; Proximal point algorithm; Minimization problem

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1 Introduction

Let (X, d) be a geodesic metric space, K be a subset of X, $K \neq \emptyset$ and the self mapping T on K be a nonlinear. The set $F(T) := \{x : Tx = x\}$ is called the set of all fixed points of T. Among the significant analytical issues are ones that relate to fixed points for certain nonlinear mappings. Now, our attention is on nonlinear problems such convex minimization problems and common fixed problems in CAT(1) spaces under some mild conditions.

In 1976, the concept of Δ-convergence in general metric spaces was first discussed by the result in, [1]. Let $\kappa \in R$. Then, a geodesic space that has a geodesic triangle that is sufficiently thinner than the comparable comparison triangle in a model space with curvature κ is said to be a CAT(κ) space.

The result in, [2], originally investigated the fixed point theory in CAT(κ) spaces in 2003. Later, many researchers expanded on the concept of CAT(κ) provided in, [3], by mainly concentrating on CAT(0) spaces. Since each CAT(κ) space is a CAT(κ') space for any $\kappa' \geq \kappa$, the results of a CAT(0) space can be applied to any CAT(κ) space with $\kappa \leq 0$ (see in, [4]). However, many researchers have studied CAT(κ) spaces for $\kappa > 0$ (e.g., [5],[6],[7],[8],[9]).

Now, we introduce some iterative algorithms for approximating common fixed point as follows. In 2021, the result in, [10], suggested the new iteration approach for approximating the common fixed point of three nonexpansive mappings. Let the self mappings on J, G_1, G_2, G_3 be three nonexpansive, then the sequence $\{c_n\}$ is generated by $c_1 \in J$ and

$$\begin{cases} a_n = (1-\kappa_n)c_n + \kappa_n G_1 c_n, \\ b_n = (1-\delta_n)a_n + \delta_n G_2 a_n, \\ c_{n+1} = (1-\mu_n)G_2 a_n, \\ \qquad +\mu_n G_3 b_n + \mu_n G_3 b_n \end{cases} \tag{1}$$

where $\{\mu_n\}$, $\{\delta_n\}$ and $\{\kappa_n\}$ are real sequences in $(0, 1)$.

On the other hand, let $f : X \to (-\infty, \infty]$ be a proper and convex function and (X, d) be a geodesic metric space. The main optimization problem objective is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y).$$

Let $\operatorname{argmin}_{y \in X} f(y)$ be the set of minimizers of f. In 1970, the proximal point algorithm(PPA) was first developed by the result in, [11]. It is an efficient technique for tackling this problem. Later on in 1976, the result in, [12], showed that the PPA converges to the convex problem's solution in Hilbert spaces. Let f be a proper, convex and lower semicontinuous function on a Hilbert space H. The PPA is generated by $x_1 \in H$ and

$$x_{n+1} = \text{argmin}_{y\in H}\left[f(y) + \frac{1}{2\lambda_n} \parallel y - x_n \parallel^2\right]$$

where for all $n \in N$ and $\lambda_n > 0$. It was proved that $\{x_n\}$ converges weakly to a minimizer of f provided $\Sigma_{n=1}^{\infty}\lambda_n = \infty$. However, the PPA does not always strongly converge, as demonstrated by the result in, [13]. The PPA and Halpern's algorithm, [14], were merged in 2000 by the result in, [15], who proved the guarantee of strong convergence.

The asymptotic behavior of the sequences generated by the PPA for a convex function in geodesic spaces with curvature constrained above was first suggested by the result in, [16], in 2017. Additionally, they introduced the PPA in the following way in a $CAT(1)$ space:

$$\begin{cases} x_1 \in X, \\ x_{n+1} = \text{argmin}_{y\in X}[g(y)+ \\ \dfrac{1}{\lambda_n}\tan(d(y,x_n))\sin(d(y,x_n))] \end{cases} \tag{2}$$

where for all $n \in N$ and $\lambda_n > 0$. By the Fejér monotonicity, it was proved that, if f has a minimizer and $\Sigma_{n=1}^{\infty}\lambda_n = \infty$, then $\{x_n\}$ Δ-converges to its minimizer, [17]. A version of split for the PPA was employed in 2014 by the result in, [18], to minimize the sum of convex functions in for CAT(0) spaces. Additional intriguing outcomes can also be studied in the result in, [19].

Several PPA convergence results have recently been extended to the context of manifolds from the usual linear spaces, including the Euclidean, Hilbert and Banach spaces(see in, [19], [20], [21], [22], [23]). In analysis and geometry branch, the minimizers of the objective convex functional in the nonlinear spaces are extremely important.

The result in, [23], introduced the result of PPA in $CAT(1)$ spaces X as follows:

$$\begin{cases} x_1 \in X, \\ w_n = \text{argmin}_{y\in X}[g(y)+ \\ \dfrac{1}{\lambda_n}\tan(d(y,x_n))\sin(d(y,x_n))], \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tw_n \end{cases} \tag{3}$$

where $\{\alpha_n\}$ is a real sequences in the interval $[0, 1]$, $\forall n \geq 1$.

We present a newly modified PPA that is motivated by (1), (2) and (3). Let g be a proper lower semi-continuous function from the set X to $(-\infty,\infty)$ and (X, d) be an admissible complete CAT(1) space. Consider three nonexpansive mappings $T_1, T_2, T_3 : K \to K$ such that $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Assume that for each $a_1, a_2 \in (0, 1)$, $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[a_1, a_2]$ and λ_n is a sequence where $\lambda_n \geq \lambda \geq 0$, for each $n \geq 1$ and for some λ, then the sequence $\{x_n\}$ is generated by

$$\begin{cases} w_n = \text{argmin}_{y\in X}[g(y)+ \\ \dfrac{1}{\lambda_n}\tan(d(y,x_n))\sin(d(y,x_n))], \\ z_n = (1-\kappa_n)x_n \oplus \kappa_n T_1 w_n, \\ y_n = (1-\delta_n)z_n \oplus \delta_n T_2 z_n, \\ x_{n+1} = (1-\mu_n)T_2 z_n \oplus \mu_n T_3 y_n \end{cases} \tag{4}$$

where the sequences $\{\mu_n\}$, $\{\delta_n\}$ and $\{\kappa_n\}$ are in $(0, 1)$ for all $n \in N$.

For the purpose to solve minimization problems and common fixed point problems in CAT(1) spaces, we introduce a newly PPA in this study and prove strong and Δ-convergence theorems for this algorithm in CAT(1) spaces. Additionally, a convex minimization application and a common fixed point problems on CAT(κ) spaces with the bounded positive real number κ are provided.

2 Preliminaries

Let (X, d) be a metric space. A *geodesic path* joining x to y is a map γ from a interval $[0, l] \subset R$ to the set X such that $\gamma(0) = x, \gamma(l) = y$, and $\rho(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, l]$ and $x, y \in X$. Specifically, γ is an isometry and $d(x, y) = l$. A *geodesic segment* joining x and y is a term given to the image of $\gamma([0, l])$ of γ. This geodesic segment is represented by the symbol $[x, y]$ when it is unique. Accordingly, $z \in [x, y]$ if and only if there exists $\alpha \in [0, 1]$ such that

$$d(x, z) = (1-\alpha)d(x, y) \text{ and } d(y, z) = \alpha d(x, y).$$

For this particular case, we can write $z = \alpha x \oplus (1 - \alpha)y$. If every two points of X are joined by a geodesic which every two points of distance smaller than D, then the space (X, ρ) is called a geodesic space or $D-$ *geodesic space*. If there is exactly one geodesic joining x and y for each $x, y \in X$, then X is called uniquely geodesic or $D-$ *uniquely geodesic*. If $K \subset X$ includes every geodesic segment joining any two of its points, then the set K is called convex. The set K is called *bounded* if

$$\text{diam}(K) := \sup\{d(x, y) : x, y \in K\} < \infty.$$

The model spaces M_κ^n are now introduced; the reader is referred to, [4], for more information on these spaces. Let $n \in N$. The metric space R^n with the usual Euclidean distance is denoted by the symbol E^n. We symbolize the Euclidean scalar product in Rn by the symbol $(\cdot|\cdot)$, that is,

$$(x|y) = x_1y_1 + ... + x_ny_n \text{ where } x = (x_1, ..., x_n), y = (y_1, ..., y_n).$$

Let S^n be the $n-dimensional\ sphere$ denoted by

$$S^n = \{x = x_1, ..., x_{n+1} \in R^{n+1} : (\cdot|\cdot) = 1\},$$

with metric $d_{S^n} = \arccos(x|y), x, y \in S^n$.

Let $E^{n,1}$ be the vector space R^{n+1} endowed with the symmetric bilinear form which associates to vectors $u = (u_1, ..., u_{n+1})$ and $v = (v_1, ..., v_{n+1})$ the real number $\langle u|v\rangle$ denoted by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \textstyle\sum_{i=1}^{n} u_iv_i.$$

Let H^n be the $hyperbolic\ n-space$ denoted by

$$H^n = \{u = (u_1, u_2, ..., u_{n+1}) \in E^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 1\}$$

with metric d_{H^n} such that

$$\cosh(d_{H^n}(x, y)) = -\langle x|y\rangle, x, y \in H^n.$$

Definition 2.1. *Let $\kappa \in R$, the following metric spaces are defined by M^n_κ.*
(1) *if $\kappa = 0$ then M^n_0 is the Euclidean space E^n;*
(2) *if $\kappa > 0$ then M^n_κ is obtained from the spherical space S^n by multiplying the distance function by the constant $1/\sqrt{\kappa}$;*
(3) *if $\kappa < 0$ then M^n_κ is obtained from the hyperbolic space H^n by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.*

A geodesic triangle is made up of three points in the geodesic space (X, d) (x, y, and z) and three geodesic segments between each pair of vertices. A comparison triangle for $\Delta(x, y, z)$ in (X, d) is a triangle $\Delta(\overline{x}, \overline{y}, \overline{z})$ in M^2_κ such that

$$d(x, y) = d_{M^2_\kappa}(\overline{x}, \overline{y}), d(x, z) = d_{M^2_\kappa}(\overline{x}, \overline{z}) \text{ and } \rho(z, x) = d_{M^2_\kappa}(\overline{z}, \overline{x}).$$

If $\kappa \leq 0$ then such a comparison triangle always exists in M^2_κ. If $\kappa > 0$ then such a triangle exists whenever $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{M^2_\kappa}(\overline{x}, \overline{p})$.

A geodesic triangle $\Delta(x, y, z)$ in X is said to satisfy the CAT(κ) inequality if for any $p, q \in \Delta(x, y, z)$ and for their comparison points $\overline{p}, \overline{q} \in \Delta(\overline{x}, \overline{y}, \overline{z})$, one has

$$d(p, q) \leq d_{M^2_\kappa}(\overline{p}, \overline{q}).$$

Definition 2.2. *If $\kappa \leq 0$, then X is called a CAT(κ) space if and only if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality. If $\kappa > 0$, then X is called a CAT(κ) space if and only if X is D_κ -geodesic and any geodesic triangle $\Delta(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ satisfies the CAT(κ) inequality.*

Definition 2.3. *A self mapping T on the set X is called:*
(1) *nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$.*
(2) *demi-compact if, for all $\{x_n\} \in C$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, $\{x_n\}$ has a convergent subsequence.*

Let CAT(1) space be (X, d) such that $x, y, z \in X$ satisfy $d(x, y) + d(y, z) + d(z, x) < 2D_1$. Then

$$\cos d(\alpha x \oplus (1-\alpha)y, z) \geq \alpha \cos d(x, z) + (1-\alpha) \cos d(y, z) \tag{5}$$

for all $\alpha \in [0, 1]$.

Definition 2.4. , [24], *Let (X, d) be a geodesic metric space.*
(1) *An open set U in (X, d) is said to be a C_R- domain for any $R \in [0, 2]$ if $x, y, z \in U$, any minimal geodesic $\gamma : [0, 1] \to X$ between y and z for all $\alpha \in [0, 1]$,*

$$d^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)d^2(x, y) + \alpha d^2(x, z) - \frac{R}{2}(1-\alpha)\alpha d^2(y, z). \tag{6}$$

(2) *(X, d) is said to be $R-convex$ for any $R \in [0, 2]$ if X itself a $C_R-domain$.*
(3) *(X, d) is said to be locally $R-convex$ for $R \in [0, 2]$ if every point in X included in a $C_R-domain$.*

Definition 2.5. *Let CAT(1) space be (X, d). A sequence $\{x_n\}$ in X is called $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. We write $\triangle-\lim_{n\to\infty} x_n = x$ and define $W_\triangle(x_n) := \cup\{A(\{u_n\})\}$.*

The domain of the function $g : X \to (-\infty, \infty]$ is

$$\text{Dom}(g) = \{x \in X : g(x) \in R\}.$$

If Dom(g) is nonempty, then g is called proper. If $K = \{x \in X : g(x) \leq \beta\}$ is closed in X for all $\beta \in R.$, then g is called lower semi-continuous.

A $\mathrm{CAT}(1)$ space X is called admissible if $d(v, v') < \frac{\pi}{2}$ for all $v, v' \in X$. Apart from that, the $\{x_n\}$ in a $\mathrm{CAT}(1)$ space is called spherically bounded if

$$\inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) < \frac{\pi}{2}.$$

Let g be a proper lower semi-continuous convex function. For all $\lambda > 0$, the following formulation of the resolvent of g in the admissible $\mathrm{CAT}(1)$ spaces:

$$R_\lambda(x) = \operatorname*{argmin}_{y\in X}\Big[g(y) + \frac{1}{\lambda}\tan d(y, x)\sin d(y, x)\Big]$$

for all $x \in X$. R_λ is well define for all $\lambda > 0$. More specifically, $F(R_\lambda)$ of fixed points of the resolvent associated with g coincides with the set $\operatorname{argmin}_{y\in X} g(y)$ of minimizers of g.

Lemma 2.6. *Let $g : X \to (-\infty, \infty]$ be a proper lower semi-continuous convex function and (X, d) be a admissible complete CAT(1) space. If $\lambda > 0$, $\in X$ and $u \in \operatorname{argmin}_X g$, then the following inequalities hold:*

$$\frac{\pi}{2}A(B - C) \geq \lambda(g(R_\lambda x) - g(u)) \tag{7}$$

and

$$B \geq C \tag{8}$$

where
$A = \frac{1}{\cos^2 d(R_\lambda x, x)} + 1$,
$B = \cos d(R_\lambda x, x)\cos d(u, R_\lambda x)$
and $C = \cos d(u, x)$.

Lemma 2.7. *Let (X, d) be the admissible complete CAT(1) space. If $g : X \to (-\infty, \infty]$ is a proper semi-continuous convex function, then g is $\Delta-$ lower semi-continuous.*

Lemma 2.8. *Let (X, d) be a complete CAT(1) space and $\{x_n\}$ be a spherical bounded sequence in X. If $d(d_n, \rho)$ is convergent for all $\rho \in W_\Delta(\{x_n\})$, then $\{x_n\}$ is $\Delta-$convergent.*

Corollary 2.9. *Let C be a nonempty closed convex subset of complete CAT(1) space (X, d). Let the self mapping T on C be a nonepansive. If $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = \omega$, then $\omega \in C$ and $\omega = T\omega$.*

3 Main results

The main results can be presented in the following.

Lemma 3.1. *Assume that $g : X \to (-\infty, \infty]$ is a proper lower semi-continuous convex function, let (X, d) be an admissible complete CAT(1) space. Assume that T, S and R are three nonexpansive mappings, such that $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \cap \operatorname{argmin}_{x\in X} g(x)$. Assume that $\{\mu_n\}$, $\{\delta_n\}$ and $\{\kappa_n\}$ are in $[a_1, a_2]$ for $a_1, a_2 \in (0, 1)$ and $\{\lambda_n\}$ is a sequence such that$\lambda_n \geq \lambda > 0$, for each and for some λ. Assume that for each $n \geq 1$, the sequence x_n is generated by (4). Then we have the following:*

(1) *for all $q \in \Omega$, $\lim_{n\to\infty} d(x_n, q)$ exists;*

(2) $\lim_{n\to\infty} d(x_n, z_n) = 0$;

(3) $\lim_{n\to\infty} d(x_n, T_1x_n)$
$= \lim_{n\to\infty} d(x_n, T_2x_n)$
$= \lim_{n\to\infty} d(x_n, T_3x_n)$.

Proof. First, we prove that $\{x_n\}$, $\{w_n\}$ are spherical bounded. Assume that $w_n = R_{\lambda_n}x_n$ for each $n \geq 1$. Let $q \in \Omega$. Then, by (7), we have

$$\begin{aligned}&\min\{\cos d(w_n, x_n), \cos d(q, w_n)\}\\ &\geq \cos d(w_n, x_n)\cos d(q, w_n)\\ &\geq \cos d(q, x_n)\end{aligned} \tag{9}$$

it shows that

$$\begin{aligned}&\max\{d(w_n, x_n), d(q, w_n)\}\\ &\leq d(q, x_n).\end{aligned} \tag{10}$$

Since T_1, T_2 and T_3 are three nonexpansive mappings and X is admissible, by (4), we obtain

$$\begin{aligned}&\cos d(q, z_n)\\ &= \cos d(q, (1-\kappa_n)x_n \oplus \kappa_n T_1 w_n)\\ &\geq (1-\kappa_n)\cos d(q, x_n) + \kappa_n \cos d(q, T_1 w_n)\\ &\geq (1-\kappa_n)\cos d(q, x_n) + \kappa_n \cos d(q, w_n)\\ &\geq (1-\kappa_n)\cos d(q, x_n) + \kappa_n \cos d(q, x_n)\\ &= \cos d(q, x_n),\end{aligned} \tag{11}$$

and

$$\begin{aligned}&\cos d(q, y_n)\\ &= \cos d(q, (1-\delta_n)z_n \oplus \delta_n T_2 z_n)\\ &\geq (1-\delta_n)\cos d(q, z_n) + \delta_n \cos d(q, T_2 z_n)\\ &\geq (1-\delta_n)\cos d(q, z_n) + \delta_n \cos d(q, z_n)\\ &\geq (1-\delta_n)\cos d(q, x_n) + \delta_n \cos d(q, x_n)\\ &= \cos d(q, x_n),\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\cos d(q, x_{n+1}) \qquad (13)\\
&= \cos d(q, (1-\mu_n)T_2 z_n \oplus \mu_n T_3 y_n)\\
&\geq (1-\mu_n)\cos d(q, T_2 z_n)\\
&\quad +\mu_n \cos d(q, T_3 y_n)\\
&\geq (1-\mu_n)\cos d(q, z_n) + \mu_n \cos d(q, y_n)\\
&\geq (1-\mu_n)\cos d(q, x_n) + \mu_n \cos d(q, x_n)\\
&= \cos d(q, x_n),
\end{aligned}$$

it shows that

$$\begin{aligned}
&d(q, x_{n+1}) \qquad (14)\\
&\leq d(q, x_n) \leq d(q, x_1) < \frac{\pi}{2}.
\end{aligned}$$

Thus, the sequence $\{x_n\}$ and $\{w_n\}$ are spherically bounded. Hence, assertion (1) is true. Now, we prove that

$$\sup_{n\geq 1} d(x_n, w_n) < \frac{\pi}{2}$$

and $\lim_{n\to\infty} d(q, x_n) < \frac{\pi}{2}$ exists for all $q \in \Omega$. So, we get

$$\lim_{n\to\infty} d(q, x_n) = r \geq 0. \qquad (15)$$

So, this claim that $\lim_{n\to\infty} d(x_n, q)$ exists, for all $q \in \Omega$. We now claim that $\lim_{n\to\infty} d(x_n, w_n) = 0$. By (13), it follows that

$$\begin{aligned}
&\cos d(q, x_{n+1})\\
&= \cos d(q, (1-\mu_n)T_2 z_n \oplus \mu_n T_3 y_n)\\
&\geq (1-\mu_n)\cos d(q, T_2 z_n)\\
&\quad +\mu_n \cos d(q, T_3 y_n)\\
&\geq (1-\mu_n)\cos d(q, z_n) + \mu_n \cos d(q, y_n)\\
&\geq (1-\mu_n)\cos d(q, x_n) + \mu_n \cos d(q, y_n)
\end{aligned}$$

so,

$$\begin{aligned}
&\cos d(q, x_{n+1})\\
&\geq \cos d(q, x_n) - \mu_n \cos d(q, x_n)\\
&\quad +\mu_n \cos d(q, y_n);\\
&\mu_n \cos d(q, x_n)\\
&\geq \cos d(q, x_n) - \cos d(q, x_{n+1})\\
&\quad +\mu_n \cos d(q, y_n);\\
&\cos d(q, x_n)\\
&\geq \frac{1}{\mu_n}[\cos d(q, x_n) - \cos d(q, x_{n+1})]\\
&\quad +\cos d(q, y_n).
\end{aligned}$$

Since $\mu_n \geq a_1 > 0$ for each $n \geq 1$, we get

$$\begin{aligned}
&\cos d(q, x_n) \qquad (16)\\
&\geq \frac{1}{a_1}[\cos d(q, x_n) - \cos d(q, x_{n+1})]\\
&\quad +\cos d(q, y_n).
\end{aligned}$$

So, by (15), (16), we get

$$\begin{aligned}
r &= \liminf_{n\to\infty} \cos d(q, x_n) \qquad (17)\\
&\geq \liminf_{n\to\infty} \cos d(q, y_n).
\end{aligned}$$

In contrast, we see from (12) that

$$\begin{aligned}
&\limsup_{n\to\infty} \cos d(q, y_n) \qquad (18)\\
&\geq \limsup_{n\to\infty} \cos d(q, x_n) = r.
\end{aligned}$$

So, by (17) and (18), we get

$$\lim_{n\to\infty} \cos d(q, y_n) = r. \qquad (19)$$

On the same way, by (13), it follows that

$$\begin{aligned}
&\cos d(q, x_{n+1})\\
&= \cos d(q, (1-\mu_n)T_2 z_n \oplus \mu_n T_3 y_n)\\
&\geq (1-\mu_n)\cos d(q, T_2 z_n)\\
&\quad +\mu_n \cos d(q, T_3 y_n)\\
&\geq (1-\mu_n)\cos d(q, z_n) + \mu_n \cos d(q, y_n)\\
&\geq (1-\mu_n)\cos d(q, z_n) + \mu_n \cos d(q, x_n)
\end{aligned}$$

so,

$$\begin{aligned}
&\cos d(q, x_{n+1})\\
&\geq (1-\mu_n)\cos d(q, z_n)\\
&\quad +\mu_n \cos d(q, x_n);\\
&\cos d(q, x_{n+1})\\
&\geq (1-\mu_n)\cos d(q, z_n)\\
&\quad +(1-(1-\mu_n))\cos d(q, x_n);\\
&(1-\mu_n)\cos d(q, x_n)\\
&\geq (1-\mu_n)\cos d(q, z_n)\\
&\quad +\cos d(q, x_n) - \cos d(q, x_{n+1});\\
&\cos d(q, x_n)\\
&\geq \frac{1}{1-\mu_n}[\cos d(q, x_n) - \cos d(q, x_{n+1})]\\
&\quad +\cos d(q, z_n).
\end{aligned}$$

Since $1-\mu_n \geq a_1 > 0$ for each $n \geq 1$, we get

$$\cos d(q,x_n) \geq \frac{1}{a_1}[\cos d(q,x_n) - \cos d(q,x_{n+1})] + \cos d(q,z_n). \tag{20}$$

So, by (15) and (20), we get

$$r = \liminf_{n\to\infty} \cos d(q,x_n) \geq \liminf_{n\to\infty} \cos d(q,z_n). \tag{21}$$

In contrast, we see from (11) that

$$\limsup_{n\to\infty} \cos d(q,z_n) \geq \limsup_{n\to\infty} \cos d(q,x_n) = r. \tag{22}$$

So, by (21) and (22), we get

$$\lim_{n\to\infty} \cos d(q,z_n) = r. \tag{23}$$

By (9), (10), we get

$$\begin{aligned}
&\cos d(q,z_n)\\
&= (1-\kappa_n)\cos d(q,x_n) + \kappa_n \cos d(q,T_1 w_n)\\
&\geq (1-\kappa_n)\cos d(q,x_n) + \kappa_n \cos d(q,w_n)\\
&\geq \cos d(q,x_n) - \kappa_n \cos d(q,x_n)\\
&\quad +\kappa_n \frac{\cos d(q,x_n)}{\cos d(w_n,x_n)}\\
&= \cos d(q,x_n)\\
&\quad +\kappa_n \cos d(q,x_n)[\frac{1}{\cos d(w_n,x_n)} - 1],
\end{aligned}$$

that is,

$$\frac{\cos d(q,z_n)}{\cos d(q,x_n)} - 1 \geq \kappa_n[\frac{1}{\cos d(w_n,x_n)} - 1].$$

Since $\kappa_n \geq a_1 > 0$ for each $n \geq 1$, by (15), (19) and (23), it follows that

$$1 \leq \frac{1}{\cos d(w_n,x_n)}$$

that is,

$$\lim_{n\to\infty} d(w_n,x_n) = 0.$$

Thus, we obtain

$$\lim_{n\to\infty} d(R_{\lambda_n} x_n, x_n) = 0.$$

Since $\lambda_n \geq \lambda > 0$ for each $n \geq 1$, we have

$$\lim_{n\to\infty} d(R_\lambda x_n, x_n) = 0.$$

Thus, this claim that $\lim_{n\to\infty} d(w_n,x_n) = 0$. Hence, assertion (2) is true. Finally, we prove that

$$\begin{aligned}
\lim_{n\to\infty} d(x_n,T_1x_n) &= \lim_{n\to\infty} d(x_n,T_2x_n)\\
&= \lim_{n\to\infty} d(x_n,T_3x_n)\\
&= 0.
\end{aligned}$$

By (5), we obtain

$$\begin{aligned}
&d^2(q,z_n)\\
&= d^2(q,(1-\kappa_n)x_n \oplus \kappa_n T_1 w_n)\\
&\leq (1-\kappa_n)d^2(q,x_n) + \kappa_n d^2(q,T_1w_n)\\
&\quad -\frac{R}{2}(1-\kappa_n)\kappa_n d^2(x_n,T_1w_n)\\
&\leq (1-\kappa_n)d^2(q,x_n) + \kappa_n d^2(q,w_n)\\
&\quad -\frac{R}{2}a_1a_2 d^2(x_n,T_1w_n)\\
&\leq (1-\kappa_n)d^2(q,x_n) + \kappa_n d^2(q,x_n)\\
&\quad -\frac{R}{2}a_1a_2 d^2(x_n,T_1w_n)\\
&= d^2(q,x_n) - \frac{R}{2}a_1a_2 d^2(x_n,T_1w_n),
\end{aligned}$$

it shows that

$$\begin{aligned}
&d^2(q,z_n)\\
&\leq d^2(q,x_n) - \frac{R}{2}a_1a_2 d^2(x_n,T_1w_n);\\
&\frac{R}{2}a_1a_2 d^2(x_n,T_1w_n)\\
&\leq d^2(q,x_n) - d^2(q,z_n);\\
&d^2(x_n,T_1w_n)\\
&\leq \frac{2}{Ra_1a_2}[d^2(q,x_n) - d^2(q,z_n)].
\end{aligned}$$

This yields

$$\lim_{n\to\infty} d(x_n,T_1w_n) = 0.$$

So, by the triangle inequality, we have

$$\begin{aligned}
d(x_n,T_1x_n) &\leq d(x_n,T_1w_n) + d(T_1w_n,T_1x_n)\\
&\leq d(x_n,T_1w_n) + d(w_n,x_n)\\
&\to 0,\ \ as\ \ n\to\infty
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(x_n,T_1x_n) = 0.$$

Next, we have

$$\begin{aligned}
&d^2(q, y_n)\\
&= d^2(q, (1-\delta_n)z_n \oplus \delta_n T_2 z_n)\\
&\le (1-\delta_n)d^2(q, z_n) + \delta_n d^2(q, T_2 z_n)\\
&\quad -\frac{R}{2}(1-\delta_n)\delta_n d^2(z_n, T_2 z_n)\\
&\le (1-\delta_n)d^2(q, z_n) + \delta_n d^2(q, T_2 z_n)\\
&\quad -\frac{R}{2}a_1 a_2 d^2(z_n, T_2 z_n)\\
&\le (1-\delta_n)d^2(q, z_n) + \delta_n d^2(q, z_n)\\
&\quad -\frac{R}{2}a_1 a_2 d^2(z_n, T_2 z_n)\\
&= d^2(q, z_n) - \frac{R}{2}a_1 a_2 d^2(z_n, T_2 z_n),
\end{aligned}$$

which implies that

$$\begin{aligned}
d^2(q, y_n) &\le d^2(q, z_n)\\
&\quad -\frac{R}{2}a_1 a_2 d^2(z_n, T_2 z_n);\\
\frac{R}{2}a_1 a_2 d^2(z_n, T_2 z_n) &\le d^2(q, x_n)\\
&\quad -d^2(q, y_n);\\
d^2(z_n, T_2 z_n) &\le \frac{2}{Ra_1 a_2}[d^2(q, x_n)\\
&\quad -d^2(q, y_n)].
\end{aligned}$$

This gives

$$\lim_{n\to\infty} d(z_n, T_2 z_n) = 0.$$

By the triangle inequality, we get

$$\begin{aligned}
&d(x_n, T_2 x_n)\\
&\le d(x_n, z_n) + d(z_n, T_2 x_n)\\
&\le d(x_n, T_2 z_n) + d(T_2 z_n, z_n)\\
&\quad +d(z_n, T_2 z_n) + d(T_2 z_n, T_2 x_n)\\
&\le d(x_n, z_n) + d(T_2 z_n, z_n)\\
&\quad +d(z_n, T_2 z_n) + d(z_n, x_n)\\
&\to 0, \ \ as \ \ n \to \infty.
\end{aligned}$$

Lastly, we have

$$\begin{aligned}
&d^2(q, x_{n+1})\\
&= d^2(q, (1-\mu_n)T_2 z_n \oplus \mu_n T_3 y_n)\\
&\le (1-\mu_n)d^2(q, T_2 z_n) + \mu_n d^2(q, T_3 y_n)\\
&\quad -\frac{R}{2}(1-\mu_n)\mu_n d^2(T_2 z_n, T_3 y_n)\\
&\le (1-\mu_n)d^2(q, z_n) + \mu_n d^2(q, y_n)\\
&= -\frac{R}{2}a_1 a_2 d^2(z_n, T_3 y_n)\\
&\le (1-\mu_n)d^2(q, x_n) + \mu_n d^2(q, x_n)\\
&= -\frac{R}{2}a_1 a_2 d^2(z_n, T_3 y_n)\\
&= d^2(q, x_n) - \frac{R}{2}a_1 a_2 d^2(z_n, T_3 y_n),
\end{aligned}$$

which implies that

$$\begin{aligned}
d^2(q, x_{n+1}) &\le d^2(q, x_n)\\
&\quad -\frac{R}{2}a_1 a_2 d^2(z_n, T_3 y_n);\\
\frac{R}{2}a_1 a_2 d^2(z_n, T_3 y_n) &\le d^2(q, x_n)\\
&\quad -d^2(q, x_{n+1});\\
d^2(z_n, T_3 y_n) &\le \frac{2}{Ra_1 a_2}[d^2(q, x_n)\\
&\quad -d^2(q, x_{n+1})].
\end{aligned}$$

Thus, we get

$$\lim_{n\to\infty} d(z_n, T_3 y_n) = 0.$$

It follows that

$$\begin{aligned}
&d(z_n, x_n)\\
&\le d((1-\kappa_n)x_n \oplus \kappa_n T_1 w_n, x_n)\\
&\le (1-\kappa_n)d(x_n, x_n) + \kappa_n d(T_1 w_n, x_n)\\
&\to 0, \ \ as \ \ n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
&d(y_n, x_n)\\
&\le d((1-\delta_n)z_n \oplus \delta_n T_2 z_n, x_n)\\
&\le (1-\delta_n)d(z_n, x_n) + \delta_n d(T_2 z_n, x_n)\\
&\le d(z_n, x_n)\\
&\to 0, \ \ as \ \ n \to \infty.
\end{aligned}$$

By the triangle inequality, we get

$$\begin{aligned}&d(x_n, T_3x_n)\\&\leq d(x_n, z_n) + d(z_n, T_3x_n)\\&\leq d(x_n, z_n) + d(z_n, T_3y_n)\\&\quad +d(T_3y_n, T_3x_n)\\&\leq d(x_n, z_n) + d(z_n, T_3y_n)\\&\quad +d(y_n, x_n)\\&\to 0,\ \ as\ \ n \to \infty.\end{aligned}$$

Hence, the assertion 3) is true. The proof is now complete. □

Next, suppose that Lemma 3.1's conclusion is true. Following are some $\Delta-$ convergence results that we prove.

Theorem 3.2. *Assume that $g : X \to (-\infty, \infty]$ is a proper lower semi-continuous convex function, let (X, d) be an admissible complete CAT(1) space. Then $\{x_n\}$ generated by (4) $\Delta-$ converges to an element of $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \cap argmin_{x\in X} g(x)$.*

Proof. Let $\omega \in \Omega$ and assume that $w_n = R_{\lambda_n} x_n$ for each $n \geq 1$. Then, for each $n > 1$, we have $g(\omega) \leq g(w_n)$. From Lemma 2.6, we have

$$D \geq \lambda_n(g(w_n) - g(\omega)) \geq 0 \tag{24}$$

where

$$\begin{aligned}D \ &= \ \frac{\pi}{2}(\frac{1}{cos^2 d(w_n, x_n)}\\&\quad +1)(cosd(w_n, x_n)cosd(\omega, w_n)\\&\quad -cosd(\omega, x_n))\end{aligned}$$

Due to the fact that $\lambda_n > \lambda > 0$ for each $n \geq 1$ and by Lemma 3.1, we can prove that

$$\begin{aligned}&d(w_n, x_n) \to 0\ \ as\ \ n \to \infty, \qquad (25)\\&\lim_{n\to\infty} d(\omega, x_n) \text{ and}\\&\lim_{n\to\infty} d(\omega, w_n) \text{ exist.}\end{aligned}$$

By (24), we get

$$\lim_{n\to\infty} g(w_n) = \inf g(X). \tag{26}$$

Next, we prove that $W_\Delta(\{x_n\}) \subset \Omega$. Let $w^* \in W_\Delta(\{x_n\})$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which $\Delta-$converges to w^*. Since $\lim_{n\to\infty} d(w_n, x_n)$, we can observe that the subsequence w_{n_i} of w_n also Δ-converges to the point w^* according to the definition of the Δ-convergence. Lemma 2.7 and (26) provide

$$\begin{aligned}g(w^*) \ &\leq \ \liminf_{i\to\infty} g(w_{n_i})\\&\leq \ \lim_{n\to\infty} g(w_n)\\&= \ \inf g(X).\end{aligned}$$

Hence, $w^* \in argmin_{x\in X} g(x)$ and so $W_\Delta(\{x_n\}) \subset \text{argmin}_{x\in X} g(x)$. Moreover, since

$$\begin{aligned}\lim_{n\to\infty} d(x_n, T_1x_n) \ &= \ \lim_{n\to\infty} d(x_n, T_2x_n)\\&= \ \lim_{n\to\infty} d(x_n, T_3x_n)\\&= \ 0,\end{aligned}$$

and $\{x_n\}$ $\Delta-$converges to w^*, it follows from Corollary 2.9 that $w^* \in F(T_1)$. So, we conclude that $W_\Delta(\{x_n\}) \subset \Omega$, we can see that for any $w^* \in W_\Delta(\{x_n\})$, $d(w^*, x_n)$ is convergent. By Lemma 2.8, $\{x_n\}$ is $\Delta-$convergent to element in Ω. Lemma 2.8 shows that $\{x_n\}$ is $\Delta-$convergent to element in Ω. The proof is now complete. □

Theorem 3.3. *Assume that $g : X \to (-\infty, \infty]$ is a proper lower semi-continuous convex function, let (X, d) be an admissible complete CAT(1) space. Consequently, these are equivalent.*
(A) Strong convergence arises to an element of Ω for the sequence x_n generated by (4).
(B) If $d(x, \Omega) = \inf\{d(x, x^*) : q \in \Omega\}$, then $\liminf_{n\to\infty} d(x_n, \Omega) = 0$.

Proof. We start by proving that $(A) \Rightarrow (B)$. It is obvious.
Furthermore, we prove that $(B) \Rightarrow (A)$. Assume that $\liminf_{n\to\infty} d(x_n, \Omega) = 0$. Since $d(x_{n+1}, q) \leq d(x_n, q)$ for all $q \in \Omega$, we get

$$d(x_{n+1}, \Omega) \leq d(x_n, \Omega).$$

Thus, $\lim_{n\to\infty} d(x_n, \Omega) = 0$. Then, using the methods in, [25], we get that $\{x_n\}$ is a Chauchy sequence in X. This implies that $\{x_n\}$ converges to point $c \in X$ and thus $d(c, \Omega) = 0$. Since Ω is closed, $c \in \Omega$. The proof is now complete. □

The mappings T_1, T_2, T_3 are called to satisfy the condition Q if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(k) \geq 0$ for all $k \in (0, \infty)$ such that

$$d(x, T_1x) \geq h(d(x, H)),$$

or

$$d(x, T_2x) \geq h(d(x, H)),$$

or

$$d(x, T_3x) \geq h(d(x, H)),$$

for all $x \in X$, where $H = H(T_1) \cap H(T_2) \cap H(T_3)$.

Applying the condition Q yields the following result.

Theorem 3.4. *Assume that $g : X \to (-\infty, \infty]$ is a proper lower semi-continuous convex function, let (X, d) be an admissible complete CAT(1) space. If R_λ, T_1 and T_2 satisfy the condition Q, then $\{x_n\}$ generated by (4) strongly converges to an element of Ω.*

Proof. We prove that $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in \Omega$ by using Lemma 3.1. Additionally, it follows that $\lim_n d(x_n, \Omega)$ exists. Applying the condition Q, we obtain

$$\lim_{n\to\infty} h(d(x_n, \Omega)) \leq \lim_{n\to\infty} d(x_n, R_\lambda x_n) = 0,$$

or

$$\lim_{n\to\infty} h(d(x_n, \Omega)) \leq \lim_{n\to\infty} d(x_n, T_1 x_n) = 0,$$

or

$$\lim_{n\to\infty} h(d(x_n, \Omega)) \leq \lim_{n\to\infty} d(x_n, T_2 x_n) = 0,$$

or

$$\lim_{n\to\infty} h(d(x_n, \Omega)) \leq \lim_{n\to\infty} d(x_n, T_3 x_n) = 0.$$

Thus, we obtain

$$\lim_{n\to\infty} h(d(x_n, \Omega)) = 0$$

which by using the property of h, results in $\lim_{n\to\infty} d(x_n, \Omega) = 0$. Also, by the remained proof can be followed by the proof in Theorem 3.3 and hence, the desired result follows. The proof is now complete. □

Theorem 3.5. *Assume that $g : X \to (-\infty, \infty]$ is a proper lower semi-continuous convex function, let (X, d) be an admissible complete CAT(1) space. If R_λ or T_1 or T_2 is demi-compact, then $\{x_n\}$ generated by (4) strongly converges to an element of Ω.*

Proof. By Lemma 3.1, we obtain

$$\begin{aligned} &\lim_{n\to\infty} d(x_n, R_\lambda x_n) \qquad (27)\\ =\ &\lim_{n\to\infty} d(x_n, T_1 x_n)\\ =\ &\lim_{n\to\infty} d(x_n, T_2 x_n)\\ =\ &\lim_{n\to\infty} d(x_n, T_3 x_n)\\ =\ &0 \end{aligned}$$

as $n \to \infty$. Without loss of generality, we assume that T_1, T_2, T_3 or R_λ is demi-compact. Therefore, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $\rho^* \in X$. Hence, from (27) and the nonexpansiveness of mappings T_1, T_2, T_3, R_λ, it followed that

$$\begin{aligned} d(\rho^*, R_\lambda \rho^*) &= d(\rho^*, T_1 \rho^*)\\ &= d(\rho^*, T_2 \rho^*)\\ &= d(\rho^*, T_3 \rho^*)\\ &= 0, \end{aligned}$$

which denote that ρ^* is in Ω. Later, we can prove the strong convergence of $\{x_n\}$ to an element of Ω. The proof is now complete. □

4 Some Applications

Applications for the common fixed point in CAT(κ) with the bounded positive real number κ and some convex optimization problems, are demonstrated in this section.

The following assumptions are made throughout this section:

(A_1) X is a complete CAT(κ) space such that $d(v, v') < \frac{D_\kappa}{2}$;

(A_2) κ is a positive real number and $D_x = \frac{\pi}{\sqrt{\kappa}}$;

(A_3) $g : X \to (-\infty, \infty]$ be a proper lower semi-continuous convex function;

(A_4) $\widehat{R}_\lambda$ is the resolvent mapping on X defined by

$$\widehat{R}_\lambda(x) = \operatorname{argmin}_{y\in X}[g(y) + \frac{1}{\lambda}\tan(\sqrt{\kappa}d(y, x))\sin(\sqrt{\kappa}d(y, x))]$$

for all $\lambda > 0$ and $x \in X$.

The mapping $\widehat{R}_\lambda$ is well-defined since $(X, \sqrt{\kappa}d)$ is the admissible complete CAT(1) space, according to the result in, [26]. From Theorem 3.2, 3.3, 3.4 and 3.5 and assume that assumptions A_1, A_2, A_3 and A_4 hold, we get some Corollaries as follows.

Corollary 4.1. *Assume that assumptions A_1, A_2, A_3 and A_4 are hold. Let the mappings $T_1, T_2, T_3 : C \to C$ are nonexpansive such that $\Omega \neq \emptyset$. Suppose that the sequence $\{\delta_n\}, \{\kappa_n\}, \{\mu_n\} \subseteq [a_1, a_2]$ for some $a_1, a_2 \in (0, 1)$. Let $\{\lambda_n\}$ be the sequence such that for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ. For any*

$x_1 \in X$, generate the sequence $\{x_n\} \in C$ by

$$\begin{cases} w_n = \mathrm{argmin_{y \in X}[g(y)+} \\ \quad + \dfrac{1}{\lambda_n} \tan(\sqrt{\kappa} d(y, x_n)) \sin(\sqrt{\kappa} d(y, x_n))], \\ z_n = (1-\kappa_n)x_n \oplus \kappa_n T_1 w_n, \\ y_n = (1-\delta_n) z_n \oplus \delta_n T_2 z_n, \\ x_{n+1} = (1-\mu_n) T_2 z_n \oplus \mu_n T_3 y_n, \end{cases} \tag{28}$$

for each $n \geq 1$. Then $\{x_n\}$ $\Delta-$converges to an element of Ω.

Corollary 4.2. *Assume that assumptions A_1, A_2, A_3 and A_4 are hold. Let the mappings $T_1, T_2, T_3 : C \to C$ are nonexpansive such that $\Omega \neq \emptyset$. Suppose that the sequence $\{\delta_n\}, \{\kappa_n\}, \{\mu_n\} \subseteq [a_1, a_2]$ for some $a_1, a_2 \in (0, 1)$. Let $\{\lambda_n\}$ be the sequence such that for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ. Consequently, these are equivalent.*

1) *The $\{x_n\}$ generated by (28) converges strongly to an element of Ω.*
2) *$\liminf_{n\to\infty} d(x_n, \Omega) = 0$ where $d(x, \Omega) = \inf\{d(x,^*) : q \in \Omega\}$.*

Corollary 4.3. *Assume that assumptions A_1, A_2, A_3 and A_4 are hold. Let the mappings $T_1, T_2, T_3 : C \to C$ are nonexpansive such that $\Omega \neq \emptyset$. Suppose that the sequence $\{\delta_n\}, \{\kappa_n\}, \{\mu_n\} \subseteq [a_1, a_2]$ for some $a_1, a_2 \in (0, 1)$. Let $\{\lambda_n\}$ be the sequence such that for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ. If the mappings R_λ, T_1, T_2, T_3 satisfy the condition(Q) then $\{x_n\}$ generated by (28) converges strongly to an element of Ω.*

Corollary 4.4. *Suppose that A_1, A_2, A_3 and A_4 are hold. Let the mappings $T_1, T_2, T_3 : C \to C$ are nonexpansive such that $\Omega \neq \emptyset$. Suppose that the sequence $\{\delta_n\}, \{\kappa_n\}, \{\mu_n\} \subseteq [a_1, a_2]$ for some $a_1, a_2 \in (0, 1)$. Let $\{\lambda_n\}$ be the sequence such that for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ. Let $\{\lambda_n\}$ be a sequence such that, for each $n \geq 1$, $\lambda_n \geq \lambda > 0$ for some λ. If R_λ or T_1 or T_2 or T_3 is demi-compact, then $\{x_n\}$ generated by (28) converges strongly to an element of Ω.*

5 Conclusion

In this paper, for the minimization problem and the common fixed point problem in CAT(1) spaces, we prove a strong and Δ-convergence theorems. Our main results are a generalization of the results of various researchers in the literature review (see in, [17], [18], [19], [20], [21], [22], [23]). Additionally, we discussed about various applications to the common fixed point problem and the convex minimization problem in CAT(κ) spaces with the bounded positive real number κ. We further expanded on the results from the work of Kimura et al., [16], regarding the asymptotic behavior of sequences produced by the proximal point algorithm for a convex function in geodesic spaces with curvature bounded above.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Chatuphol Khaofong carried out conceptualization, methodology, supervision, writing and editing manuscript preparation. Phachara Saipara carried out conceptualization, methodology, supervision, writing original draft, editing manuscript preparation and submitting the manuscript to Journal. Suphot Srathonglang carried out writing original draft, writing and editing the manuscript preparation. Anantachai Padcharoen carried out methodology, investigation and validation.
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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.