A Note on non-Newtonian Isometry

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Abstract: In this article, we introduce non-Newtonian isometry and examine some of its basic properties. We also give a characterization of the relationship between real isometry and non-Newtonian isometry. Finally, we show that the $\nu-$measure of $\nu-$measurable sets is invariant for every generator under $\nu-$-isometries.

Key-Words: Non-Newtonian calculus, geometric calculus, isometry, ν-isometry, measure, ν-measure, ν-inner measure, ν-outer mesaure

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1 Introduction

Non-Newtonian calculus, which is used in many fields such as engineering, mathematics, finance, economics, medicine and biomedicine, was developed between 1967 and 1970 as an alternative to the classical analysis of Newton and Leibnitz, [1, 2]. The book 'Non-Newtonian Calculus', which forms the basis of non-Newtonian calculus, was published in 1972, [3]. The derivative and integral were investigated in the metacalculus, [4]. Geometric calculus and their applications were investigated in [5]. Some basic topological properties of the real non-Newtonian axis were investigated in [6]. The non-Newtonian Lebesgue measure for non-Newtonian open sets was defined in [7]. Finally, the non-Newtonian measure for closed non-Newtonian sets was defined and some related theorems were given in [8]. For more details see, [9],[10],[11],[12],[13],[14],[15],[16],[17],[18],[19], [20].

Let ν be a generator, which means that ν is a bijection function from $\mathbb{R}$ to a subset $\mathbb{A}$ of $\mathbb{R}$. Let $\dot{p}, \dot{q} \in A$. Then the $\nu-$ arithmetic is defined as follows;

$\nu-$ *addition*	$\dot{p}\dot{+}\dot{q} = \nu\{\nu^{-1}(\dot{p}) + \nu^{-1}(\dot{q})\}$
$\nu-$ *subtraction*	$\dot{p}\dot{-}\dot{q} = \nu\{\nu^{-1}(\dot{p}) - \nu^{-1}(\dot{q})\}$
$\nu-$ *multiplicative*	$\dot{p}\dot{\times}\dot{q} = \nu\{\nu^{-1}(\dot{p}) \times \nu^{-1}(\dot{q})\}$
$\nu-$ *division* $(\nu^{-1}(\dot{q}) \neq 0)$	$\dot{p}\dot{/}\dot{q} = \nu\{\nu^{-1}(\dot{p}) / \nu^{-1}(\dot{q})\}$
$\nu-$ *order*	$\dot{p}\dot{\leq}\dot{q} \Leftrightarrow \nu^{-1}(\dot{p}) \leq \nu^{-1}(\dot{q})$

The set of $\nu-$integers is

$$\mathbb{Z}_\nu = \mathbb{Z}(N) = \ldots, \nu(-1), \nu(0), \nu(1), \ldots.$$

The set $\mathbb{R}_\nu = \mathbb{R}(N) = \{\nu(a) : a \in \mathbb{R}\}$ is called the set of non-Newtonian real numbers.

The absolute value of non-Newtonian number $\dot{a} \in A \subset \mathbb{R}_\nu$ is denoted by $|\dot{a}|_N$ and define as follows;

$$|\dot{a}|_\nu = \begin{cases} \dot{a} & , \dot{a}\dot{>}\nu(0) \\ \nu(0) & , \dot{a} = \nu(0) \\ \nu(0)\dot{-}\dot{a} & , \dot{a}\dot{<}\nu(0) \end{cases}$$

Accordingly,

$$\sqrt{\dot{a}^{2_N}}^N = |\dot{a}|_N = \nu\left\{|\nu^{-1}(\dot{a})|\right\}$$

is written for each $\dot{u}$ in the set $A \subset \mathbb{R}_\nu$, [21].

Definition 1. *The non-Newtonian outer measure of a non-empty $\nu-$bounded set K is the greatest lower bound of the measures of all $\nu-$bounded, $\nu-$open sets containing the set K. So it is defined by*

$$m_N^* K =^\nu \inf_{K \subset G} \{m_N G\},$$

[22].

Definition 2. *The non-Newtonian interior measure of a nonempty $\nu-$bounded set K is the smallest upper bound on the measures of all $\nu-$closed sets contained in the set K. So it is defined by*

$$m_{*N} K =^\nu \sup_{F \subset K} \{m_N F\},$$

[22].

Theorem 1. *Given a $\nu-$bounded set K. If Δ is a $\nu-$open set containing the set K, then we have the following equation;*

$$m_N^* K \dot{+} m_{*N}\left[C_\Delta^K\right] = m_N \Delta,$$

[22].

Definition 3. *If the non-Newtonian inner and outer measure of a $\nu-$-bounded set K are equal, the set K is called a non-Newtonian Lebesgue-measurable set or simply the $\nu-$-measurable set, [22].*

Theorem 2. *If the set K is the $\nu-$measurable set in $\mathbb{R}_\nu$, then $\nu^{-1}(K)$ is the measurable set in $\mathbb{R}$, [22].*

Theorem 3. *Given a $\nu-$bounded set E. If the set E can be written as a combination of finite or countably infinite sets of pairwise disjoint $\nu-$measurable sets E_k, then E is $\nu-$measurable and*

$$m_N E =_\nu \sum_k m_N E_k$$

equality is fulfilled, [23].

2 Main Results

In this section, we introduce non-Newtonian isometry and examine some of its basic properties. We also give a characterization of the relationship between real isometry and non-Newtonian isometry. Finally, we show that the $\nu-$measure of $\nu-$measurable sets is invariant for every generator under $\nu-$isometries.

Definition 4. *Let $\varphi_\nu : \mathbb{R}_\nu \to \mathbb{R}_\nu$ be a function such that*

$$|\varphi_\nu(x) \dot{-} \varphi_\nu(y)|_N = |x \dot{-} y|_N$$

for every $x, y \in \mathbb{R}_\nu$, then the function φ_ν is called a non-Newtonian isometry or $\nu-$isometry.

Example 1. *Let $\varphi_\nu : \mathbb{R}_\nu \to \mathbb{R}^+(N)$ be a $\nu-$isometry and let the generator ν be the function* exp. *Hence, we have*

$$\begin{aligned}\left|\varphi_\nu(x) \dot{-} \varphi_\nu(y)\right|_N &= \nu\left\{\left|\nu^{-1}(\varphi_\nu(x)) - \nu^{-1}(\varphi_\nu(y))\right|\right\} \\ &= \exp\{|\ln(\varphi_\nu(x)) - \ln(\varphi_\nu(y))|\} \\ &= \exp\left\{\left|\ln\frac{\varphi_\nu(x)}{\varphi_\nu(y)}\right|\right\}.\end{aligned}$$

Also, we can write that

$$\begin{aligned}\left|x \dot{-} y\right|_N &= \nu\left\{\left|\nu^{-1}(x) - \nu^{-1}(y)\right|\right\} \\ &= \exp\{|\ln(x) - \ln(y)|\} \\ &= \exp\left\{\left|\ln\frac{x}{y}\right|\right\}.\end{aligned}$$

Thus, we get

$$\exp\left\{\left|\ln\frac{\varphi_\nu(x)}{\varphi_\nu(y)}\right|\right\} = \exp\left\{\left|\ln\frac{x}{y}\right|\right\}$$

and so

$$\left|\ln\frac{\varphi_\nu(x)}{\varphi_\nu(y)}\right| = \left|\ln\frac{x}{y}\right|.$$

Theorem 4. *Let $\nu : A \subseteq \mathbb{R} \to \mathbb{R}_\nu$ be the generator function and let $\varphi_\nu : \mathbb{R}_\nu \to \mathbb{R}_\nu$ be a non-Newtonian function. If φ_ν is an $\nu-$ isometry, then the function $\nu^{-1} \circ \varphi_\nu \circ \nu$ is an isometry in $A \subseteq \mathbb{R}$.*

Proof. Since the function φ_ν is an $\nu-$isometry, we have

$$\left|\varphi_\nu(x) \dot{-} \varphi_\nu(y)\right|_N = \left|x \dot{-} y\right|_N.$$

Thus, we write

$$\begin{aligned}&\nu\left\{\left|\nu^{-1}(\varphi_\nu(x)) - \nu^{-1}(\varphi_\nu(y))\right|\right\} \\ &\quad = \nu\left\{\left|\nu^{-1}(x) - \nu^{-1}(y)\right|\right\}\end{aligned}$$

and

$$\left|\nu^{-1}(\varphi_\nu(x)) - \nu^{-1}(\varphi_\nu(y))\right| = \left|\nu^{-1}(x) - \nu^{-1}(y)\right|.$$

This gives

$$\begin{aligned}&\left|\left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)\nu^{-1}(x) - \left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)\nu^{-1}(y)\right| \\ &= \left|\nu^{-1}(x) - \nu^{-1}(y)\right|\end{aligned}$$

which completes the proof. □

Theorem 5. *Let φ_ν be a $\nu-$isometry. Then, we have the following properties;*

a) If $A \subset B$, then $\varphi_\nu(A) \subset \varphi_\nu(B)$,

b) $\varphi_\nu\left(\bigcup_k E_k\right) = \bigcup_k \varphi_\nu(E_k)$,

c) $\varphi_\nu\left(\bigcap_k E_k\right) = \bigcap_k \varphi_\nu(E_k)$.

d) If E_0 is an empty set, then $\varphi_\nu(E_0) = E_0$.

Proof. Since φ_ν is a $\nu-$isometry then $\nu^{-1} \circ \varphi_\nu \circ \nu$ is an isometry.

a) If $A \subset B$, then $\nu^{-1}(A) \subset \nu^{-1}(B)$. Thus, we have

$$\begin{aligned}&\Rightarrow \left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)\left(\nu^{-1}(A)\right) \subset \left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)\left(\nu^{-1}(B)\right) \\ &\Rightarrow \left(\nu^{-1} \circ \varphi_\nu\right)(A) \subset \left(\nu^{-1} \circ \varphi_\nu\right)(B) \\ &\Rightarrow \varphi_\nu(A) \subset \varphi_\nu(B).\end{aligned}$$

b) Since $\nu^{-1}\left(\bigcup_k E_k\right) = \bigcup_k \nu^{-1}(E_k)$, then we get

$$\Rightarrow \left(\nu^{-1}\circ\varphi_\nu\circ\nu\right)\left(\nu^{-1}\left(\bigcup_k E_k\right)\right)$$
$$= \bigcup_k \left(\nu^{-1}\circ\varphi_\nu\circ\nu\right)\left(\nu^{-1}(E_k)\right)$$
$$\Rightarrow \left(\nu^{-1}\circ\varphi_\nu\right)\left(\bigcup_k E_k\right) = \bigcup_k \left(\nu^{-1}\circ\varphi_\nu\right)(E_k)$$
$$\Rightarrow \nu^{-1}\left(\varphi_\nu\left(\bigcup_k E_k\right)\right) = \nu^{-1}\left(\bigcup_k \varphi_\nu(E_k)\right)$$
$$\Rightarrow \varphi_\nu\left(\bigcup_k E_k\right) = \bigcup_k \varphi_\nu(E_k).$$

c) Since $\nu^{-1}\left(\bigcap_k E_k\right) = \bigcap_k \nu^{-1}(E_k)$ and $\nu^{-1}\circ\varphi_\nu\circ\nu$ is an isometry, we have

$$\Rightarrow \left(\nu^{-1}\circ\varphi_\nu\circ\nu\right)\left(\nu^{-1}\left(\bigcap_k E_k\right)\right)$$
$$= \bigcap_k \left(\nu^{-1}\circ\varphi_\nu\circ\nu\right)\left(\nu^{-1}(E_k)\right)$$
$$\Rightarrow \left(\nu^{-1}\circ\varphi_\nu\right)\left(\bigcap_k E_k\right) = \bigcap_k \left(\nu^{-1}\circ\varphi_\nu\right)(E_k)$$
$$\Rightarrow \nu^{-1}\left(\varphi_\nu\left(\bigcap_k E_k\right)\right) = \nu^{-1}\left(\bigcap_k \varphi_\nu(E_k)\right)$$
$$\Rightarrow \varphi_\nu\left(\bigcap_k E_k\right) = \bigcap_k \varphi_\nu(E_k).$$

d) Since $\nu^{-1}\circ\varphi_\nu\circ\nu$ is an isometry, we get

$$\Rightarrow \left(\nu^{-1}\circ\varphi_\nu\circ\nu\right)(E_0) = E_0$$
$$\Rightarrow \varphi_\nu\left(\nu(E_0)\right) = \nu(E_0)$$
$$\Rightarrow \varphi_\nu(E_0) = E_0.$$

□

Example 2. *Consider the geometric arithmetic generated by the function $\nu(x) = e^x$. If φ_ν is a $\nu-$isometry for $x, c \in \mathbb{R}^+$, then either*

$$\varphi_\nu(x) = x\dot{+}c = x.c$$

or

$$\varphi_\nu(x) = c\dot{-}x = \frac{c}{x}\ .$$

Proof. Let $\varphi_\nu(1) = c$. Then for every x,

- If $\varphi_\nu(x) = x.c$ then

$$\begin{aligned}|\varphi_\nu(x)\dot{-}\varphi_\nu(1)|_N &= |x.c\dot{-}c|_N \\ &= |\nu\left\{\nu^{-1}(x.c) - \nu^{-1}(c)\right\}|_N \\ &= |\exp\ \{\ln x\}|_N \\ &= |\exp\{\ln x - \ln 1\}|_N \\ &= |x\dot{-}1|_N\end{aligned}$$

and

- If $\varphi_\nu(x) = \frac{c}{x}$ then we have

$$\begin{aligned}|\varphi_\nu(x)\dot{-}\varphi_\nu(1)|_N &= \left|\frac{c}{x}\dot{-}c\right|_N \\ &= \left|\nu\left\{\nu^{-1}\left(\frac{c}{x}\right) - \nu^{-1}(c)\right\}\right|_N \\ &= \left|\exp\left\{\ln\frac{1}{x}\right\}\right|_N \\ &= |\exp\{\ln 1 - \ln x\}|_N \\ &= |\exp\{\ln x - \ln 1\}|_N \\ &= |x\dot{-}1|_N.\end{aligned}$$

Let define the following function;

$$\varphi_\nu(x) = x^{(-1)^{\sigma(x)}}.c \quad [\sigma(x) = 0, 1].$$

Let take x and y such that $x \neq 1$,$y \neq 1$ and $x \neq y$. Thus, we have

$$\begin{aligned}&\varphi_\nu(x)\dot{-}\varphi_\nu(y) = x^{(-1)^{\sigma(x)}}.c\dot{-}y^{(-1)^{\sigma(y)}}.c \\ &= \nu\left\{\nu^{-1}\left(x^{(-1)^{\sigma(x)}}.c\right) - \nu^{-1}\left(y^{(-1)^{\sigma(y)}}.c\right)\right\} \\ &= \exp\left\{\ln\left(x^{(-1)^{\sigma(x)}}.c\right) - \ln\left(y^{(-1)^{\sigma(y)}}.c\right)\right\} \\ &= \exp\left\{\ln\frac{x^{(-1)^{\sigma(x)}}}{y^{(-1)^{\sigma(y)}}}\right\} \\ &= \frac{x^{(-1)^{\sigma(x)}}}{y^{(-1)^{\sigma(y)}}} \\ &= \left(\frac{x}{y^p}\right)^{(-1)^{\sigma(x)}}\end{aligned}$$

Here, since $p = (-1)^{\sigma(y)-\sigma(x)}$, $p = -1 or 1$.
By the last equality, we get

$$\left|\left(\frac{x}{y^p}\right)^{(-1)^{\sigma(x)}}\right|_N = |x\dot{-}y|_N$$

and

$$\begin{aligned}\exp\left\{\left|\ln\left(\frac{x}{y^p}\right)^{(-1)^{\sigma(x)}}\right|\right\} &= \exp\left\{\left|(-1)^{\sigma(x)}.\ln\frac{x}{y^p}\right|\right\} \\ &= \exp\left\{\left|\ln\frac{x}{y^p}\right|\right\}.\end{aligned}$$

Also, we have

$$\exp\left\{\left|\ln\frac{x}{y^p}\right|\right\} = \exp\left\{\left|\ln\frac{x}{y}\right|\right\}$$
$$\left|\ln\frac{x}{y^p}\right| = \left|\ln\frac{x}{y}\right|$$

and so we get

$$\ln\frac{x}{y^p} = \ln\frac{x}{y}$$

or

$$\ln\frac{x}{y^p} = -\ln\frac{x}{y}.$$

But the second equality is impossible since

$$\begin{aligned}&\Rightarrow \ln x - \ln y^p = -\ln x + \ln y\\ &\Rightarrow 2\ln x = \ln y^p + \ln y\\ &\Rightarrow \ln x^2 = \ln y^{p+1}.\end{aligned}$$

which gives if $p = -1$, $x = y$ and if $p = 1$, $x = y$ which is a contradiction. Thus; we get

$$\begin{aligned}&\Rightarrow (-1)^{\sigma(y)-\sigma(x)} = 1\\ &\Rightarrow \sigma(y) - \sigma(x) = 0\\ &\Rightarrow \sigma(x) = \sigma(y)\end{aligned}$$

Therefore, the function $\sigma(x)$ be as follows;

$$\sigma(x) = \sigma \quad (\sigma = 0, 1), foreveryx \neq 1$$

Finally, since

$$\varphi_\nu(x) = x^{(-1)^\sigma}.c$$

and $\varphi_\nu(1) = c$ we get $x = 1$. □

Theorem 6. *Let $x \in \mathbb{R}_\nu$ and let φ_ν be a $\nu-$isometry. Then, there are some $c \in \mathbb{R}_\nu$ such that*

$$\varphi_\nu(x) = x\dot{+}c$$

or

$$\varphi_\nu(x) = c\dot{-}x\ .$$

Proof. Since φ_ν is a $\nu-$isometry, then the function $\nu^{-1} \circ \varphi_\nu \circ \nu$ is a isometry. Thus, we have $\left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)\left(\nu^{-1}(x)\right) = \nu^{-1}(x) + d$ or $\left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)\left(\nu^{-1}(x)\right) = -\nu^{-1}(x) + d$ for some $d \in \mathbb{R}$. Let $d = \nu^{-1}(c)$. Then, we get

$$\begin{aligned}&\Rightarrow \left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)\left(\nu^{-1}(x)\right) = \nu^{-1}(x) + \nu^{-1}(c)\\ &\Rightarrow \left(\nu^{-1} \circ \varphi_\nu\right)(x) = \nu^{-1}(x) + \nu^{-1}(c)\\ &\Rightarrow \nu\left(\left(\nu^{-1} \circ \varphi_\nu\right)(x)\right) = \nu\left\{\nu^{-1}(x) + \nu^{-1}(c)\right\}\\ &\Rightarrow \varphi_\nu(x) = x\dot{+}c\end{aligned}$$

or

$$\begin{aligned}&\Rightarrow \left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)\left(\nu^{-1}(x)\right) = -\nu^{-1}(x) + \nu^{-1}(c)\\ &\Rightarrow \left(\nu^{-1} \circ \varphi_\nu\right)(x) = \nu^{-1}(c) - \nu^{-1}(x)\\ &\Rightarrow \nu\left(\left(\nu^{-1} \circ \varphi_\nu\right)(x)\right) = \nu\left\{\nu^{-1}(c) - \nu^{-1}(x)\right\}\\ &\Rightarrow \varphi_\nu(x) = c\dot{-}x\end{aligned}$$

which completes the proof. □

Theorem 7. *If the function φ_ν is an $\nu-$ isometry, then its inverse is an $\nu-$ isometry.*

Proof. Since φ_ν is a $\nu-$isometry, then $\nu^{-1} \circ \varphi_\nu \circ \nu$ is an isometry. Since the inverse of isometry is also isometry; $\left(\nu^{-1} \circ \varphi_\nu \circ \nu\right)^{-1} = \nu^{-1} \circ \varphi_\nu^{-1} \circ \nu$ is an isometry. Thus, we get φ_ν^{-1} is a $\nu-$isometry. □

Theorem 8. *Under a $\nu-$isometry the following is true;*

a) Every $\nu-$open interval maps to an $\nu-$-open interval of the same measure, and the endpoints of the image interval are images of the endpoints of the original interval.

b) The image of a $\nu-$-bounded set is a $\nu-$-bounded set.

Proof.

a) Let $\Delta = (a, b)_N$ be a $\nu-$open interval. Let $\varphi_\nu(x) = x\dot{+}c$. Then, we have $\varphi_\nu(\Delta) = (a\dot{+}c, b\dot{+}c)_N$. Thus, we get

$$\begin{aligned}m_N\varphi_\nu(\Delta) &= m_N(a\dot{+}c, b\dot{+}c)_N\\ &= (b\dot{+}c)\dot{-}(a\dot{+}c)\\ &= \nu\{\nu^{-1}(b\dot{+}c) - \nu^{-1}(a\dot{+}c)\}\\ &= \nu\{\nu^{-1}(\nu(\nu^{-1}(b) + \nu^{-1}(c)))\\ &\quad - \nu^{-1}(\nu(\nu^{-1}(b) + \nu^{-1}(c)))\}\\ &= \nu\{\nu^{-1}(b) + \nu^{-1}(c) - \nu^{-1}(a) - \nu^{-1}(c)\}\\ &= \nu\{\nu^{-1}(b) - \nu^{-1}(a)\}\\ &= b\dot{-}a\\ &= m_n\Delta.\end{aligned}$$

Let $\varphi_\nu(x) = c\dot{-}x$. Then $\varphi_\nu(\Delta) = (c\dot{-}b, c\dot{-}a)_N$.

thus, we get

$$\begin{aligned}
&m_N\varphi_\nu(\Delta) = m_N(c\dot{-}b, c\dot{-}a)_N \\
&= (c\dot{-}a)\dot{-}(c\dot{-}b) \\
&= \nu\{\nu^{-1}(c\dot{-}a) - \nu^{-1}(c\dot{-}b)\} \\
&= \nu\{\nu^{-1}(\nu(\nu^{-1}(c) - \nu^{-1}(a))) \\
&- \nu^{-1}(\nu(\nu^{-1}(c) - \nu^{-1}(b)))\} \\
&= \nu\{\nu^{-1}(c) - \nu^{-1}(a) - \nu^{-1}(c) + \nu^{-1}(b)\} \\
&= \nu\{\nu^{-1}(b) - \nu^{-1}(a)\} \\
&= b\dot{-}a \\
&= m_n\Delta.
\end{aligned}$$

In both cases, we get

$$m_N\varphi_\nu(\Delta) = b\dot{-}a = m_n\Delta\ .$$

b) Let E be a $\nu-$bounded set and let Δ be a $\nu-$open interval contaning the set E. Then, we have

$$\varphi_\nu(E) \subset \varphi_\nu(\Delta)$$

and so the set $\varphi_\nu(E)$ is a $\nu-$bounded. Indeed, Since E is $\nu-$bounded, we have $|x|_N\dot{<}k$ for every $x \in E$. Then, for every $y \in \varphi_\nu(E)$, if $\varphi_\nu(x) = x\dot{+}c$, then

$$|y|_N = |x\dot{+}c|_N\dot{<}|x|_N\dot{+}|c|_N < k\dot{+}|c|_N$$

and if $\varphi_\nu(x) = c\dot{-}x$, then

$$|y|_N = |c\dot{-}x|_N = |x\dot{-}c|_N\dot{<}|x|_N\dot{+}|c|_N < k\dot{+}|c|_N.$$

which gives that the set $\varphi_\nu(E)$ is $\nu-$bounded. □

Theorem 9. *Under a $\nu-$isometry the following properties are true;*

a) The image of a $\nu-$-closed set is a $\nu-$-closed set.

b) The image of a $\nu-$-open set is a $\nu-$-open set.

Proof.

a) Let $\varphi_\nu(F)$ be the image of $\nu-$closed set F. Let y_0 be a $\nu-$limit point of the set $\varphi_\nu(F)$ and let y_n be a sequence such that

$${}^{\nu}\lim y_n = y_0 \qquad y_n \in \varphi_\nu(f_\nu).$$

Also, let define

$$x_0 = \varphi_\nu^{-1}(y_0), \qquad x_n = \varphi_\nu^{-1}(y_n)$$

and so $(x_n) \subset F$.
Since $\varphi_\nu$is a $\nu-$isometry, φ_ν^{-1} is a $\nu-$isometry. Thus, we have

$$\begin{aligned}
&\Rightarrow |x_n\dot{-}x_0|_N = |\varphi_\nu^{-1}(y_n)\dot{-}\varphi_\nu^{-1}(y_0)|_N \\
&\Rightarrow |x_n\dot{-}x_0|_N = |y_n\dot{-}y_0|_N
\end{aligned}$$

and so

$$x_n \overset{\nu}{\to} x_0.$$

Since F is a $\nu-$closed, $x_0 \in F$ and thus

$$y_0 = \varphi_\nu(x_0) \in \varphi_\nu(F).$$

which completes the proof.

b) Let G be a $\nu-$open set and let define

$$F = G^c.$$

Then, F is a $\nu-$closed set and

$$G \cup F = \mathbb{R}_\nu, \quad G \cap F = \emptyset.$$

Thus, we get

$$\varphi_\nu(G \cup f_\nu) = \varphi_\nu(\mathbb{R}_\nu), \quad \varphi_\nu(G \cap f_\nu) = \varphi_\nu(\emptyset)$$

$$\varphi_\nu(G) \cup \varphi_\nu(f_\nu) = \mathbb{R}_\nu, \quad \varphi_\nu(G) \cap \varphi_\nu(f_\nu) = \emptyset.$$

which shows that $\varphi_\nu(G)$ is complemet of $\varphi_\nu(F)$ $\nu-$closed. This completes the proof. □

Theorem 10. *The $\nu-$measure of a $\nu-$bounded open set is invariant under all $\nu-$isometries.*

Proof. Let G be a $\nu-$bounded open set. Then, $\varphi_\nu(G)$ is a $\nu-$bounded open set. Let $\delta_k = (a_k, b_k)_N$ $(k = 1, 2, \cdots)$ and let define

$$G = \bigcup_k \delta_k.$$

Thus, we have

$$\varphi_\nu(G) = \varphi_\nu\left(\bigcup_k \delta_k\right) = \bigcup_k \varphi_\nu(\delta_k).$$

Therefore, we get

$$\begin{aligned}
m_N\varphi_\nu(G) &=_N \sum_k m_N(\varphi_\nu(\delta_k)) \\
&=_N \sum_k m_N\varphi_\nu((a_k, b_k)_N) \\
&=_N \sum_k b_k\dot{-}a_k \\
&=_N \sum_k m_N\delta_k \\
&= m_N G.
\end{aligned}$$

This gives that

$$\begin{aligned}
m_N\varphi_\nu(G) &=_N \sum_k m_N\varphi_\nu(\delta_k) \\
&=_N \sum_k m_N\delta_k \\
&= m_N G.
\end{aligned}$$

□

Theorem 11. *$\nu-$isometries do not change the $\nu-$outer and $\nu-$inner measures of a $\nu-$bounded set.*

Proof.

a) Let E be a $\nu-$bounded set. For every $\epsilon\dot{>}\dot{0}$, there is a $\nu-$bounded open set G such that

$$G \subset E, \quad m_N G \dot{<} m_N^* E \dot{+} \epsilon.$$

Then, $\varphi_\nu(G)$ is $\nu-$bounded open set containing the set $\varphi_\nu(E)$. Thus, we get

$$m_N^* \varphi_\nu(E) \dot{\leq} m_N \varphi_\nu(G) = m_N G \dot{<} m_N^* E \dot{+} \epsilon$$

which gives

$$m_N^* \varphi_\nu(E) \dot{\leq} m_N^* E.$$

This shows that the $\nu-$outer measure of a $\nu-$-bounded set does not increase under a $\nu-$isometry. Otherwise, the $\nu-$inverse isometry is non-decreasing since it leads to an increase in the $\nu-$outer measure. Therefore we get

$$m_N^* \varphi_\nu(E) = m_N^* E.$$

b) Let Δ be a $\nu-$open interval contaning E. Then, $\varphi_\nu(\Delta)$ is a $\nu-$open interval contaning $\varphi_\nu(E)$. Let

$$A = C_\Delta^E.$$

Since

$$E \cup A = \Delta, \quad E \cap A = \emptyset$$

we have

$$\varphi_\nu(E) \cup \varphi_\nu(A) = \varphi_\nu(\Delta), \quad \varphi_\nu(E) \cap \varphi_\nu(A) = \emptyset.$$

Thus, we write

$$m_N^* \varphi_\nu(A) \dot{+} m_{*N} \varphi_\nu(E) = m_N \varphi_\nu(\Delta)$$

and so

$$m_N^* A \dot{+} m_{*N} \varphi_\nu(E) = m_N \Delta$$

This shows that

$$m_{*N} \varphi_\nu(E) = m_N \Delta \dot{-} m_N^* \left[C_\Delta^E\right].$$

Finally, we get

$$\Rightarrow m_{*N} \varphi_\nu(E) = m_N^* \left[C_\Delta^E\right] \dot{+} m_{*N} E \dot{-} m_N^* \left[C_\Delta^E\right]$$
$$\Rightarrow m_{*N} \varphi_\nu(E) = m_{*N} E.$$

□

3 Conclusion

In this article, we have introduced $\nu-$isometry and gave some of its properties using examples. First, we showed that the necessary and sufficient condition for the function φ_ν to be a $\nu-$isometry is that the function $\nu^{-1} \circ \varphi_\nu \circ \nu$ is a real isometry. Using this theorem, we showed that the inverse of a $\nu-$isometry is a $\nu-$isometry. Finally, we show that the $\nu-$measure of $\nu-$measurable sets is invariant for every generator under $\nu-$isometries.

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