A New Method for Solving a Neutral Functional-Differential Equation with Proportional Delays

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Abstract: - This study presents and implements a new hybrid technique that combines the Sawi transform (ST) and Homotopy perturbation method (HPM) to solve neutral functional-differential equations with proportional delays. Some of the important properties of the method are established and validated. We start the method by first applying ST to obtain the recurrence relation. We, next, implement HPM to find convergent series solutions of the recurrence relation. The series is free of assumptions and restrictions, highlighting its adaptability and robustness. Moreover, the convergence of the method is established through convincing proof. To demonstrate its effectiveness and applicability, we provide five examples. The method yields accurate approximate solutions, or in some cases exact solutions, with a few number of iterations, reinforcing its reliability and validity. Moreover, the performance of the method is compared with some available methods and demonstrates its superiority and efficiency.

Key-Words: - Sawi transform, Integral transform, Homotopy perturbation method, Proportional delay, Pantograph equations, Neutral functional-differential equations.

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1 Introduction

Functional–differential equations (FDEs) with proportional delays are usually indicated as pantograph equations. The term "pantograph" was first introduced by Ockendon and Tayler in their study, [1]. These equations frequently appear in industry and studies based on economy, biology, electrodynamics, and control theory among others, [2]. One noteworthy characteristic of such equations is the presence of compactly supported solutions, [3]. Pantograph equations play a significant role in describing various phenomena and are distinguished by the presence of a linear functional argument. They become especially essential when the ODEs-based models fail. Recently, numerous methods have been established for solving pantograph differential equations including the Taylor operation method, perturbation iteration algorithms, and the Adomian decomposition method, [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27].

HPM, [28], [29], is an efficient and reliable technique for solving enormous classes of differential problems. HPM does not require any discretization, perturbation, or linearization. This strategy yields the solution in the form of a polynomial. Lately, HPM has been utilized to obtain approximate solutions for numerous classes of differential problems, [30], [31], [32]. It also has been demonstrated that HPM is a fast and trustworthy method compared to other methods. More details regarding HPM can be found in, [33], and references within.

In 2019, the work of, [34] proposed a new integral transform, known as the Sawi transform (ST). This transform is very simple to implement, requiring no assumptions in its procedure, [35]. Recently, the Sawi transform has been widely utilized for solving various integral and differential equations, [35], [36], [37], [38], [39], [40], [41].

In this study, we combined the ST and HPM to create the strategy of the Sawi homotopy perturbation transform method (SHPTM) and find the analytic results of FDEs with proportional delays, in convergent series form. HPM is utilized to handle delay components. ST is used to minimize the computational work and to provide more accurate results. We observe that HPM is an efficient method for addressing various phenomena. Outcomes demonstrate that the approach is simple and easy to employ.

2 Sawi Transform

In this section, we outline the fundamental concepts and properties of the Sawi transform, [42], [43].

Definition 1. The Sawi transform for a function $\rho(\tau)$ is given by

$$S[\rho(\tau)] = \wp[\vartheta] = \frac{1}{\vartheta^2}\int_0^\infty \rho(\tau)\, e^{-\frac{\tau}{\vartheta}} d\tau,$$

where S is designated as Sawi transform. If $\wp[\vartheta]$ is the Sawi transform of a function $\rho(\tau)$, then $\rho(\tau)$ is the inverse of $\wp[\vartheta]$ so that,

$\wp^{-1}[\vartheta] = \rho(\tau)$, where $\wp^{-1}$ is said to be inverse Sawi transform.

Definition 2. If $S[\rho_1(\tau)] = \wp_1[\vartheta]$ and $S[\rho_2(\tau)] = \wp_2[\vartheta]$, then

$$S[\delta\rho_1(\tau) + \varepsilon\rho_2(\tau)] = \delta\wp_1[\vartheta] + \varepsilon\wp_2[\vartheta],$$

where δ and ε are arbitrary constants.

Table 1. Sawi Transforms of Some Fundamental Functions

	$\rho(\tau)$	$S[\rho(\tau)] = \wp[\vartheta]$
1.	1	$1/\vartheta$
2.	τ	1
3.	τ^2	2ϑ
4.	τ^n	$n!\,\vartheta^{n-1}$
5.	τ^γ	$\Gamma(\gamma+1)\vartheta^{\gamma-1}$
6.	$e^{\delta\tau}$	$1/(\vartheta(1-\delta\vartheta))$
7.	$\sin\delta\tau$	$\delta/((1+\delta^2\vartheta^2))$
8.	$\cos\delta\tau$	$1/(\vartheta(1+\delta^2\vartheta^2))$
9.	$\sinh\delta\tau$	$\delta/((1-\delta^2\vartheta^2))$
10.	$\cosh\delta\tau$	$1/(\vartheta(1-\delta^2\vartheta^2))$

Definition 3. If $S[\rho(\tau)] = \wp[\vartheta]$, we can consider the following differential properties as

i. $S[\rho'(\tau)] = \frac{\wp[\vartheta]}{\vartheta} - \frac{\rho(0)}{\vartheta^2}$,

ii. $S[\rho''(\tau)] = \frac{\wp[\vartheta]}{\vartheta^2} - \frac{\rho(0)}{\vartheta^3} - \frac{\rho'(0)}{\vartheta^2}$,

iii. $S\left[\rho^{(m)}(\tau)\right] = \frac{\wp[\vartheta]}{\vartheta^m} - \sum_{r=0}^{m-1}\frac{\rho^{(r)}(0)}{\vartheta^{m+1-r}}$.

Table 1 provides Sawi transforms of various essential functions that are useful in solving significant issues in the fields of science and engineering.

3 Method of Solution

In this section, we introduce the new SHPTM to solve the following FDEs with proportional delays, [23],

$$\left(\rho(\tau) + a(\tau)\rho(\zeta_\nu\tau)\right)^{(\nu)} = \alpha\rho(\tau) + \sum_{r=0}^{\nu-1} b_r(\tau)\rho^{(r)}(\zeta_r\tau) + f(\tau), \tau \geq 0, \tag{1}$$

with the initial conditions

$\rho^{(r)}(0) = \lambda_r, r = 0,1,\cdots,\nu-1,$

where a, f, and b_r ($r = 0,1,\cdots,\nu-1$) denote given analytic functions, and $\alpha, \zeta_r, \lambda_r$ designate given constants, with $0 < \zeta_r < 1$ for $r = 0,1,\cdots,\nu$. Notably, this approach does not rely on integration or any assumptions in its formulation. We express (1) as:

$$\rho^{(\nu)}(\tau) = \alpha\rho(\tau) - \left(a(\tau)\rho(\zeta_\nu\tau)\right)^{(\nu)} + \sum_{r=0}^{\nu-1} b_r(\tau)\rho^{(r)}(\zeta_r\tau) + f(\tau), \tau \geq 0. \tag{2}$$

Applying ST to (2) yields:

$$S\left[\rho^{(\nu)}(\tau)\right] = S\Big[\alpha\rho(\tau) - \left(a(\tau)\rho(\zeta_\nu\tau)\right)^{(\nu)} + \sum_{r=0}^{\nu-1} b_r(\tau)\rho^{(r)}(\zeta_r\tau) + f(\tau)\Big] \tag{3}$$

Using ST properties, we have:

$$\frac{\wp[\vartheta]}{\vartheta^m} - \frac{\rho(0)}{\vartheta^{m+1}} - \frac{\rho'(0)}{\vartheta^m} - \cdots - \frac{\rho^{(m-1)}(0)}{\vartheta^2} = \alpha\wp[\vartheta] + S\Big[-\left(a(\tau)\rho(\zeta_\nu\tau)\right)^{(\nu)} + \sum_{r=0}^{\nu-1} b_r(\tau)\rho^{(r)}(\zeta_r\tau) + f(\tau)\Big]. \tag{4}$$

Thus, using the initial conditions in (4), we obtain:

$$\wp[\vartheta] = \frac{\lambda_0}{\vartheta(1-\alpha\vartheta^m)} + \frac{\lambda_1}{(1-\alpha\vartheta^m)} + \cdots + \frac{\vartheta^{m-1}}{(1-\alpha\vartheta^m)}\lambda_{m-1} + \frac{\vartheta^m}{(1-\alpha\vartheta^m)} S\Big[-\left(a(\tau)\rho(\zeta_\nu\tau)\right)^{(\nu)} + \sum_{r=0}^{\nu-1} b_r(\tau)\rho^{(r)}(\zeta_r\tau) + f(\tau). \tag{5}$$

Operating inverse ST on (5), we get:

$$\rho(\tau) = \mu(\tau) + S^{-1}\left[\frac{\vartheta^m}{(1-\alpha\vartheta^m)} S\left[\left(a(\tau)\rho(\zeta_\nu\tau)\right)^{(\nu)} + \sum_{r=0}^{\nu-1} b_r(\tau)\rho^{(r)}(\zeta_r\tau)\right]\right] \tag{6}$$

where

$$\mu(\tau) = S^{-1}\left[\frac{\lambda_0}{\vartheta(1-\alpha\vartheta^m)} + \frac{\lambda_1}{(1-\alpha\vartheta^m)} + \cdots + \frac{\vartheta^{m-1}}{(1-\alpha\vartheta^m)}\lambda_{m-1}\right] + \frac{\vartheta^m}{(1-\alpha\vartheta^m)} S[f(\tau)].$$

Let us introduce HPM as

$$\rho(\tau) = \sum_{r=0}^{\infty} p^r \rho_r(\tau). \tag{7}$$

Substituting (7) in (6) and matching terms with the same power of p, we obtain:

$p^0: \rho_0(\tau) = \mu(\tau)$,

$p^1: \rho_1(\tau) = S^{-1}\left[\frac{\vartheta^m}{(1-\alpha\vartheta^m)} S\left[-\left(a(\tau)\rho_0(\zeta_\nu\tau)\right)^{(\nu)} + \sum_{r=0}^{\nu-1} b_r(\tau){\rho_0}^{(r)}(\zeta_r\tau)\right]\right]$,

$p^2: \rho_2(\tau) = S^{-1}\left[\frac{\vartheta^m}{(1-\alpha\vartheta^m)} S\left[-\left(a(\tau)\rho_1(\zeta_\nu\tau)\right)^{(\nu)} + \sum_{r=0}^{\nu-1} b_r(\tau){\rho_1}^{(r)}(\zeta_r\tau)\right]\right]$,

$p^3: \rho_3(\tau) = S^{-1}\left[\frac{\vartheta^m}{(1-\alpha\vartheta^m)} S\left[-\left(a(\tau)\rho_2(\zeta_\nu\tau)\right)^{(\nu)}\right.\right.$

$+\sum_{r=0}^{\nu-1} b_r(\tau)\rho_2{}^{(r)}(\zeta_r\tau)\big]\,\big],$
$\vdots$

Following this process, we can calculate $\rho_0(\tau), \rho_1(\tau), \rho_2(\tau)$, and so on. These functions can be combined to derive a solution for (1) as:

$$\rho(\tau) = \lim_{p\to 1} \sum_{r=0}^{\infty} p^r \rho_r(\tau) = \rho_0(\tau) + \rho_1(\tau) + \cdots \quad (8)$$

To prove the convergence of the solution in (8), we illustrate the following theorem.

Theorem. Assume that Ω and Υ are Banach spaces and $\mathrm{K}: \Omega \longrightarrow \Upsilon$ is a contraction function, that is
$\forall \delta, \tilde{\delta} \in \Omega;\ \|\mathrm{K}(\delta) - \mathrm{K}(\tilde{\delta})\| \le \varepsilon\|\delta - \tilde{\delta}\|, 0 < \varepsilon < 1.$
Then, according to Banach's fixed point theorem, the existence of a unique fixed point γ is guaranteed. Moreover, suppose that the sequence produced by the HPM can be written as
$\mu_i = \mathrm{K}(\mu_{i-1}),\ \mu_{i-1} = \sum_{j=0}^{i-1} \rho_j, \rho_j \in \Omega, i = 1,2,\cdots,$
and suppose that $\mu_0 = \rho_0 \in B_\alpha(\gamma)$ where $B_\alpha(\gamma) = \{\rho^* \in \Omega: \|\rho^* - \gamma\| \le \alpha\}$, then we have

a. $\|\mu_i - \gamma\| \le \varepsilon^i \|\rho_0 - \gamma\|$,
b. $\mu_i \in B_\alpha(\gamma)$,
c. $\lim_{i\to\infty} \mu_i = \rho$.

Proof.
a. By the mathematical induction method on i, for $i = 1$ we have
$\|\mu_1 - \gamma\| = \|\mathrm{K}(\rho_0) - \mathrm{K}(\gamma)\| \le \varepsilon\|\rho_0 - \gamma\|.$
Assume that $\|\mu_i - \gamma\| \le \varepsilon^i\|\rho_0 - \gamma\|$ for some $i \in \mathbb{N}$ as an induction assumption, then
$$\|\mu_{i+1} - \gamma\| = \|\mathrm{K}(\rho_i) - \mathrm{K}(\gamma)\| \le \varepsilon\|\rho_i - \gamma\| \le \varepsilon\, \varepsilon^i\|\rho_0 - \gamma\| = \varepsilon^{i+1}\|\rho_0 - \gamma\|.$$
Hence
$$\|\mu_{i+1} - \gamma\| \le \varepsilon^{i+1}\|\rho_0 - \gamma\|.$$
b. Using (a), we have
$\|\mu_i - \gamma\| \le \varepsilon^i\|\rho_0 - \gamma\| \le \varepsilon^i\alpha < \alpha.$
Therefor, $\mu_i \in B_\alpha(\gamma)$.
c. Since $\|\mu_i - \gamma\| \le \varepsilon^i\|\rho_0 - \gamma\|$, and $\lim_{i\to\infty} \varepsilon^i = 0$ (as $0 < \varepsilon < 1$), we have $\lim_{i\to\infty}\|\mu_i - \gamma\| = 0$, that is, $\lim_{i\to\infty} \mu_i = \rho$. ■

4 Numerical Examples

In this section, we provide several examples to demonstrate the effectiveness of the method introduced in Section 3. All examples are implemented using *MATHEMATICA* 12.

Example 1. Consider the following first-order neutral FDEs with proportional delays:

$$\rho'(\tau) = -\rho(\tau) + \frac{1}{2}\rho\left(\frac{\tau}{2}\right) + \frac{1}{2}\rho'\left(\frac{\tau}{2}\right), \quad (9)$$
$$0 < \tau < 1, \rho(0) = 1$$

Applying ST to (9), we get:
$$S[\rho'(\tau)] = S\left[-\rho(\tau) + \frac{1}{2}\rho\left(\frac{\tau}{2}\right) + \frac{1}{2}\rho'\left(\frac{\tau}{2}\right)\right]. \quad (10)$$

Using the properties of ST, we have:
$$\frac{\wp[\vartheta]}{\vartheta} - \frac{1}{\vartheta^2} = -\wp[\vartheta] + S\left[\frac{1}{2}\rho\left(\frac{\tau}{2}\right) + \frac{1}{2}\rho'\left(\frac{\tau}{2}\right)\right]. \quad (11)$$

Thus, $\wp[\vartheta]$ yields:
$$\wp[\vartheta] = \frac{1}{\vartheta(1+\vartheta)} + \frac{\vartheta}{(1+\vartheta)}\, S\left[\frac{1}{2}\rho\left(\frac{\tau}{2}\right) + \frac{1}{2}\rho'\left(\frac{\tau}{2}\right)\right]. \quad (12)$$

Operating inverse ST on (12), we obtain:
$$\rho(\tau) = e^{-\tau} + S^{-1}\left[\frac{\vartheta}{(1+\vartheta)}\, S\left[\frac{1}{2}\rho\left(\frac{\tau}{2}\right) + \frac{1}{2}\rho'\left(\frac{\tau}{2}\right)\right]\right]. \quad (13)$$

Substituting (7) into (13), we get:
$$\sum_{r=0}^{\infty} p^r\rho_r(\tau) = e^{-\tau} + S^{-1}\left[\frac{\vartheta}{(1+\vartheta)}\, S\left[\frac{1}{2}\sum_{r=0}^{\infty} p^r\rho_r\left(\frac{\tau}{2}\right) + \frac{1}{2}\sum_{r=0}^{\infty} p^r\rho_r'\left(\frac{\tau}{2}\right)\right]\right] \quad (14)$$

Equating the coefficients of p that have the same exponent leads to:
$p^0: \rho_0(\tau) = e^{-\tau}$,
which is the analytical solution of (9).

Example 2. Consider the following first-order neutral FDEs with proportional delays:
$$\rho'(\tau) = -\rho(\tau) + 0.1\rho(0.8\tau) + 0.5\rho'(0.8\tau) + (0.32\tau - 0.5)e^{-0.8\tau} + e^{-\tau},\ \tau \ge 0, \rho(0) = 0. \quad (15)$$

Applying ST on (15), we have:
$$S[\rho'(\tau)] = S[-\rho(\tau) + 0.1\rho(0.8\tau) + 0.5\rho'(0.8\tau) + (0.32\tau - 0.5)e^{-0.8\tau} + e^{-\tau}]. \quad (16)$$

Using the properties of ST, we get:
$$\frac{\wp[\vartheta]}{\vartheta} = -\wp[\vartheta] + \frac{1}{2\vartheta} - \frac{1}{1+\vartheta} + \frac{0.5}{1.25+\vartheta} + \frac{0.5}{(1.25+\vartheta)^2} + S[0.1\rho(0.8\tau) + 0.5\rho'(0.8\tau)]. \quad (17)$$

Thus, $\wp[\vartheta]$ yields:
$$\wp[\vartheta] = \frac{\vartheta}{1+\vartheta}\left(\frac{1}{2\vartheta} - \frac{1}{1+\vartheta} + \frac{0.5}{1.25+\vartheta} + \frac{0.5}{(1.25+\vartheta)^2}\right) + \frac{\vartheta}{1+\vartheta} S[0.1\rho(0.8\tau) + 0.5\rho'(0.8\tau)]. \quad (18)$$

Operating inverse ST on (18), we get:
$$\rho(\tau) = (10.5 + \tau)e^{-\tau} + (-10.5 + 1.6\tau)e^{-0.8\tau} + S^{-1}\left[\frac{\vartheta}{1+\vartheta} S[0.1\rho(0.8\tau) + 0.5\rho'(0.8\tau)]\right]. \quad (19)$$

Substituting (7) into (19), we get:

$$\sum_{r=0}^{\infty} p^r \rho_r(\tau) = (10.5+\tau)e^{-\tau} + (-10.5 + 1.6\tau)e^{-0.8\tau} + S^{-1}\left[\frac{\vartheta}{1+\vartheta} S\left[0.1\sum_{r=0}^{\infty} p^r \rho_r(0.8\tau) + \frac{\tau}{2}\sum_{r=0}^{\infty} p^r \rho_r'(0.8\tau)\right]\right]. \tag{20}$$

Matching terms with the same power of p, we obtain:

p^0: $\rho_0(\tau) = (10.5+\tau)e^{-\tau} + (-10.5 + 1.6\tau)e^{-0.8\tau}$,

p^1: $\rho_1(\tau) - 3.43519e^{-\tau} + e^{-0.8\tau}(-10.5 - 1.6\tau) + e^{-0.64\tau}(13.9352 - 1.06667\tau)$,

p^2: $\rho_2(\tau) = -2.27171e^{-\tau} + 6.87037e^{-0.8\tau} + e^{-0.512\tau}(-8.16347 + 0.384699\tau) + e^{-0.64\tau}(3.56481 + 1.06667\tau)$,

$\vdots$

In Table 2 we compare the absolute errors of the 5 and 6-term solutions with those of the VIM, [23], HPM, [24], two-stage order-one Runge-Kutta method (RK method), [25], and the one-legθ-method (legθ-method), [26], [27]. In Figure 1 we show the comparison of the 3, 4, and 5-term solutions with the exact solution $\rho(\tau) = \tau e^{-\tau}$.

Example 3. Consider the following second-order neutral FDEs with proportional delays:

$$\rho''(\tau) = \rho'\left(\frac{\tau}{2}\right) - \frac{1}{2}\tau\rho''\left(\frac{\tau}{2}\right) + 2, 0 < \tau < 1, \quad \rho(0) = 1, \rho'(0) = 0. \tag{21}$$

Applying ST on (21), we have:

$$S[\rho''(\tau)] = S\left[\rho'\left(\frac{\tau}{2}\right) - \frac{1}{2}\tau\rho''\left(\frac{\tau}{2}\right) + 2\right]. \tag{22}$$

Utilizing the properties of ST, we get:

Table 2. Comparison of the absolute errors for Example 2.

τ_i	legθ-method	RK method	VIM		HPM		SHPTM	
			$n=5$	$n=6$	$n=5$	$n=6$	5-term solution	6-term solution
0.1	4.65×10^{-3}	8.68×10^{-4}	2.62×10^{-3}	1.30×10^{-3}	2.17×10^{-3}	1.06×10^{-3}	1.29×10^{-3}	6.45×10^{-4}
0.2	1.45×10^{-2}	1.49×10^{-3}	4.36×10^{-3}	2.14×10^{-3}	2.87×10^{-3}	1.35×10^{-3}	2.14×10^{-3}	1.05×10^{-3}
0.3	2.57×10^{-2}	1.90×10^{-3}	5.40×10^{-3}	2.63×10^{-3}	2.63×10^{-3}	1.18×10^{-3}	2.62×10^{-3}	1.28×10^{-3}
0.4	3.60×10^{-2}	2.16×10^{-3}	5.89×10^{-3}	2.84×10^{-3}	1.83×10^{-3}	7.61×10^{-4}	2.83×10^{-3}	1.37×10^{-3}
0.5	4.43×10^{-2}	2.28×10^{-3}	5.96×10^{-3}	2.83×10^{-3}	7.67×10^{-4}	2.32×10^{-4}	2.83×10^{-3}	1.35×10^{-3}
0.6	5.03×10^{-2}	2.31×10^{-3}	5.71×10^{-3}	2.67×10^{-3}	3.33×10^{-4}	2.98×10^{-4}	2.66×10^{-3}	1.25×10^{-3}
0.7	5.37×10^{-2}	2.27×10^{-3}	5.23×10^{-3}	2.39×10^{-3}	1.35×10^{-3}	7.64×10^{-4}	2.39×10^{-3}	1.10×10^{-3}
0.8	5.47×10^{-2}	2.17×10^{-3}	4.59×10^{-3}	2.04×10^{-3}	2.20×10^{-3}	1.12×10^{-3}	2.03×10^{-3}	9.20×10^{-4}
0.9	5.35×10^{-2}	2.03×10^{-3}	3.84×10^{-3}	1.64×10^{-3}	2.82×10^{-3}	1.37×10^{-3}	1.63×10^{-3}	7.11×10^{-4}
1.0	5.03×10^{-2}	1.86×10^{-3}	3.04×10^{-3}	1.22×10^{-3}	1.22×10^{-3}	3.21×10^{-3}	1.21×10^{-3}	4.93×10^{-4}

Table 3. Comparison of the absolute errors for Example 4.

τ_i	legθ-method	RK method	VIM		HPM		SHPTM	
			$n=5$	$n=6$	$n=5$	$n=6$	5-term solution	6-term solution
0.1	6.10×10^{-3}	1.00×10^{-3}	3.34×10^{-4}	1.67×10^{-4}	3.33×10^{-4}	1.67×10^{-4}	1.66×10^{-4}	8.34×10^{-5}
0.2	2.58×10^{-2}	2.02×10^{-3}	1.43×10^{-3}	7.15×10^{-4}	1.42×10^{-3}	7.15×10^{-4}	7.13×10^{-4}	3.57×10^{-4}
0.3	6.47×10^{-2}	3.07×10^{-3}	3.45×10^{-3}	1.73×10^{-3}	3.44×10^{-3}	1.18×10^{-3}	1.71×10^{-3}	8.59×10^{-4}
0.4	1.37×10^{-1}	4.17×10^{-3}	6.58×10^{-3}	3.30×10^{-3}	6.57×10^{-3}	7.61×10^{-4}	3.26×10^{-3}	1.63×10^{-3}
0.5	2.81×10^{-1}	5.34×10^{-3}	1.11×10^{-2}	5.55×10^{-3}	1.10×10^{-2}	5.55×10^{-3}	5.45×10^{-3}	2.73×10^{-3}

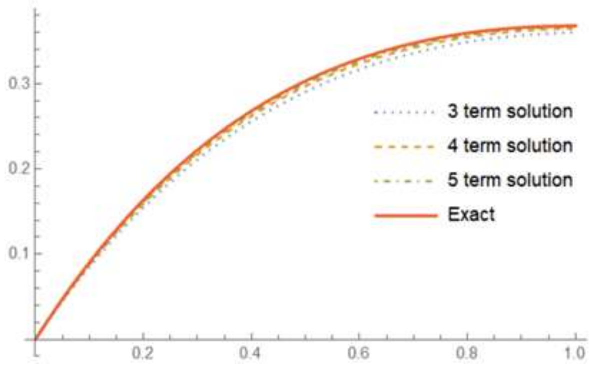

Fig. 1: Comparison of the exact solution with the approximate solutions for Example 2.

$$\frac{\wp[\vartheta]}{\vartheta^2}-\frac{1}{\vartheta^3}=\frac{2}{\vartheta}+S\left[\rho'\left(\frac{\tau}{2}\right)-\frac{1}{2}\tau\rho''\left(\frac{\tau}{2}\right)\right]. \quad (23)$$

Therefore, $\wp[\vartheta]$ yields:

$$\wp[\vartheta]=\frac{1}{\vartheta}+2\vartheta+\vartheta^2 S\left[\rho'\left(\frac{\tau}{2}\right)-\frac{1}{2}\tau\rho''\left(\frac{\tau}{2}\right)\right]. \quad (24)$$

Upon applying inverse ST to (24), we obtain:

$$\rho(\tau)=1+\tau^2+S^{-1}\left[\vartheta^2 S\left[\rho'\left(\frac{\tau}{2}\right)-\frac{1}{2}\tau\rho''\left(\frac{\tau}{2}\right)\right]\right]. \quad (25)$$

Substituting (7) into (25) and equating the coefficients of p with the same exponent gives:
$p^0: \rho_0(\tau)=1+\tau^2$,
which coincides with the analytical solution of (21).

Example 4. Consider the following second-order neutral FDEs with proportional delays:

$$\rho''(\tau)=\frac{3}{4}\rho(\tau)+\rho\left(\frac{\tau}{2}\right)+\rho'\left(\frac{\tau}{2}\right)+\frac{1}{2}\rho''\left(\frac{\tau}{2}\right)-\tau^2-\tau+1, 0<\tau<1, \rho(0)=0, \rho'(0)=0. \quad (26)$$

Using ST on (26), and utilizing the properties of ST, we have:

$$\frac{\wp[\vartheta]}{\vartheta^2}=\frac{3}{4}\wp[\vartheta]-2\vartheta-1+\frac{1}{\vartheta}+S\left[\rho\left(\frac{\tau}{2}\right)+\rho'\left(\frac{\tau}{2}\right)+\frac{1}{2}\rho''\left(\frac{\tau}{2}\right)\right]. \quad (27)$$

Thus, $\wp[\vartheta]$ yields:

$$\wp[\vartheta]=\frac{4\vartheta^2}{4-3\vartheta^2}\left(-2\vartheta-1+\frac{1}{\vartheta}\right)+\frac{4\vartheta^2}{4-3\vartheta^2}S\left[\rho\left(\frac{\tau}{2}\right)+\rho'\left(\frac{\tau}{2}\right)+\frac{1}{2}\rho''\left(\frac{\tau}{2}\right)\right]. \quad (28)$$

Applying inverse ST to (28), we obtain:

$$\rho(\tau)=\frac{4}{9}\left(3\tau^2+3\tau+5-2\sqrt{3}\sinh\frac{\sqrt{3}\tau}{2}-5\cosh\frac{\sqrt{3}\tau}{2}\right)+S^{-1}\left[\frac{4\vartheta^2}{4-3\vartheta^2}S\left[\rho\left(\frac{\tau}{2}\right)+\rho'\left(\frac{\tau}{2}\right)+\frac{1}{2}\rho''\left(\frac{\tau}{2}\right)\right]\right]. \quad (29)$$

Substituting (7) into (29) and matching the coefficients of p with the same exponent results in:

$p^0: \rho_0(\tau)=\frac{4}{9}\left(3\tau^2+3\tau+5-2\sqrt{3}\sinh\frac{\sqrt{3}\tau}{2}-5\cosh\frac{\sqrt{3}\tau}{2}\right)$,

$p^1: \rho_1(\tau)=-\frac{4}{81}\left(9\tau^2+54\tau+156-84\sqrt{3}\sinh\frac{\sqrt{3}\tau}{4}+6\sqrt{3}\sinh\frac{\sqrt{3}\tau}{4}-158\cosh\frac{\sqrt{3}\tau}{2}+2\cosh\frac{\sqrt{3}\tau}{2}\right)$,

$p^2: \rho_2(\tau)=\frac{4}{3645}\left(135\tau^2+2160\tau+13500-8408\sqrt{3}\sinh\frac{\sqrt{3}\tau}{8}+740\sqrt{3}\sinh\frac{\sqrt{3}\tau}{4}+292\sqrt{3}\sinh\frac{\sqrt{3}\tau}{2}-15092\cosh\frac{\sqrt{3}\tau}{8}+940\cosh\frac{\sqrt{3}\tau}{4}+652\cosh\frac{\sqrt{3}\tau}{2}\right)$,

$\vdots$

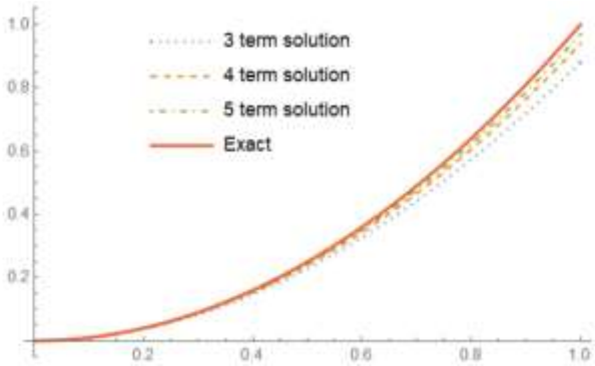

Fig. 2: Comparison of the exact solution with the approximate solutions for Example 4.

In Table 3 we compare the absolute errors of the 5 and 6-term solutions with the VIM, [23], HPM, [24], RK method, [25], and the legθ-method, [26], [27]. In Figure 2 we show the comparison of the 3, 4, and 5-term solutions with the exact solution $\rho(\tau)=\tau^2$.

Example 5. Consider the following third-order neutral FDEs with proportional delays:

$$\rho'''(\tau)=2\tau\rho'\left(\frac{\tau}{2}\right)-\frac{\tau^3}{6}\rho'''\left(\frac{\tau}{4}\right)+24\tau, \quad 0<\tau<1, \rho(0)=\rho'(0)=\rho''(0)=0. \quad (30)$$

Applying ST to (30), and using the properties of ST, we have:

$$\frac{\wp[\vartheta]}{\vartheta^3}=S\left[2\tau\rho'\left(\frac{\tau}{2}\right)-\frac{\tau^3}{6}\rho'''\left(\frac{\tau}{4}\right)+24\tau\right]. \quad (31)$$

Thus, $\wp[\vartheta]$ yields:

$$\wp[\vartheta]=\vartheta^3 S\left[2\tau\rho'\left(\frac{\tau}{2}\right)-\frac{\tau^3}{6}\rho'''\left(\frac{\tau}{4}\right)+24\tau\right]. \quad (32)$$

Operating inverse ST on (32), we obtain:

$$\rho(\tau) = \tau^4 + S^{-1}\left[\vartheta^3 \, S\left[2\tau\rho'^{\left(\frac{\tau}{2}\right)} - \frac{\tau^3}{6}\rho''^{\left(\frac{\tau}{4}\right)}\right]\right]. \tag{33}$$

Substituting (7) into (33) and equating the coefficients of p that have the same exponent leads to:
$p^0: \rho_0(\tau) = \tau^4,$

which is the analytical solution of (30).

Moreover, we demonstrate that the proposed method is quite simple and efficient in solving such problems. The graphical illustrations expose that the obtained results are extremely close, and in some cases are identical, to the exact results.

5 Conclusion

In this study, we confirm the capability of the SHPTM for solving neutral FDEs with proportional delays. This approach does not rely on integration or any assumptions in its formulation. The approach starts by first applying ST to the considered problem and using HPM to generate a series solution. This series yields accurate approximations, or in some cases exact solutions, with a few number of iterations. All the computations and the graphical illustration are made using *MATHEMATICA* 12. The proposed examples demonstrate that the results of the SHPTM agree excellently with the exact solution and with those of some other methods. Furthermore, the findings seem to indicate that the SHPTM is an efficient and convenient approach to approximate the solution of such problems. We expect that this method can be easily used as a viable alternative for various problems in science and engineering that lack exact solutions.

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