On Generic Automorphisms

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Abstract: In this article we investigate generic automorphisms of countable models. Hodges et al. 1993 introduces the notion of SI (small index) generic automorphisms. They used the existence of small index generics to show the small index property of the model. Truss 1989 defines the notion of Truss generic automorphisms. An automorphism f of M is called Truss generic if its conjugacy class is comeagre in the automorphism group of M. We study the relationship between these two types of generic automorphisms. We show that either the countable random graph or a countable arithmetically saturated model of True Arithmetic have both SI generic and Truss generic automorphisms. We prove that the dense linear order has the small index property and Truss-generic automorphisms but it does not have SI generic automorphisms. We also construct an example of a countable structure which has SI generics but it does not have Truss generics.

Key-Words: The small index property, countable models, dense linear order, generic automorphism, random graph, True Arithmetic, comeagre conjugacy class.

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1 Introduction

Throughout this paper M is to be a countable infinite model. We can consider its automorphism group $\mathrm{Aut}(M)$ as a topological group by letting the stabilizers of finite subsets of M be the basic open subgroups.

We will use the notation $G = \mathrm{Aut}(M)$. If $A \subset M$ we write

$$G_A = \{g \in G : g(a) = a \text{ for all } a \in A\},$$

so G_A is the pointwise stabilizer of A.

The paper [1] introduces the following definition of generic automorphism.

Definition 1. *[1] An automorphism f of M is called Truss generic if its conjugacy class $[f]_G = \{f^g = g^{-1}fg : g \in G\}$ is comeagre in G.*

A subgroup H of G is said to have *small index* in G if $|G : H| < 2^\omega$, and *large* otherwise. If $a \in M$, then there is a bijection between the set of the right cosets of G_a with $\{g(a) : g \in G\}$. Hence G_a, and so any open subgroup of G, has small index in G. We say that M has the *small index property* if the converse holds: that is, every subgroup $H \leq G$ of small index is open in G.

If G has the small index property, then the topology on G can be recovered from its abstract group structure. This has applications in reconstructing a structure from its automorphism group.

The paper [2] introduces the notion of SI generic automorphisms, which are used to show that M has the small index property.

Definition 2. *[2] A base for M is a set $\mathbb{B}(M)$ of subsets of M satisfying:*

1. *G_A is open in G for all $A \in \mathbb{B}(M)$,*
2. *if $A \in \mathbb{B}(M)$ and $g \in G$ then $g(A) \in \mathbb{B}(M)$.*

Definition 3. *[2] Let $\mathbb{B}(M)$ be a base for M, and let $0 < n < \omega$. We say that $(g_1, \ldots, g_n) \in G^n$ is $\mathbb{B}(M)$-automorphism, if the following hold.*

1. *If $A \in \mathbb{B}(M)$ then $\{G_B : A \subseteq B \in \mathbb{B}(M), g_i(B) = B \text{ for all } i \leq n\}$ is a base of open neighborhoods of 1 in G.*
2. *Let $A \in \mathbb{B}(M)$ be such that $g_i(A) = A, 1 \leq i \leq n$. Let $B \in \mathbb{B}(M), A \subseteq B$, and let $h_i \in \mathrm{Aut}(B)$ extend $g_i|_A$ $(1 \leq i \leq n)$. Then there is $\alpha \in G_A$ such that $g_i^\alpha = \alpha^{-1}g_i\alpha$ extends h_i $(1 \leq i \leq n)$.*

If it is needed one can add extra requirements on automorphisms $(g_1, \ldots, g_n)$ in Definition 3. For example, when working with models of Peano Arithmetic, [3] defines $(g_1, \ldots, g_n) \in G^n$ to be $\mathbb{B}(M)$-automorphism with additional conditions: $g_i|_B$ is existentially closed in the part 1 of the definition and $g_i|_A$ is existentially closed in the part 2.

We will need the following property of $\mathbb{B}(M)$-automorphisms.

Lemma 4. *[2] Let $\mathbb{B}(M)$ be a base for M, and let $(g_1, \ldots, g_n)$, $(h_1, \ldots, h_n)$ be $\mathbb{B}(M)$-automorphisms $(0 < n < \omega)$. Let $B \in \mathbb{B}(M)$ and suppose that $g_i|_B = h_i|_B$ for each $i \leq n$. Then there is $f \in G_B$ such that $g_i^f = h_i$ for all $i \leq n$.*

Definition 5. *[2] We say that M has SI generic automorphisms, if there exists a base $\mathbb{B}(M)$ for M such that for all non-zero $n < \omega$, the set of $\mathbb{B}(M)$-automorphisms of G^n is comeagre in G^n.*

Theorem 6. *[2] If M is a countable model with SI generic automorphisms, then M has the small index property.*

The paper [2] and [4] show existence of SI generic (and therefore the small index property) for the countable random graph and for ω-stable ω-categorical structures. The paper [3] is using SI generics to show that countable arithmetically saturated models of Peano Arithmetic have the small index property.

In [5] it is noticed that if M has both SI and Truss generic automorphisms, then they coincide (because they both are comeagre). Hence there are four possibilities for a countable model: to have both SI and Truss generics; to have SI but not Truss generics; to have Truss but not SI generics; and to have neither SI nor Truss generics.

The paper [2] shows that the countable random graph has SI generic automorphisms and [1] proves that the countable random graph has Truss generic automorphisms.

The paper [3] proves that if M is a countable arithmetically saturated model of Peano Arithmetic then M has SI generic automorphisms. The paper [5] shows that if M is a countable arithmetically saturated model of True Arithmetic then M has Truss generics. Hence we have the following result:

Example 7. *Let M be either the countable random graph or a countable arithmetically saturated model of True Arithmetic. Then M has both SI generic and Truss generic automorphisms.*

It is an open question whether there exists a countable arithmetically saturated model of Peano Arithmetic which does not have Truss generic automorphisms. The paper [5] shows that existence of Truss-generics for arithmetically saturated models of Peano Arithmetic is very closely tied to Hedetniemi's Conjecture – a well known open conjecture in the chromatic theory of graphs (see for example [6]).

In [7] the notion of SI generic automorphisms is introduced. They show that every uncountable saturated structure has SI generics.

2 Dense Linear Order

It is known that the countable model of the dense linear order without end-points has the small index property and has Truss generic automorphisms ([1], [8]). In this section we show that such model has no SI generic automorphisms.

Let $\langle \mathbb{Q}, < \rangle$ be the countable model of the dense linear order without end-points. It is not difficult to show the next lemma.

Lemma 8. *Let $B \subset \mathbb{Q}$ be finite and let $a \in \mathbb{Q} \setminus B$. Then there exists $f \in \mathrm{Aut}(\mathbb{Q})$ such that $f \in G_B$ and $f(a) \neq a$.*

The following result is not difficult either.

Lemma 9. *Let $B \subset \mathbb{Q}$ be such that G_B is open. Then B is finite.*

Proof. Assume not, that B is infinite. Since G_B is open there is a finite set $A \subset \mathbb{Q}$ such that G_A is a subgroup of G_B. Let $b \in B \setminus A$. By Lemma 8 there exists $f \in G_A$ such that $f(b) \neq b$. Hence $f \notin G_B$, but $f \in G_A$ which is contradiction. □

Corollary 10. *If $\mathbb{B}$ is a base for $\mathbb{Q}$ and if $B \in \mathbb{B}$ then B is finite.*

Proof. Definition 2 requires G_B to be open for every $B \in \mathbb{B}$. Thus by Lemma 9 every $B \in \mathbb{B}$ is finite. □

Lemma 11. *Let $B \subset \mathbb{Q}$ be finite and let $f \in G$ be such that $f(B) = B$. Then $f(b) = b$ for every $b \in B$.*

Proof. Assume that there is $b_0 \in B$ such that $f(b_0) \neq b_0$. Then there is a finite sequence $b_0, b_1, \ldots, b_{n-1} \in B$ such that

$$f(b_0) = b_1, f(b_1) = b_2, \ldots, f(b_{n-2}) = b_{n-1}, f(b_{n-1)} = b_0.$$

Now, if $b_0 < b_1$ then $b_0 < b_1 < b_2 < \ldots < b_{n-1} < b_0$ which is contradiction. The other case when $b_0 > b_1$ is similar. □

Statement 12. *$\langle \mathbb{Q}, < \rangle$ does not have SI generic automorphisms.*

Proof. Let $\mathbb{B}$ be a base for $\langle \mathbb{Q}, < \rangle$. Assume $f \in G$ is $\mathbb{B}$-generic and $f(A) = A$ for some $A \in \mathbb{B}$. By Definition 3

$$\mathbb{H} = \{G_B | A \subset B \in \mathbb{B}, f(B) = B\}$$

is a base of open neighborhoods of 1 in G. If $B \in \mathbb{B}$ and $f(B) = B$ then by Lemma 10 B is finite and by Lemma 11 $f \in G_B$. Hence

$$\mathbb{H} = \{G_B | A \subset B \in \mathbb{B}, f \in G_B\}.$$

Because $\mathbb{H}$ is a base of open neighborhoods of 1 in G: for every finite $C \subset \mathbb{Q}$ there exists $B \subset \mathbb{Q}$ such that $G_C \geq G_B \in \mathbb{H}$. Hence for every finite $C \subset \mathbb{Q}$ there exists $B \in \mathbb{B}$ such that $A, C \subseteq B, G_B \in \mathbb{H}$. Therefore

$$\bigcup_{A \subset B \in \mathbb{B}, f \in G_B} B = \mathbb{Q}.$$

Thus f is the identity map and the proposition follows. □

The papers [1], [8] prove the next result about the countable model of the dense linear order without end-points.

Statement 13. *[1], [8] $\langle \mathbb{Q}, < \rangle$ has the small index property and Truss generic automorphisms.*

By combining Statement 12 and Statement 13 we conclude.

Theorem 14. *$\langle \mathbb{Q}, < \rangle$ has the small index property and Truss-generic automorphisms but $\langle \mathbb{Q}, < \rangle$ does not have SI generic automorphisms.*

Notice that we define SI generic for n-tuples of automorphisms, $n < \omega$. Similarly, we can also say that an n-tuple $(g_1, g_2, ..., g_n) \in G^n$ is Truss generic if the set $\{(g_1^f, g_2^f, \ldots, g_n^f) : f \in G\}$ of its conjugates is a comeager in G^n. An unpublished result of I.Hodkinson shows that $\langle \mathbb{Q}, < \rangle$ has no Truss generic 2-tuples of automorphisms, by Statement 12 it does not have SI generic 2-tuples of automorphisms either.

3 A Few Examples

As we have mentioned earlier, there are four possibilities for a countable model: to have both SI and Truss generics; to have SI but not Truss generics; to have Truss but not SI generics; and to have neither SI nor Truss generics. We have already found models with both SI and Truss generics: the countable random graph or a countable arithmetically saturated model of True Arithmetic (Example 7). In the previous chapter we have shown that $\langle \mathbb{Q}, < \rangle$ has Truss-generic automorphisms but $\langle \mathbb{Q}, < \rangle$ does not have SI generic automorphisms. In this chapter we address the other two remaining possibilities.

Let us consider a model $\mathbb{Q} \times \{0, 1\}$ where each copy of $\mathbb{Q}$ is the countable model of the dense linear order without end-points. An automorphism of $\mathbb{Q} \times \{0, 1\}$ preserves the partition $\{\mathbb{Q} \times \{0\}, \mathbb{Q} \times \{1\}\}$ and is order preserving on each copy of $\mathbb{Q}$. Then every automorphism of $\mathbb{Q} \times \{0, 1\}$ either fixes each copy of $\mathbb{Q}$ setwise or sends $\mathbb{Q} \times \{0\}$ to $\mathbb{Q} \times \{1\}$. We notice that $\mathbb{Q} \times \{0, 1\}$ cannot have Truss generics. By a similar argument as in the previous section we could show $\mathbb{Q} \times \{0, 1\}$ has no SI generics. Therefore we have an example of structure with neither Truss nor SI generics.

Example 15. *$\mathbb{Q} \times \{0, 1\}$ has neither Truss nor SI generic automorphisms.*

Notice that in Example 15 instead of $\mathbb{Q} \times \{0, 1\}$ we might have considered $\mathbb{Q} \times \{0, 1, \ldots, n-1\}, n < \omega$. By a similar argument $\mathbb{Q} \times \{0, 1, \ldots, n-1\}, n < \omega$ does not have Truss generics. On the other hand $\mathbb{Q} \times \omega$ behaves differently, $\mathbb{Q} \times \omega$ has Truss generics. See [1] for the details.

Our next goal is to find a model with SI generics but without Truss generics. We need the following result.

Theorem 16. *[2] Let M be a countable ω-stable ω-categorical structure. Then M has SI generics.*

Example 17. *Let M be a countable ω-stable ω-categorical structure. Then $M \times \{0, 1\}$ has SI generics but it does not have Truss generics.*

Proof. Let M be a countable ω-stable ω-categorical structure. Since M is countable ω-stable ω-categorical then $M \times \{0, 1\}$ is also countable ω-stable ω-categorical. By Theorem 16 $M \times \{0, 1\}$ has SI generics. An automorphism of $M \times \{0, 1\}$ preserves the partition $\{M \times \{0\}, M \times \{1\}\}$. Thus every automorphism of $M \times \{0, 1\}$ either fixes each copy of M setwise or sends $M \times \{0\}$ to $M \times \{1\}$. We notice that $M \times \{0, 1\}$ cannot have Truss generics. □

[2] and [4] show that the countable random graph has an amalgamation base. [2] proves that if M is countable ω-categorical with an amalgamation base then it has SI generics. Similar argument might be applied to $R \times \{0, 1\}$. It can be shown that the base consisting of finite sets is an amalgamation base for $R \times \{0, 1\}$. Since $R \times \{0, 1\}$ is ω-categorical, then, using the mentioned result from [2], $R \times \{0, 1\}$ has SI generics. $R \times \{0, 1\}$ does not have Truss generics by the same argument as before. Hence we obtain the following result.

Example 18. *Let R be the countable random graph. Then $R \times \{0, 1\}$ has SI generics but it does not have Truss generics.*

4 Conclusion

Two types of generic automorphisms are defined somewhat similar. Indeed if a model has both of SI generics and Truss generics, they coincide. If M is either the countable random graph or a countable arithmetically saturated model of True Arithmetic, then M has both SI generic and Truss generic automorphisms. We prove that the dense linear order has the small index property and Truss-generic automorphisms but it does not have SI generic automorphisms. We also construct examples: a countable structures which has SI generics but it does not have Truss generics; a countable structure which has no SI generics but it has Truss generics; a countable structures which has neither SI generics nor Truss generics.

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